Math 52H: Solutions to the Final Exam

March 19, 2012

1. Let $T \subset \mathbb{R}^3$ be the torus defined by the parametric equations

$$x = (a + R\cos\theta)\cos\phi$$
$$y = (a + R\cos\theta)\sin\phi$$
$$z = R\sin\theta,$$

where $0 < R < a$ are constants. the parameters ϕ, θ take values in $[0, 2\pi]$

a) Compute the area of T;

b) Compute the volume of the domain bounded by the torus T.

a) Let us compute the pull-back of the area form σ of the torus T by the parametrization map

$$\Phi(\theta, \phi) = (a + R\cos\theta)\cos\phi, (a + R\cos\theta)\sin\phi, R\sin\theta).$$

We have

$$\frac{\partial\Phi}{\partial\theta} = (-R\sin\theta\cos\phi, -R\sin\theta\sin\phi, R\cos\theta),$$
$$\frac{\partial\Phi}{\partial\phi} = (-(a + R\cos\theta)\sin\phi, (a + R\cos\theta)\cos\phi, 0).$$

Then

$$E = \left\| \frac{\partial \Phi}{\partial \theta} \right\|^2 = R^2,$$
$$E = \left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \phi} \right\rangle = 0,$$
$$G = \left\| \frac{\partial \Phi}{\partial \phi} \right\|^2 = (a + R\cos\theta)^2.$$

Hence,

$$\sqrt{EG - F^2} = R(a + R\cos\theta),$$

and therefore,

$$\Phi^*\sigma = R(a + R\cos\theta)d\theta \wedge d\phi$$

and

$$\text{Area}(T) = \int\limits_T \sigma = \int\limits_{0\leq\phi,\theta\leq 2\pi} \Phi^*\sigma = \int\limits_0^{2\pi}\int\limits_0^{2\pi} R(a + R\cos\theta)d\theta d\phi = 4\pi^2 aR.$$

b) Denote by U the domain bounded by T. By Stokes' theorem we have

$$\text{Vol}(U) = \int\limits_U dx \wedge dy \wedge dz = \int\limits_T z dx \wedge dy.$$

Using the parameterization Φ we get

$$\int\limits_T z dx \wedge dy = \int\limits_{0\leq\phi,\theta\leq 2\pi} \Phi^*(z dx \wedge dy)$$
$$= R\sin\theta\,(-R\sin\theta\cos\phi d\theta - (a + R\cos\theta)\sin\phi d\phi) \wedge (-R\sin\theta\sin\phi d\theta + (a + r\cos\theta)\cos\phi)d\phi)$$
$$= R^2(a + R\cos\theta)\sin^2\theta d\phi \wedge d\theta.$$

Thus,

$$\text{Vol}(U) = \int\limits_{0\leq\phi,\theta\leq 2\pi} R^2(a + R\cos\theta)\sin^2\theta d\phi \wedge d\theta = 2\pi^2 aR^2.$$

2. Let u, v be two smooth functions on the unit disc $D = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$. Suppose that

- everywhere in D one has

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x};$$

- $u = x, v = y$ when $x^2 + y^2 \leq \frac{1}{2}$;

- $u^2 + v^2 \neq 0$ in $D \setminus 0$.

Prove that the differential 1-form

$$\alpha = \frac{udy - vdx}{u^2 + v^2}$$

is closed in D and compute $\int\limits_{\partial D} \alpha$. Here ∂D is oriented counter-clockwise.

We will denote partial derivatives by u_x, u_y, v_x, v_y. Thus we have $u_x = v_y, u_y = -v_x$. We compute

$$d\alpha = \frac{(du \wedge dy - dv \wedge dx)}{u^2 + v^2} - 2\frac{(udu + vdv) \wedge (udy - vdx)}{(u^2 + v^2)^2}$$
$$= \frac{1}{(u^2 + v^2)^2}\big((u^2 + v^2)(u_x + v_y)dx \wedge dy - 2(uu_x dx + uu_y dy + vv_x dx + vv_y dy) \wedge (udy - vdx)\big)$$
$$= \frac{1}{(u^2 + v^2)^2}\big(u^2 u_x + v^2 u_x + u^2 v_y + v^2 v_y - 2(u^2 u_x + uvv_x + uvu_y + uvu_y + v^2 v_y)\big)\, dx \wedge dy$$
$$== \frac{1}{(u^2 + v^2)^2}\big(u^2(-u_x + v_y) + v^2(u_x - v_y) - 2uv(v_x + u_y)\big)\, dx \wedge dy = 0$$

in view of the equations for the partial derivatives.

Thus the form α is closed in $\mathbb{R}^2 \setminus 0$. Denote $D' = \{x^2 + y^2 \leq \frac{1}{2}\}$. Then

$$\int\limits_{\partial D} \alpha = \int\limits_{\partial D'} \alpha = \int\limits_{\partial D'} \frac{xdy - ydx}{x^2 + y^2} = 2\pi.$$

3. Consider $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2). Denote $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Let

$H : \mathbb{R}^4 \to \mathbb{R}$ be a smooth function equal to x_2 outside the ball $B_1(0)$ of radius 1 centered at 0. Suppose a vector vield $\mathbf{v}$ satisfies

$$\mathbf{v} \lrcorner \omega = dH.$$

Compute the flux of $\mathbf{v}$ through the 3-dimensional disc

$$D = \{y_2 = 0,\ x_1^2 + y_1^2 + x_2^2 \le 2\},$$

co-oriented by the normal vector $(0,0,0,1)$ at the origin.

$\mathrm{Flux}_D\, \mathbf{v} = \int\limits_D \eta$, where $\eta = \mathbf{v} \lrcorner \Omega$, $\Omega = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$. Let us compute η. we have

$$\eta = \mathbf{v} \lrcorner \Omega = \frac{1}{2}(\mathbf{v} \lrcorner \omega^2) = (\mathbf{v} \lrcorner \omega) \wedge \omega = dH \wedge \omega = d(H\omega).$$

Applying Stokes' theorem we find

$$\int\limits_D \eta = \int\limits_{\partial D} H\omega.$$

Recall that by assumption $H|_{\partial D} = x_2$. Using again Stokes' theorem we conclude that

$$\int\limits_{\partial D} H\omega = \int\limits_{\partial D} x_2\omega = \int\limits_D dx_2 \wedge \omega = \int\limits_D dx_1 \wedge dy_1 \wedge dx_2.$$

The absolute value of the latter integral is just the volume of the Euclidean ball of radius $\sqrt{2}$, i.e. it is equal to $\frac{8\pi\sqrt{2}}{3}$. However, the orientation of D is determined by the co-orientation of D by the vector $(0,0,0,1)$ at the origin is opposite to the orientation given by coordinates x_1, y_1, x_2. Hence,

$$\mathrm{Flux}_D\, \mathbf{v} = -\frac{8\pi\sqrt{2}}{3}.$$

4. Let us consider the complex vector space $\mathbb{C}^2$ with coordinates $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$. We can also view $\mathbb{C}^2$ as the real space $\mathbb{R}^4$ with cooedinates (x_1, y_1, x_2, y_2). Denote

$$\alpha := \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2.$$

Take any vector $c = (c_1, c_2) \in \mathbb{C}^2$ of length 1, i.e. $|c_1|^2 + |c_2|^2 = 1$. Denote by Γ_c the circle $\Gamma_c(t) = ce^{2\pi it}$, $t \in [0, 1]$. Compute $\int\limits_{\Gamma_c} \alpha$.

We compute the integral directly. Denote $c_1 = a_1 + ib_1, c_2 = a_2 + ib_2$. We have

$$\Gamma_c(t) = ce^{2\pi it} = (c_1e^{2\pi it}, c_2e^{2\pi it})$$
$$= (a_1 \cos 2\pi it - b_1 \sin 2\pi it, a_1 \sin 2\pi it + b_1 \cos 2\pi it, a_2 \cos 2\pi it - b_2 \sin 2\pi it, a_2 \sin 2\pi it + b_2 \cos 2\pi it).$$

Hence,

$$\Gamma_c^*\alpha = \pi\big((a_1 \cos 2\pi it - b_1 \sin 2\pi it)^2 + (a_1 \sin 2\pi it + b_1 \cos 2\pi it)^2$$
$$+(a_2 \cos 2\pi it - b_2 \sin 2\pi it)^2 + (a_2 \sin 2\pi it + b_2 \cos 2\pi it)^2\big)dt = \pi(|c_1|^2 + |c_2|^2)dt = \pi dt.$$

Thus,

$$\int\limits_{\Gamma_c} = \pi.$$

5. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. Verify whether the map $f : S^{n-1} \to S^{n-1}$ given by the formula $f(x) = -x$ is homotopic to the identity map. Consider the volume form on S^{n-1}:

$$\sigma = \sum_1^n (-1)^{i-1} x_i dx_1 \wedge \ldots \overset{i}{\vee} \cdots \wedge dx_n.$$

If we orient the sphere is the co-orientation outward to the unit ball that $\int\limits_{S^{n-1}} \sigma > 0$. On the other hand, $f^*\sigma = (-1)^n\sigma$. Thus, $f^*\sigma = \sigma$ if n is even and $f^*\sigma = -\sigma$ if n is odd. Thus, for n odd the diffeomorphism f *is not homotopic to the identity map*. Indeed, given two homotopic maps $f, g : S^{n-1} \to S^{n-1}$ we must have $\int\limits_{S^{n-1}} f^*\sigma = \int\limits_{S^{n-1}} g^*\sigma$, because the form σ is closed on S^{2n-1}, but in our case $\int\limits_{S^{n-1}} f^*\sigma < 0$ while $\int\limits_{S^{n-1}} \sigma > 0$.

If $n = 2k$ is even then the map f is homotopic to the identity. Indeed, we can think about $\mathbb{R}^n = \mathbb{R}^{2k}$ as $\mathbb{C}^k$. Then the required homotopy $f_t : S^{2k-1} \to S^{2k-1}$ can be defined by the formula

$$f_t(z) = e^{\pi it}z,$$

for $z = (z_1, \ldots, z_k) \in S^{2k-1}$ and $t \in [0, 1]$.