Math 52H Homework 6 Solutions

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1. a. The integral is

$$\begin{aligned}
\int_0^1 \int_{2y}^{y+1} x - y dx dy &= \int_0^1 \frac{(y+1)^2}{2} - (y+1)y - \frac{(2y)^2}{2} + 2y^2 dy \\
&= \int_0^1 \frac{-y^2+1}{2} dy \\
&= 1/2 - 1/6 \\
&= 1/3.
\end{aligned}$$

For a, b, c either 0 or 1, the rest of the questions all integrate $\int_B x^a y^b z^c dV = \int_0^1 \int_0^1 \int_0^{xy} x^a y^b z^c dz dx dy = \int_0^1 \int_0^1 x^a y^b (xy)^{c+1}/(c+1) dx dy = \frac{1}{(c+1)(a+c+2)(b+c+2)}$. The numerical answers follow from the above formula, and are (in order) $1/6, 1/6, 1/18$, and $1/9$.

2. a. Using polar coordinates, the given bound translates to $r^2 \leq 2r\cos\theta$, whence $0 \leq r \leq 2\cos\theta$. Note that this implies that $\cos\theta \geq 0$, so that $-\pi/2 \leq \theta \leq \pi/2$. (Alternatively, you can complete the square to see that the equation of the circle is $(x-1)^2 + y^2 = 1$, which is on the right side of the y-axis.) We get that the integral is

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^3 dr d\theta &= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta \\
&= 4\int_{-\pi/2}^{\pi/2} \cos^2\theta(1 - \sin^2\theta) d\theta \\
&= 4\int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta + 1}{2} d\theta - \int_{-\pi/2}^{\pi/2} \sin^2 2\theta d\theta \\
&= 2\pi - \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 4\theta}{2} d\theta \\
&= \frac{3\pi}{2}.
\end{aligned}$$

b. Change coordinates to $u = x/a$ and $v = y/b$ which has Jacobian ab. The integral then becomes $ab \int_D \sqrt{1-u^2-v^2}dV$ where $D = \{u^2+v^2 \leq 1\}$. Now change to polar coordinates to get

$$\begin{aligned} ab\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}rdrd\theta &= ab\int_0^{2\pi} 1d\theta[-1/3(1-r^2)^{3/2}]_0^1 \\ &= 2ab\pi/3. \end{aligned}$$

3. Let the region in question be D. We calculate $\frac{\partial u}{\partial x} = \frac{2x}{y}$, $\frac{\partial u}{\partial y} = -\frac{x^2}{y^2}$, $\frac{\partial v}{\partial x} = -\frac{y^2}{x^2}$, and $\frac{\partial v}{\partial y} = \frac{2y}{x}$. The Jacobian determinant is $\frac{\partial(u,v)}{\partial(x,y)} = 3$. Thus we have $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}$. The bounds given in the question translate to $a \leq u \leq b$, and $\alpha \leq v \leq \beta$.

Thus, the area of D is

$$\int_D 1dV = \frac{1}{3}\int_a^b\int_\alpha^\beta 1dvdu = \frac{(b-a)(\beta-\alpha)}{3}.$$

4. Use cylindrical coordinates, so that the hyperboloid becomes $r^2 - z^2 = a^2$, and the region bounded by the sphere is described by $r^2 + z^2 \leq 3a^2$. We calculate the volume of the region given by $r^2 - z^2 \geq a^2$. These bounds imply that $a \leq r \leq \sqrt{3}a$. We divide this into two regions. Let A_1 denote the area of region given by $a \leq r \leq \sqrt{2}a$, and A_2 denote the area of the region given by $\sqrt{2}a \leq r \leq \sqrt{3}a$. Then, by symmetry

$$\begin{aligned} \frac{A_1}{2} &= \int_0^{2\pi}\int_a^{\sqrt{2}a}\int_0^{\sqrt{r^2-a^2}} rdzdrd\theta \\ &= 2\pi\int_a^{\sqrt{2}a} \sqrt{r^2-a^2}rdr \\ &= 2\pi[(r^2-a^2)^{3/2}/3]_a^{\sqrt{2}a} \\ &= \frac{2\pi a^3}{3}. \end{aligned}$$

Similarly,

$$\begin{aligned}\frac{A_2}{2} &= \int_0^{2\pi}\int_{\sqrt{2}a}^{\sqrt{3}a}\int_0^{\sqrt{3a^2-r^2}} r dz dr d\theta \\ &= 2\pi\int_{\sqrt{2}a}^{\sqrt{3}a}\sqrt{3a^2-r^2} r dr \\ &= 2\pi[-(3a^2-r^2)^{3/2}/3]_{\sqrt{2}a}^{\sqrt{3}a} \\ &= \frac{2\pi a^3}{3}.\end{aligned}$$

As a fraction of the total volume of the ball, this is

$$\frac{A_1+A_2}{4/3\pi(\sqrt{3}a)^3} = \frac{2}{3\sqrt{3}}.$$

The ratio is therefore $2 : (3\sqrt{3}-2)$.

5. We prove the contrapositive: Assume that Vol $A > 0$ and that A is Riemann measurable. We show that Int A is nonempty. From our definition of interior, it suffices to show that there is some open set contained in A. Because the appropriate context for Riemann integration is the real line, we assume for this problem that $A \subset \mathbb{R}$. A set is Riemann measurable if the indicator function $\chi_A(x)$ is Riemann integrable. Let $\mathcal{P}$ be any partition of A. Then we have that $\sup_{\mathcal{P}} \mathcal{L}(\chi_A, \mathcal{P}) = \inf_{\mathcal{P}} \mathcal{U}(\chi_A, \mathcal{P}) > 0$. So, this says that there exists a partition $\mathcal{P}_0$ so that

$$\sum_{\mathcal{P}} \inf_{x_i \in [t_i, t_{i+1}]} \chi_A(x_i)(t_{i+1}-t_i) > 0.$$

Thus, by positivity, there must be some closed interval $[t_i, t_{i+1}]$ for which $\inf_{x_i \in [t_i, t_{i+1}]} \chi_A(x_i)$, i.e. $[t_i, t_{i+1}] \subset A$. So A contains the open set (t_i, t_{i+1}). So it has nonempty interior, and is not nowhere dense. Thus we have shown the first part.

Secondly, take a "positive measure cantor set", say $\mathcal{K}$ as an example. To construct $\mathcal{K}$, take $\mathcal{K}_1$ to be $[0, 1/3] \cup [2/3, 1]$, i.e. remove the middle 1/3. At the second stage, we remove the middle 1/9:

$$\mathcal{K}_2 = [0, 4/27] \cup [5/27, 9/27] \cup [18/27, 22/27] \cup [23/27, 1].$$

At the $\mathcal{K}_3$ stage, remove the middle 1/27 of every remaining interval.

Similarly to the usual Cantor set, $\mathcal{K}$ is an intersection of closed sets, and hence is closed. Thus $\mathcal{K}$ is its own closure. Suppose $\mathcal{K}$ contained some open interval U. Then U must be contained in every $\mathcal{K}_n$. But U is connected, so must lie entirely in only one of the connected components of $\mathcal{K}_n$ for any n. But any open interval has some length ℓ. Take n so large so that

$$\prod_{i=1}^{n}\left(1-\frac{1}{3^i}\right)/2^n < \ell,$$

i.e. each of the subintervals of $\mathcal{K}_n$ are shorter than U. Contradiction. Thus $\mathcal{K}$ contains no open sets, and hence is nowhere dense.

Now we show that $\mathcal{K}$ is not Riemann measurable. On the one hand, $\mathcal{K}$ contains no open intervals, hence contains no closed intervals, hence

$$\mathcal{L}(\chi_{\mathcal{K}},\mathcal{P}) := \sum_{\mathcal{P}} \inf_{x_i\in[t_i,t_{i+1}]} \chi_A(x_i)(t_{i+1}-t_i) = 0$$

for all partitions! So we have that $\sup_{\mathcal{P}} \mathcal{L}(\chi_{\mathcal{K}},\mathcal{P}) \leq 0$. On the other hand, we see that the total length of $\mathcal{K}$ is given by

$$\prod_{i=1}^{\infty}\left(1-\frac{1}{3^i}\right) > 0,$$

so that

$$\mathcal{U}(\chi_{\mathcal{K}},\mathcal{P}) := \sum_{\mathcal{P}} \sup_{x_i\in[t_i,t_{i+1}]} \chi_A(x_i)(t_{i+1}-t_i) \geq \prod_{i=1}^{\infty}\left(1-\frac{1}{3^i}\right)$$

for all partitions. Hence, we have that $\inf_{\mathcal{P}} \mathcal{U}(\chi_{\mathcal{K}},\mathcal{P}) \geq \prod_{i=1}^{\infty}\left(1-\frac{1}{3^i}\right)$. Thus, $\chi_{\mathcal{K}}$ is not Riemann integrable, i.e. $\mathcal{K}$ is not Riemann measurable.