Math 52H Homework 1 Solutions

January 19, 2012

1. To verify that $\langle \cdot, \cdot \rangle$ is an inner product we check the axioms. First, $\langle X, Y \rangle = \mathrm{Tr}(XY) = \mathrm{Tr}(YX) = \langle Y, X \rangle$. Also $\langle \alpha X + Y, Z \rangle = \mathrm{Tr}((\alpha X + Y)Z) = \alpha \mathrm{Tr}(XZ) + \mathrm{Tr}(YZ) = \alpha \langle X, Z \rangle + \langle Y, Z \rangle$. Finally, for X symmetric, recall that the trace is invariant under change of basis. Because X is symmetric, X is diagonalizable over $\mathbb{R}$ with real eigenvalues λ_1 and λ_2. Thus $\langle X, X \rangle = \mathrm{Tr}\langle X^2 \rangle = \lambda_1^2 + \lambda_2^2 \geq 0$. Equality holds if and only if $\lambda_1 = \lambda_2 = 0$ which is equivalent to $X = 0$.

Let $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $A_3 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It is easy to check that $A_i A_j$ has zeros on the diagonal unless $i = j$. Moreover $\mathrm{Tr}(A_i^2) = 1$ if $i = 1, 3$ and $\mathrm{Tr}(A_2^2) = 2$. Thus the matrix for the biliear form is $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

2. Pick the standard basis $\{e_1, \ldots, e_n\}$ for $V = \mathbb{R}^n$. The basis of V^* is then given by the coordinate functions $x_1, \ldots, x_n$. The columns of the matrix of $\mathcal{D}$ are then given by expanding $\mathcal{D}e_i$ in the basis $x_1, \ldots, x_n$. We have as a function of x, $\mathcal{D}e_i(x) = \langle e_i, x \rangle$. Expanded in the basis for V, $e_i = (0, \ldots, 1, \ldots, 0)$, with the 1 appearing in the i place, and in the basis for V^*, $x = (x_1, \ldots, x_n)$. Then

$$\mathcal{D}e_i(x) = \langle e_i, x \rangle = i \cdot 1 \cdot x_i.$$

So the i-th column of the matrix of $\mathcal{D}$ with respect to these bases is 0 everywhere, except for an i in the i-th place. Hence the matrix is

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & n \end{pmatrix}.$$

3. Let $\{e_1, ..., e_n\}$ be the usual basis for $\mathbb{R}^n$, dual to $x_1, ..., x_n$. We have

$$f \otimes g = \sum_{1 \leq i_1, i_2, i_3, i_4 \leq n} f \otimes g(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}) x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4}$$

[Proof: Let

$$H = \sum_{1 \le i_1, i_2, i_3, i_4 \le n} f \otimes g(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}) x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4}.$$

Then $H(e_{j_1}, e_{j_2}, e_{j_3}, e_{j_4}) = f \otimes g(e_{j_1}, e_{j_2}, e_{j_3}, e_{j_4})$. Equality at general v_1, v_2, v_3, v_4 then follows by multilinearity of both H and $f \otimes g$.]

But

$$f \otimes g(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}) = f(e_{i_1}, e_{i_2}) g(e_{i_3}, e_{i_4}) = a_{i_1 i_2} b_{i_3 i_4}$$

so

$$f \otimes g = \sum_{1 \le i_1, i_2, i_3, i_4 \le n} a_{i_1 i_2} b_{i_3 i_4} x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4}.$$

4. (Assume $\beta \neq 0$.) Let $\alpha = a_1 x_2 \wedge x_3 + a_2 x_3 \wedge x_1 + a_3 x_1 \wedge x_2$ and $\beta = b_1 x_1 + b_2 x_2 + b_3 x_3$. Then $\alpha \wedge \beta = (a_1 b_1 + a_2 b_2 + a_3 b_3) x_1 \wedge x_2 \wedge x_3$ so

$$\alpha \wedge \beta = 0 \quad \Leftrightarrow \quad \langle a, b \rangle = 0,$$

where we consider a, b as elements of $\mathbb{R}^3$, and $\langle \ , \ \rangle$ is the standard dot product. This is to say that a and b are orthogonal vectors in $\mathbb{R}^3$, so we can use the standard cross product to construct a c so that $a = b \times c$ for some $c \in \mathbb{R}^3$. Calling $c = (c_1, c_2, c_3)$, and writing out the formulae for the cross product, we have

$$a_1 = b_2 c_3 - b_3 c_2, \quad a_2 = b_3 c_1 - b_1 c_3, \quad a_3 = b_1 c_2 - b_2 c_1$$

which says exactly that if $\gamma := c_1 x_1 + c_2 x_2 + c_3 x_3$ then

$$\alpha = (b_1 x_1 + b_2 x_2 + b_3 x_3) \wedge (c_1 x_1 + c_2 x_2 + c_3 x_3) = \beta \wedge \gamma.$$

5. We have $\theta = \sum_{i=1}^{n-1} x_i \otimes x_{i+1} - \sum_{i=1}^{n-1} x_{i+1} \otimes x_i$ so

$$\begin{aligned} \theta(A, B) &= \sum_{i=1}^{n-1} A_i B_{i+1} - \sum_{i=1}^{n-1} A_{i+1} B_i \\ &= \sum_{i=1}^{n-1} (-1)^{i+1} - \sum_{i=1}^{n-1} (-1)^i = \left\{ \begin{array}{ll} 2 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{array} \right. . \end{aligned}$$

6. a. First think about what ω is: if $\alpha, \beta \in \mathbb{R}^{2n}$, then ω is the function of two vector inputs given by

$$\omega(\alpha, \beta) = \sum_{i=1}^{n} x_i(\alpha) y_i(\beta) - y_i(\alpha) x_i(\beta).$$

Now, note that $(L_A)^* : \Lambda^2((\mathbb{R}^{2n})^*) \to \Lambda^2((\mathbb{R}^n)^*)$, so that for any two $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n) \in \mathbb{R}^n$,

$$\begin{aligned}(L_A)^*\omega(v,w) &= \omega(L_A v, L_A w) = \omega((v, Av), (w, Aw)) \\ &= \sum_{i=1}^n x_i((v, Av))y_i((w, Aw)) - y_i((v, Av))x_i((w, Aw)) \\ &= \sum_{i=1}^n v_i \sum_{j=1}^n a_{ij} w_j - w_i \sum_{j=1}^n a_{ij} v_j.\end{aligned}$$

Basically, this says that if we call the bilinear form associated to A by $f_A(v,w)$, then $(L_A)^*\omega(v,w) = f_A(v,w) - f_A(w,v)$. This is 0 iff $f_A(v,w) = f_A(w,v)$ for all v, w, i.e. iff A is a symmetric matrix.

b. Call the standard basis for V by $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ and the dual basis for V^* by $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ as in part a. The first n columns of the matrix of $\mathcal{C}_\omega$ are given by expanding $\mathcal{C}_\omega(e_k)$ in the basis $x_1, \dots, y_n$, and the second n columns are given by expanding $\mathcal{C}_\omega(f_k)$. If $Z = (z_1, \dots, z_{2n})$ is an arbitrary element of V, we have

$$\mathcal{C}_\omega(e_k)(Z) = \omega(e_k, Z) = \sum_{i=1}^n x_i(e_k)y_i(Z) - y_i(e_k)x_i(Z) = y_k(Z) = z_{n+k}.$$

Meanwhile,

$$\mathcal{C}_\omega(f_k)(Z) = \omega(f_k, Z) = \sum_{i=1}^n x_i(f_k)y_i(Z) - y_i(f_k)x_i(Z) = -x_k(Z) = -z_k.$$

So $\mathcal{C}_\omega(e_k) \in V^*$ is the linear function that returns the $n+k$-th coordinate of Z, i.e. expanded in the dual basis, $\mathcal{C}_\omega(e_k) = (0, \dots, 0, 1, 0, \dots, 0)$, with the 1 appearing in the $n+k$-th place. Likewise, $\mathcal{C}_\omega(f_k) = (0, \dots, 0, -1, 0, \dots, 0)$, with a -1 in the k-th place. Thus, the $2n \times 2n$ matrix for $\mathcal{C}_\omega$ is $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where each of A, B, C, D represents the $n \times n$ block given by $A = 0, B = -I, C = I$ and $D = 0$.

c. The determinant is the only skew-symmetric $2n$-form on $\mathbb{R}^{2n}$ up to scaling. So this motivates us to take the n-fold wedge product of ω with itself. We compute that it is given by

$$(\omega)^{\wedge n} = \underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ times}} = n!(x_1 \wedge y_1) \wedge (x_2 \wedge y_2) \wedge \dots \wedge (x_n \wedge y_n).$$

Lemma:

$$F^*(\omega^{\wedge n}) = (F^*\omega)^{\wedge n}.$$

Proof. This follows by repeatedly applying (i.e. with induction) the identity

$$F^*(\omega_1 \wedge \omega_2) = (F^*\omega_1) \wedge (F^*\omega_2)$$

valid for any $\omega_1 \in \Lambda^k(V^*)$, $\omega_2 \in \Lambda^l(V^*)$. To prove the identity, write

$$\begin{aligned}
&F^*(\omega_1 \wedge \omega_2)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) \\
&\qquad = \omega_1 \wedge \omega_2(Fv_1, ..., Fv_k, Fv_{k+1}, ..., Fv_{k+l}) \\
&\qquad = \sum_{i_1<...<i_k, i_{k+1}<...<i_{k+l}} (-1)^{inv(i_1,...,i_{k+l})} \omega_1(Fv_{i_1}, ..., Fv_{i_k}) \omega_2(Fv_{i_{k+1}}, ..., Fv_{i_{k+l}}) \\
&\qquad = \sum_{i_1<...<i_k, i_{k+1}<...<i_{k+l}} (-1)^{inv(i_1,...,i_{k+l})} F^*\omega_1(v_{i_1}, ..., v_{i_k}) F^*\omega_2(v_{i_{k+1}}, ..., v_{i_{k+l}}) \\
&\qquad = (F^*\omega_1) \wedge (F^*\omega_2)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}).
\end{aligned}$$

□

So the lemma and the assumption $F^*\omega = \omega$ give us that

$$n!F^*((x_1 \wedge y_1) \wedge ... \wedge (x_n \wedge y_n)) = n!(x_1 \wedge y_1) \wedge ... \wedge (x_n \wedge y_n).$$

On the other hand, since the space of skew-symmetric $2n$ forms on $\mathbb{R}^{2n}$ is 1-dimensional, we must have

$$F^*((x_1 \wedge y_1) \wedge ... \wedge (x_n \wedge y_n)) = c(x_1 \wedge y_1) \wedge ... \wedge (x_n \wedge y_n)$$

for some constant c. We can compute c by evaluating on the standard basis $\{e_1, \ldots, e_{2n}\}$,

$$\begin{aligned}
c &= F^*((x_1 \wedge y_1) \wedge ... \wedge (x_n \wedge y_n))(e_1, ..., e_{2n}) \\
&= ((x_1 \wedge y_1) \wedge ... \wedge (x_n \wedge y_n))(Fe_1, ..., Fe_{2n}) \\
&= \det \begin{pmatrix} | & & | \\ Fe_1 & ... & Fe_{2n} \\ | & & | \end{pmatrix} \\
&= \det F.
\end{aligned}$$

Thus we are forced to have $\det F = 1$. Note: The third equality above can be checked by induction starting with the case of a 2×2 determinant.