Math 52H Homework 5 Solutions

February 15, 2012

1. Let $\alpha = e^{ix}$, $\beta = e^{-ix}$ and observe that we have $\alpha\beta = 1$. Then

$$\begin{aligned}\frac{\sin nx}{\sin x} = \frac{e^{inx} - e^{-inx}}{e^{ix} - e^{-ix}} = \frac{\alpha^n - \beta^n}{\alpha - \beta} &= \alpha^{n-1} + \alpha^{n-2}\beta + \alpha^{n-3}\beta^2 + \cdots + \alpha\beta^{n-2} + \beta^{n-1} \\ &= \alpha^{n-1} + \alpha^{n-3} + \cdots + \alpha^{-(n-1)},\end{aligned}$$

so that there are two cases:

$$\frac{\sin nx}{\sin x} = \begin{cases} 2\left(\cos(n-1)x + \cos(n-3)x + \cdots + \cos x\right) & n \text{ even} \\ 2\left(\cos(n-1)x + \cos(n-3)x + \cdots + \cos 2x\right) + 1 & n \text{ odd.} \end{cases}$$

2.a. We have

$$\begin{aligned}\sum_{k=0}^{n} \cos k\theta = \operatorname{Re} \sum_{k=0}^{n} e^{ik\theta} &= \operatorname{Re} \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \\ &= \operatorname{Re} \frac{e^{i(n+1)\theta/2}}{e^{i\theta/2}} \frac{e^{i(n+1)\theta/2} - e^{-i(n+1)\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} \\ &= \frac{\sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}} \operatorname{Re} e^{in\theta/2} \\ &= \frac{\sin \frac{(n+1)\theta}{2} \cos \frac{n\theta}{2}}{\sin \frac{\theta}{2}}.\end{aligned}$$

Likewise using imaginary parts, one derives

$$\sum_{k=1}^{n} \sin k\theta = \frac{\sin \frac{(n+1)\theta}{2} \sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}}.$$

b. This is very similar to what we did in class on monday. Recall that the binomial theorem works for complex numbers. We find

$$(1+1)^n = 1 + \binom{n}{1} + \binom{n}{2} + \cdots$$

$$(1-1)^n = 1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots,$$

so that

$$2^{n-1} = 1 + \binom{n}{2} + \binom{n}{4} + \cdots.$$

Likewise, we have

$$(1+i)^n = 1 + \binom{n}{1}i - \binom{n}{2} - \binom{n}{3}i + \binom{n}{4} + \cdots$$

and

$$(1-i)^n = 1 - \binom{n}{1}i - \binom{n}{2} + \binom{n}{3}i + \cdots$$

from which we find

$$\frac{(1+i)^n + (1-i)^n}{2} = 1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots.$$

Summing again we have

$$\frac{(1+i)^n + (1-i)^n + 2^n}{4} = 1 + \binom{n}{4} + \binom{n}{8} + \binom{n}{12} + \cdots.$$

Using $re^{i\theta}$ form of complex numbers we find that

$$(1+i)^n = \sqrt{2}^n e^{i\pi n/4},$$

and

$$(1-i)^n = \sqrt{2}^n e^{-i\pi n/4},$$

so that we have

$$2^{\frac{n}{2}-2}\cos\frac{\pi}{4}n + 2^{n-2} = 2^{\frac{n}{2}-2}\left(2^{\frac{n}{2}} + \cos\frac{\pi}{4}n\right) = 1 + \binom{n}{4} + \binom{n}{8} + \binom{n}{12} + \cdots.$$

3.a. Write $\alpha = \frac{xdx+ydy+zdz}{(x^2+y^2+z^2)^{3/2}}$ and let $G : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be defined by $G(x,y,z) = -(x^2+y^2+z^2)^{\frac{-1}{2}}$. Then

$$dG = -(x^2+y^2+z^2)^{\frac{-3}{2}}(\frac{-1}{2})(2xdx + 2ydy + 2zdz) = \alpha$$

so α is exact on $\mathbb{R}^3 \setminus \{0\}$, which contains the curve γ. It follows that

$$\int_\gamma \alpha = G(\gamma(1)) - G(\gamma(0)) = (\frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{3}}).$$

b. Write $\alpha = \frac{xdy - ydx}{x^2+y^2}$. On the set $\{(x,y) : x > 0\}$ we have $\alpha = d\arctan(\frac{y}{x})$, so γ is exact on this domain. Since $\gamma_x(t) = 1 + \frac{1}{2}\sin\frac{t^2}{\pi} > 0$ for all t, the curve γ is contained in $\{(x,y) : x > 0\}$. Hence

$$\int_\gamma \alpha = \arctan(\frac{\gamma_y(\pi)}{\gamma_x(\pi)}) - \arctan(\frac{\gamma_y(0)}{\gamma_x(0)}) = 0 - \frac{\pi}{4} = \frac{-\pi}{4}.$$

4. a. Write $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$. Assume that there is a primitive F. If we integrate α_1 with respect to x, we get

$$F = x^3/3 - x^2/2 + xy + xz - xyz + f(y,z),$$

for some function $f(y,z)$ depending only on y and z. Differentiate this w.r.t. y and set it equal to α_2 to get

$$x - xz + \frac{\partial f}{\partial y} = y^2 + x - y + z - xz,$$

which is equivalent to $f = y^3/3 - y^2/2 + yz + g(z)$ for some function g depending only on z. Now differentiate the expression we have for F w.r.t. z and set equal to α_3 to get

$$x - xy + y + g'(z) = z^2 + x + y - z - xy,$$

whence $g(z) = z^3/3 - z^2/2 + C$ for some constant C. We thus have

$$F(x,y,z) = \frac{x^3+y^3+z^3}{3} - \frac{x^2+y^2+z^2}{2} + xy + xz + yz - xyz + C.$$

Remark: That we were able to find a solution to the equations above implies that α is exact with F as its primitive. In fact, the equations we solved above imply that $dF = \alpha$. Also, note that finding a primitive for an exact 1-form is the same as finding a potential function for a vector field. Finally, we remark here that one can guess the form of F very easily by symmetry.

b. Since $\alpha = dF$, we have that

$$\int_\gamma \alpha = F(\gamma(\pi)) - F(\gamma(0)) = 0,$$

since $\gamma(\pi) = \gamma(0)$. In other words, γ is a closed path, and so the integral of the exact form α along γ is 0.