Math 52H: Homework N3

Due to Friday, February 3

1. Given a parallelepiped $P(v_1, v_2, v_3, v_4) \subset \mathbb{R}^4$, compute the 3-dimensional volume of each of its 3-dimensional face. Here

$$v_1 = (1, 1, 1, 1),$$
$$v_2 = (1 - 1, 1, 1),$$
$$v_3 = (1, 1, -1, 1),$$
$$v_4 = (1, 1, 1, -1).$$

Compare the orientation of $\mathbb{R}^4$ given by the basis v_1, v_2, v_3, v_4 with the orientation given by its standard basis e_1, e_2, e_3, e_4.

2. A vector subspace $L \subset V$ of a vector space V is called *invariant* with respect to a linear operator $\mathcal{A} : V \to V$ if $\mathcal{A}(v) \in L$ for each vector $v \in L$.

Let $\mathcal{A} : V \to V$ be a linear operator, and $l_1, \dots, l_k \in V^*$ be linear independent vectors from the dual space V^*. Suppose that

$$\mathcal{A}^*(l_1 \wedge \dots \wedge l_k) = c\, l_1 \wedge \dots \wedge l_k,$$

for some real number $c \in \mathbb{R}$. Prove that the vector subspace $\mathrm{Span}(l_1, \dots, l_k)$ is invariant with respect to the dual operator $\mathcal{A}^* : V^* \to V^*$.

3. Let η be an exterior 2-form on a vector space V. Suppose that $\eta \wedge \eta = 0$. Prove that there exists two 1-forms $\alpha, \beta \in V^*$ such that $\eta = \alpha \wedge \beta$.

4. In $\mathbb{R}^2$ with the standard dot-product consider a basis $v_1 = (1, 0), v_2 = (1, 1)$. Let (y_1, y_2) be coordinates dual to this basis. Given a function $f(y_1, y_2)$ compute its gradient in these coordinates, i.e. find functions $g_1(y_1, y_2), g_2(y_1, y_2)$ such that $\nabla f = g_1 \frac{\partial}{\partial y_1} + g_2 \frac{\partial}{\partial y_2}$. However, check that the differential of the function f has the form

$$df = \frac{\partial f}{\partial y_1} dy_1 + \frac{\partial f}{\partial y_2} dy_2.$$

All problems and subproblems are 10 points.