Math 52H: Solutions to Practice problems for the Final Exam

1. Prove that if S is a closed surface in $\mathbb{R}^3$, $\mathbf{n}$ its unit normal vector field and $\mathbf{l}$ any fixed vector then

$$\iint\limits_S \langle \mathbf{n}, \mathbf{l} \rangle dS = 0.$$

$$\iint\limits_S \langle \mathbf{n}, \mathbf{l} = \mathrm{Flux}_S(\mathbf{l}) = \int\limits_{\mathbf{U}} \mathrm{div}(\mathbf{l}) = \mathbf{0}.$$

Here U is the domain bounded bu S.

2. Given a function $u : U \to \mathbb{R}$, where U is an open domain in $\mathbb{R}^n$ we denote by Δu the *Laplace operator*

$$\Delta u = \sum_1^n \frac{\partial^2 u}{\partial x_j^2}.$$

A function u is called *harmonic* in U if $\Delta u = 0$. Suppose that $n = 2$, i.e. U is a planar domain.

a) Prove that u is harmonic in U if and only if for any compact subdomain $\Omega \subset U$ with smooth boundary Γ one has

$$\oint\limits_\Gamma \frac{\partial u}{\partial \mathbf{n}} ds = 0, \tag{1}$$

where n is a unit normal vector field to Γ and $\frac{\partial u}{\partial \mathbf{n}} = du(\mathbf{n})$ is the directional derivative.

We have $\frac{\partial u}{\partial \mathbf{n}} = \langle \nabla u, \mathbf{n} \rangle$. We also note that $\mathrm{div} \nabla u = \Delta u$. Hence, harmonicity of u is equivalent to the fact that $\mathrm{div} \nabla u = 0$. Hence, if u is harmonic then the divergence theorem implies that $\oint\limits_{\Gamma} \frac{\partial u}{\partial \mathbf{n}} ds = 0$, Conversely, applying (1) to circles $S_\epsilon(a)$ of radius ϵ centered at a point $a \in U$

$$\Delta u(a) = \mathrm{div} \nabla u(a) = \lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \oint\limits_{S_\epsilon(a)} \frac{\partial u}{\partial \mathbf{n}} ds = 0.$$

b) Prove that for any C^2-smooth function $u : U \to \mathbb{R}$ one has

$$\iint\limits_S \left(\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 \right) dx_1 dx_2 = -\int\limits_S u \Delta u dx_1 dx_2 + \oint\limits_\Gamma u \frac{\partial u}{\partial \mathbf{n}} ds,$$

where $S \subset U$ is any compact domain with boundary Γ.

We have

$$\oint\limits_\Gamma u \frac{\partial u}{\partial \mathbf{n}} ds = \mathrm{Flux}_\Gamma(u \nabla u).$$

Furthermore,

$$\mathrm{div}(u \nabla u) = u \Delta u + \left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2.$$

Hence, the required formula is just the divergence theorem for $u \nabla u$.

c) Let S and Γ be as in the previous problem. Prove that for any two C^2-functions $u, v : U \to \mathbb{R}$ one has the following identity:

$$\iint\limits_S \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dx_1 dx_2 = \oint\limits_\Gamma \begin{vmatrix} \frac{\partial u}{\partial \mathbf{n}} & \frac{\partial v}{\partial \mathbf{n}} \\ u & v \end{vmatrix} ds\,.$$

This is the divergence theorem for the vector field $v \nabla u - u \nabla v$.

3. Compute the integral

$$\iint_S (x^2+y^2)dS,$$

where S is the boundary of the domain $\{\sqrt{x^2+y^2} \le z \le 1\}$.

The surface S is the union of the surface $P = \{z = \sqrt{x^2+y^2}; x^2+y^2 \le 1\}$ and the disc $\Delta = \{z = 1; x^2+y^2 \le 1\}$. Let us coordinatize both surfaces via the projection to the plane (x,y). Then the area form on Δ is just $\sigma_\Delta = dx \wedge dy$ and to compute σ_P we use the parametrization $\Phi(x,y) = x, y, r = \sqrt{x^2+y^2}$. Then $\Phi_x = (1, 0, \frac{x}{r})$, $\Phi_y = (0, 1, \frac{y}{r})$. Thus $E = 1 + \frac{x^2}{r^2}, G = 1 + \frac{y^2}{r^2}$ and $F = \frac{xy}{r^2}$. Hence

$$EG - F^2 = (1 + \frac{x^2}{r^2})(1 + \frac{y^2}{r^2}) - \frac{x^2y^2}{r^4} = 2.$$

Thus $\sigma_P = \sqrt{2}dx \wedge dy$. Denote $D = \{x^2+y^2 \le 1\} \subset \mathbb{R}^2$. We need to compute two integrals

$$I_1 = \int_D (x^2+y^2)dx \wedge dy = \int_0^{2\pi}\int_0^1 r^3 dr d\phi = \frac{\pi}{2}$$

and

$$I_2 = \sqrt{2}\int_D (x^2+y^2)dxdy = \frac{\pi}{\sqrt{2}}.$$

Hence the answer is $\frac{\pi(1+\sqrt{2})}{2}$.

4. Compute

$$\int_S \frac{dy \wedge dz}{x} + \frac{dz \wedge dx}{y} + \frac{dx \wedge dy}{z},$$

where S is the ellipsoid

$$S = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

co-oriented by the outward normal to the domain which it bounds.

Denote $\eta := \frac{dy \wedge dz}{x} + \frac{dz \wedge dx}{y} + \frac{dx \wedge dy}{z}$. Let us rescale variables:

$$X = \frac{x}{a}, Y = \frac{y}{b}, Z = \frac{z}{c}.$$

The S becomes the unit sphere $X^2 + Y^2 + Z^2 = 1$ and the form η can be written as $\frac{A}{X} dY \wedge dZ + \frac{B}{Y} dZ \wedge dX + \frac{C}{Z} dX \wedge dY$, where

$$A = \frac{bc}{a}, B = \frac{ca}{b}, C = \frac{ab}{c}.$$

We note that $\int_S \eta = \text{Flux}_S \, \mathbf{v}$, where $\mathbf{v}$ is the vector field with coordinate functions $\frac{A}{X}, \frac{B}{Y}, \frac{C}{Z}$. Hence,

$$\int\limits_S \eta = \int\limits_S (A + B + C) dS = (A + B + C) \text{Area}(S) = 4\pi \frac{(ab)^2 + (bc)^2 + (ca)^2}{abc}.$$

5. Consider a differential form $\omega = \sum\limits_1^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$.

a) Find a vector field $\mathbf{v}$ on $\mathbb{R}^{2n}$ such that

$$d(\mathbf{v} \lrcorner \omega) = \omega.$$

(This problem has infinitely many solutions. Find any of them.)

b) Compute $\text{Flux}_S \, \mathbf{v}$, where S is an ellipsoid

$$\left\{ \sum_1^n \frac{x_i^2 + y_i^2}{a_i^2} = 1 \right\}$$

cooriented by the outward normal vector field. Explain why the answer is independent of the choice of $\mathbf{v}$ in Part a).

a) One of the solutions is $\mathbf{v} = \sum\limits_1^n y_i \frac{\partial}{\partial y_i}$. Indeed, $\mathbf{v} \lrcorner \omega = -\sum\limits_1^n y_i dx_i$ and $d(\mathbf{v} \lrcorner \omega) = \omega$. Recall that the volume form $\Omega = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ is equal to $\frac{1}{n!}\omega^n$. Hence, we have

$$\mathbf{v} \lrcorner \Omega = \frac{1}{n!} \mathbf{v} \lrcorner \omega^n = \frac{1}{(n-1)!} \mathbf{v} \lrcorner \omega \wedge \omega^{n-1}.$$

In particular,

$$d(\mathbf{v} \lrcorner \Omega) = \frac{1}{(n-1)!} \omega^n = n\Omega.$$

b) By Stokes theorem we have

$$\mathrm{Flux}_S\,\mathbf{v} = \int\limits_S \mathbf{v} \lrcorner \Omega = \int\limits_U d(\mathbf{v} \lrcorner \Omega) = n \int\limits_U \Omega = n\mathrm{Vol}U,$$

where we denote by U the solid ellipsoid bounded by S.

Note, that $\mathrm{Vol}U = a_1^2 \dots a_n^2 \mathrm{Vol}B_1$ where B_1 is the unit ball in $\mathbb{R}^{2n}$. We recall that $\mathrm{Vol}B_1 = \frac{\pi^n}{n!}$

6. Consider a 4-dimensional submanifold with boundary in $\mathbb{R}^8$:

$$\Gamma = \big\{(x_1, \dots, x_8) \in \mathbb{R}^8;\ x_5 = x_1 \cos\alpha + x_2 \sin\alpha, x_6 = -x_1 \sin\alpha + x_2 \cos\alpha,$$
$$x_7 = 2x_3 - x_4, x_8 = -x_3 + x_4,\ x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 1\big\}.$$

Suppose that Γ is oriented by its parameterization by coordinates (x_1, x_2, x_3, x_4). Compute

$$\int\limits_\Gamma dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8.$$

Parameterizing Γ by coordinates x_1, x_2, x_3, x_4 and expressing $dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8$ in these coordinates we get

$$dx_5 \wedge dx_6 \wedge dx_7 \wedge dx_8 = dx_1 \wedge dx_2 wedge dx_3 \wedge dx_4.$$

Hence, the integral is equal to the volume of the init 4-ball, i.e. $\frac{\pi^2}{2}$.

7. Consider a vector field

$$\mathbf{v} = \frac{1}{r^3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right),$$

$r = \sqrt{x^2 + y^2 + z^2}$ in $\mathbb{R}^3 \setminus 0$. Let us denote

$$S := \left\{(x, y, z) \in \mathbb{R}^3;\ z = e^{x^2 + y^2 - \frac{1}{2}},\ x^2 + y^2 + z^2 \le \frac{3}{2}\right\}$$

and co-orient this surface by a normal vector field which is equal to $(0, 0, 1)$ at the point $(0, 0, \frac{1}{\sqrt{e}}) \in S$. Compute $\mathrm{Flux}_S\,\mathbf{v}$.

The equation $e^{2r^2-1}+r^2=\frac{3}{2}$ has a solution $r^2=\frac{1}{2}$, and hence the surface S is bounded by the circle $\Gamma=\{x^2+y^2=\frac{1}{2}, z=1\}$. The normal component of the vector field $\mathbf{v}$ to the unit sphere has the length $\sqrt{3}2$ (equal to the radius of the sphere). Hence, the question amonts to a computation of the area of the spherical cap bounded by Γ. Let us compute the area form. The surface given by a parameterizing map $\Phi(x,y)=(x,y,S=\sqrt{\frac{3}{2}-x^2-y^2}$. We have $\Phi_x=(1,0,-\frac{x}{S}), \Phi_y=(0,1,\frac{y}{S})$. Thus,

$$E=1+\frac{x^2}{S^2},\ G=1+\frac{y^2}{S^2},\ F=\frac{xy}{S^2}.$$

Hence,

$$EG-F^2=1+\frac{x^2}{S^2}+\frac{y^2}{S^2}=\frac{3}{3-2x^2-2y^2},$$

and

$$\text{Area}(S)=3\int\limits_{x^2+y^2\le 12}\frac{dxdy}{\sqrt{3-2x^2-2y^2}}=3\int\limits_0^{2\pi}\int\limits_0^{\frac{1}{\sqrt{2}}}\frac{rdrd\phi}{\sqrt{3-2r^2}}=3\pi\int\limits_0^{\frac{1}{2}}\frac{du}{\sqrt{3-2u}}=3\pi(\sqrt{3}-\sqrt{2}).$$

Finally to get the flux we need to multiply the area by $\sqrt{32}$.

8. Suppose that a vector field $\mathbf{v}$ in $\mathbb{R}^3$ with coordinate functions (P,Q,R) satisfies $\operatorname{curl}\mathbf{v}=0$. Find an explicit expression for a function F such that $\mathbf{v}=\nabla F$.

The equation $\operatorname{curl}\mathbf{v}=0$ is equivalent to $d\alpha=0$, where We have $\alpha:=\mathcal{D}(\mathbf{v})=Pdx+Qdy+Rdz$. In $\mathbb{R}^3$ the closed form α is exact and its primitive F (i.e. $dF=\alpha$) can be computed by the formula

$$F(u)=\int\limits_0^1\left(xP(tu)+yP(tu)+zP(tu)\right)dt.$$

where $u=(x,y,z)$. The equation $dF=\alpha$ is equivalent to $\nabla F=\mathbf{v}$.

9. Let C be the intersection of the sphere $S=\{x^2+y^2+z^2=1\}$ and the plane $P=\{x+y+z=0\}$. We orient C counter-clockwise when looking from the point $(0,0,100)$. Compute $\int\limits_C z^3dx$.

Let us use Stokes' theorem applied to the disc Δ bounded by the circle C in the plane $\{x + y + z = 0\}$. The corresponding orientation of Δ coincides with its orientation by coordinates (x, y) via the orthogonal projection. We have

$$I := \int_C z^3 dx = \int_\Delta 3z^2 dz \wedge dx.$$

Expressing in coordinates x, y we get $z = -(x + y)$ and

$$3z^2 dz \wedge dx = 3(x + y)^2 dx \wedge dy.$$

Disc D projects to the plane (x, y) as a (solid) ellipse $E = \{x^2 + y^2 + (x + y)^2 \le 1\}$. By rotating the axes by $\pi/4$, $u = \frac{\sqrt{2}}{2}(x - y), v = \frac{\sqrt{2}}{2}(x + y)$ we can rewrite the equation of the solid ellipse as $u^2 + 3v^2 \le 1$. Note that in the new coordinates $dx \wedge dy = du \wedge dv$ and $(x + y)^2 = 2v^2$.

$$I = \int_\Delta 3z^2 dx \wedge dy = 3\int_E (x + y)^2 dx \wedge dy = 6 \iint_{u^2+3v^2\le 1} v^2 du dv = 6\int_{-1}^{1}\int_{-S}^{S} v^2 dv du,$$

where we denoted $S := \frac{1}{\sqrt{3}}\sqrt{1 - u^2}$. We further have

$$I := 6\int_{-1}^{1}\int_{-S}^{S} v^2 dv du = 4\int_{-1}^{1} S^3 du = \frac{4}{3\sqrt{3}}\int_{-1}^{1}(1 - u^2)^{\frac{3}{2}} du.$$

Substituting $u = \sin t$ we get

$$I = \frac{4}{3\sqrt{3}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 t dt = \frac{1}{6\sqrt{3}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(3 + \cos 2t + 4\cos 4t)dt = \frac{\pi}{2\sqrt{3}}.$$

10. Let M be an oriented closed n-dimensional manifold, and ω be a differential $(n - 1)$-form on M. Prove that there exists a point $a \in M$ such that $(d\omega)_a = 0$.

By Stokes theorem we have $\int\limits_M \omega = 0$. The n-form ω is proportional to the volume for σ_M, $\omega = f\sigma_M$ and we have $\int\limits_M \omega = \int f dV$. Hence, the function f should change sign and thus by continuity at some point $f(a) = 0$, and hence $\omega_a = f(a)(\sigma_M) = 0$.

11. Let us view the space $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) as a complex vector space $\mathbb{C}^2$ with coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$. Consider a surface

$$S = \{(z_1, z_2) \in \mathbb{C}^2;\ z_2 = z_1^2, |z_1| \leq 1.\}$$

Compute Area(S).

Let us introduce polar coordinates on complex lines z_1, z_2, i.e. $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$. The the surface S is given by the parameterization

$$(r_1 \phi_1) \mapsto F(r_1, \phi_1) = (r_1, \phi_1, r_1^2, 2\phi_1);\ 0 \leq r_1 \leq 1, 0 \leq \phi_1 < 2\pi.$$

The tangent space to the surface is generated by vectors

$$A := \frac{\partial F}{\partial r_1} = \frac{\partial}{\partial r_1} + 2r_1 \frac{\partial}{\partial r_2},$$

$$B := \frac{\partial F}{\partial \phi_1} = \frac{\partial}{\partial \phi_1} + 2\frac{\partial}{\partial \phi_2}.$$

The basis $\frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_2}, \frac{\partial}{\partial \phi_1}, \frac{\partial}{\partial \phi_1}$ is orthogonal and we have

$$||\frac{\partial}{\partial r_1}|| = ||\frac{\partial}{\partial r_2}|| = 1$$

and

$$||\frac{\partial}{\partial \phi_1}|| = r_1,\ \ ||\frac{\partial}{\partial \phi_2}|| = r_2.$$

Hence,

$$E = \langle A, A\rangle = 1 + 4r_1^2,\ \ G = \langle B, B\rangle = r_1^2 + 4r_2^2 = r_1^2 + 4r_1^4,\ \ F = \langle A, B\rangle = 0.$$

Thus,

$$\sqrt{EG - F^2} = \sqrt{(1 + 4r_1^2)(r_1^2 + 4r_1^4)} = r_1(1 + 4r_1^2).$$

Thus, Area$(S) = \int\limits_0^{2\pi}\int\limits_0^1 r_1(1 + 4r_1^2) dr_1 d\phi_1 = 2\pi(\frac{1}{2} + \frac{4}{3}) = \frac{11\pi}{3}$.