Math 52H Homework 2 Solutions

January 26, 2012

1. (a). We can check this straight from the definitions. Let $\alpha \in \Lambda^k(V^*)$ be a k-form on V. Let $U_1, \ldots, U_{n-k}$ be any vectors. First, assume that they are linearly dependent. Then clearly $(\mathcal{A}^* \circ \star)\alpha(U_1, \ldots, U_{n-k}) = 0$ because $\star\alpha(U_1, \ldots, U_{n-k}) = 0$. But if we call $\omega = \mathcal{A}^*\alpha$, then $\star \circ \mathcal{A}^*\alpha(U_1, \ldots, U_{n-k}) = \star\omega(U_1, \ldots, U_{n-k}) = 0$ by definition of the Hodge-$\star$ operator. So we have shown the identity in the case that the vectors are linearly dependent.

Now assume $U_1, \ldots, U_{n-k}$ are linearly independent. Let $Z_1, \ldots, Z_k$ be the complementary vectors in the sense of Definition 4.1 in the notes. Then $\mathcal{A}^* \circ \star\alpha(U_1, \ldots, U_{n-k}) = \star\alpha(\mathcal{A}U_1, \ldots, \mathcal{A}U_{n-k})$. Now, because $\mathcal{A}$ is special orthogonal, we have that $\mathcal{A}Z_1, \ldots, \mathcal{A}Z_k$ is a basis of $\mathrm{span}(\mathcal{A}U_1, \ldots, \mathcal{A}U_{n-k})^\perp$ with the same volume and the same orientation. Thus

$$\mathcal{A}^* \circ \star\alpha(U_1, \ldots, U_{n-k}) = \star\alpha(\mathcal{A}U_1, \ldots, \mathcal{A}U_{n-k}) = \alpha(\mathcal{A}Z_1, \ldots, \mathcal{A}Z_k).$$

On the other hand, if we call $\omega := \mathcal{A}^*\alpha$, then

$$\star \circ \mathcal{A}^*\alpha(U_1, \ldots, U_{n-k}) = \star\omega(U_1, \ldots, U_{n-k}) = \omega(Z_1, \ldots, Z_k) = \alpha(\mathcal{A}Z_1, \ldots, \mathcal{A}Z_k).$$

(b). Choose an orthonormal basis $e_1, \ldots, e_n$ and apply the previous to the k-form $x_{i_1} \wedge \cdots \wedge x_{i_k}$. Let $j_1, \ldots j_{n-k}$ be some choice of $n-k$ indices, and let $\widehat{j}_1, \ldots, \widehat{j}_k$ be the complementary k indices. Likewise, let $\widehat{i}_1, \ldots, \widehat{i}_{n-k}$ be the complementary indices to the is. Then by the previous part,

$$(\mathcal{A}^* \circ \star)x_{i_1} \wedge \cdots \wedge x_{i_k}(e_{j_1}, \ldots e_{j_{n-k}}) = x_{i_1} \wedge \cdots \wedge x_{i_k}(Ae_{\widehat{j}_1}, \ldots, Ae_{\widehat{j}_k}),$$

which by, say, Prop 2.17 from the course notes is the minor of A given by the $i_1, \ldots, i_k$ and the $\widehat{j}_1, \ldots, \widehat{j}_k$. On the other hand, this is equal to

$$\begin{aligned}(\star \circ \mathcal{A}^*)x_{i_1} \wedge \cdots \wedge x_{i_k}(e_{j_1}, \ldots, e_{j_{n-k}}) &= \star x_{i_1} \wedge \cdots \wedge x_{i_k}(Ae_{j_1}, \ldots, Ae_{j_{n-k}}) \\ &= (-1)^{\mathrm{inv}(i_1,\ldots,i_k,\widehat{i}_1,\ldots,\widehat{i}_{n-k})} x_{\widehat{i}_1} \wedge \cdots \wedge x_{\widehat{i}_{n-k}}(Ae_{j_1}, \ldots, Ae_{j_{n-k}}),\end{aligned}$$

which is up to a ± 1 the minor of A given by the $\widehat{i}_1, \ldots, \widehat{i}_k$, and the $j_1, \ldots, j_{n-k}$. So the two minors are the same in absolute value.

2.

b. Let $e_1, ..., e_{2n}$ be basis vectors for $\mathbb{R}^{2n}$, where e_{2k-1} corresponds to x_k and e_{2k} corresponds to y_k for $1 \le k \le n$. We first find the matrix corresponding to the skew-symmetric bilinear form ω. Say that $i \le j$. Then, we have that $\omega(e_i, e_j) = 0$ unless $i = 2k-1$ and $j = 2k$ for some k in which case $\omega(e_i, e_j) = 1$. By skew-symmetry of ω, the matrix corresponding to ω is $-J$. Now $\mathcal{A}^*\omega = \omega$ implies that for $u, v \in \mathbb{R}^{2n}$ that

$$u^t(-J)v = u^t A^t(-J)Av,$$

which immediately gives $J = A^t JA$, as desired.

c. Let $\mathcal{U}$ be an orthogonal operator with matrix U. Then $\mathcal{U}$ is unitary $\Leftrightarrow UJ = JU \Leftrightarrow J = U^{-1}JU = U^tJU \Leftrightarrow \mathcal{U}$ is symplectic.

d. L is Lagrangian is equivalent to $0 = \omega(u, v) = -u^tJv$ for all $u, v \in L \Leftrightarrow u$ is orthogonal to Jv for all $u, v \in L \Leftrightarrow J(L) = L^\perp$. The last part follows since J is one to one, and L is dimension n so $J(L)$ is a dimension n subspace orthogonal to L.

3. Consider the linear operator $L = \frac{1}{2}(\star + I) : \Lambda^2((\mathbb{R}^4)^*) \to \Lambda^2((\mathbb{R}^4)^*)$; the two-form β is self-dual if and only if $L\beta = \beta$.

Since $\star^2 = I$ is the identity, $L^2 = \frac{1}{4}(\star^2 + 2\star + I) = \frac{1}{2}(\star + I) = L$ so L is a projection, that is,

$$\{v : Lv = v\} = \mathrm{im}(L).$$

We have

$$L(x_1 \wedge x_2) = \frac{1}{2}(x_1 \wedge x_2 + x_3 \wedge x_4) = L(x_3 \wedge x_4), \quad L(x_1 \wedge x_3) = \frac{1}{2}(x_1 \wedge x_3 - x_2 \wedge x_4) = -L(x_2 \wedge x_4)$$

$$L(x_1 \wedge x_4) = \frac{1}{2}(x_1 \wedge x_4 + x_2 \wedge x_3) = L(x_2 \wedge x_3),$$

so the space of self-dual forms is 3-dimensional, with basis

$$\{\frac{1}{2}(x_1 \wedge x_2 + x_3 \wedge x_4), \frac{1}{2}(x_1 \wedge x_3 - x_2 \wedge x_4), \frac{1}{2}(x_1 \wedge x_4 + x_2 \wedge x_3)\}.$$

4. Let $Y = \alpha X + \beta Z$ where $\langle X, Z\rangle = 0$. Then $X \times Y = \beta X \times Z$, since $X \times X = 0$ whereas $\mathcal{D}^{-1}(\star(\mathcal{D}(X) \wedge \mathcal{D}(Y))) = \beta \mathcal{D}^{-1}(\star(\mathcal{D}(X) \wedge \mathcal{D}(Z)))$ since $\mathcal{D}(X) \wedge \mathcal{D}(X) = 0$. It thus suffices to check that

$$X \times Z = \mathcal{D}^{-1}(\star(\mathcal{D}(X) \wedge \mathcal{D}(Z))).$$

We may assume that $X, Z \neq 0$, and by dividing the above by $||X||||Z||$, we may further assume that X and Z are orthonormal. Let X, Z, W be an orthonormal basis for $\mathbb{R}^3$, where $W = X \times Z$, so that the basis defines the standard orientation for $\mathbb{R}^3$. Note that $\mathcal{D}(X), \mathcal{D}(Z), \mathcal{D}(W)$ forms the dual basis. By Lemma 4.3 in the notes, we have that $\star(\mathcal{D}(X) \wedge \mathcal{D}(Z)) = \mathcal{D}(W)$, and so $\mathcal{D}^{-1}(\star(\mathcal{D}(X) \wedge \mathcal{D}(Z))) = W = X \times Z$ as desired.

5. We first prove the following lemma.

Lemma 1. *For any two $n \times n$ matrices A and B such that $AB = BA$, we have that*

$$\exp(A + B) = \exp(A)\exp(B).$$

Proof.

$$\begin{aligned}
\exp(A+B) &= \sum_{k\geq 0} \frac{(A+B)^k}{k!} \\
&= \sum_{j\geq 0} \sum_{m=0}^{k} \frac{A^m B^{j-m}}{m!(j-m)!} \\
&= \sum_{m,n} \frac{A^m}{m!}\frac{B^n}{n!},
\end{aligned}$$

as desired. In the above, we have used that $AB = BA$ in our binomial expansion. □

Next note that for any $n \times n$ matrix M that $\exp(M)^T = \left(\sum_{k\geq 0} \frac{M^k}{k!}\right)^T = \sum_{k\geq 0} \frac{(M^T)^k}{k!} = \exp(M^T)$. Now let A be skew-symmetric so $A^T = -A$. Then $A^T A = AA^T$ so applying the above and the Lemma gives

$$\exp(A)\exp(A)^T = \exp(A)\exp(A^T) = \exp(A + A^T) = \exp(0) = I,$$

so $\exp(A)$ is orthogonal.

Conversely, say that $\exp(tA)$ is orthogonal for all t. Then

$$\begin{aligned}
I &= \exp(tA)\exp(tA^T) \\
&= \left(\sum_{k\geq 0} \frac{t^k A^k}{k!}\right)\left(\sum_{k\geq 0} \frac{t^k (A^T)^k}{k!}\right) \\
&= I + t(A + A^T) + \frac{t^2}{2}(2AA^T + A^2 + (A^T)^2) + \ldots
\end{aligned}$$

Apply $\frac{d}{dt}\big|_{t=0}$ to the equation above. The left hand side is 0, whereas the right hand side is $(A + A^T + 2t(2AA^T + A^2 + (A^T)^2) + \ldots)\big|_{t=0} = A + A^T$. Hence $A = -A^T$, as desired.