Math 52H Homework 9 Solutions

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March 9, 2011

1. (Online text 5.3.6) Note that the gradients $\nabla F_1|_x, ..., \nabla F_{n-k}|_x$ span the normal space to $F = 0$ at x and that $dF_1 = \mathcal{D}(\nabla F_1), ..., dF_{n-k} = \mathcal{D}(\nabla F_{n-k})$ are the dual linear functionals. Let $v_1, ..., v_k$ be k vectors from the tangent space of A at x, chosen so that the sequence $\nabla F_1|_x, ..., \nabla F_{n-k}|_x, v_1, ..., v_k$ has positive orientation on $\mathbb{R}^n$. By the second characterization of duality introduced in class,

$$*(dF_1 \wedge ... \wedge dF_{n-k})(v_1, ..., v_k) = \frac{Vol(v_1, ..., v_k)}{Vol(\nabla F_1, ..., \nabla F_{n-k})} dF_1 \wedge ... \wedge dF_{n-k}(\nabla F_1, ..., \nabla F_{n-k}).$$

Now let $u_1, ..., u_{n-k}$ be an orthonormal basis for $span(\nabla F_1, ..., \nabla F_{n-k})$ preserving the orientation, and let $y_1, ..., y_{n-k}$ be the dual basis. Then

$$dF_1 \wedge ... \wedge dF_{n-k}(\nabla F_1, ..., \nabla F_{n-k}) = Vol(\nabla F_1, ..., \nabla F_{n-k}) dF_1 \wedge ... \wedge dF_{n-k}(u_1, ..., u_{n-k})$$

and since $dF_i = \mathcal{D}(\nabla F_i)$,

$$dF_1 \wedge ... \wedge dF_{n-k} = Vol(\nabla F_1, ..., \nabla F_{n-k}) dy_1 \wedge ... \wedge dy_{n-k}$$

We conclude that

$$*(dF_1 \wedge ... \wedge dF_{n-k})(v_1, ..., v_k) = \frac{Vol(v_1, ..., v_k)}{Vol(\nabla F_1, ..., \nabla F_{n-k})} Vol(\nabla F_1, ..., \nabla F_{n-k})^2$$

so that

$$\frac{*(dF_1 \wedge ... \wedge dF_{n-k})}{Vol(\nabla F_1, ..., \nabla F_{n-k})}(v_1, ..., v_k) = Vol(v_1, ..., v_k)$$

and thus

$$\frac{*(dF_1 \wedge ... \wedge dF_{n-k})}{Vol(\nabla F_1, ..., \nabla F_{n-k})} = \frac{*(dF_1 \wedge ... \wedge dF_{n-k})}{\|dF_1 \wedge ... \wedge dF_{n-k}\|}$$

gives the area form on the tangent space of A.

2. (Online text 6.2.7) Suppose that U is star-shaped with respect to the base point $r = (r_x, r_y, r_z)$. Let $F : U \times [0,1] \to U$ be defined by

$$F(x,y,z,t) = ((1-t)x + tr_x, (1-t)y + tr_y, (1-t)z + tr_z).$$

This is a contraction mapping of U onto the point r.

We calculate

$$\begin{aligned} F^*(dx) &= (1-t)dx + (r_x - x)dt \\ F^*(dy) &= (1-t)dy + (r_y - y)dt \\ F^*(dz) &= (1-t)dz + (r_z - z)dt \end{aligned}$$

so

$$\begin{aligned} F^*\alpha = P((1-t)(x,y,z)+tr)&\left[(1-t)^2 dy\wedge dz + (1-t)dt\wedge((r_y-y)dz - (r_z-z)dy)\right] \\ + Q((1-t)(x,y,z)+tr)&\left[(1-t)^2 dz\wedge dx + (1-t)dt\wedge((r_z-z)dx - (r_x-x)dz)\right] \\ + R((1-t)(x,y,z)+tr)&\left[(1-t)^2 dx\wedge dy + (1-t)dt\wedge((r_x-x)dy - (r_y-y)dx)\right] \\ = \beta(t) + dt\wedge\gamma(t)& \end{aligned}$$

with

$$\begin{aligned} \gamma(t) = (1-t)&P((1-t)(x,y,z)+tr)\left[(r_y-y)dz - (r_z-z)dy\right] \\ + (1-t)&Q((1-t)(x,y,z)+tr)\left[(r_z-z)dx - (r_x-x)dz\right] \\ + (1-t)&R((1-t)(x,y,z)+tr)\left[(r_x-x)dy - (r_y-y)dx\right] \end{aligned}$$

From the proof of Poincare's lemma we know that $\alpha = dK(F^*\alpha) = d(\int_0^1 \gamma(t)dt)$ so a primitive is given by

$$\begin{aligned} \omega = &\left\{\int_0^1 (1-t)\left[Q((1-t)(x,y,z)+tr)(r_z-z) - R((1-t)(x,y,z)+tr)(r_y-y)\right]dt\right\}dx \\ &+ \left\{\int_0^1 (1-t)\left[R((1-t)(x,y,z)+tr)(r_x-x) - P((1-t)(x,y,z)+tr)(r_z-z)\right]dt\right\}dy \\ &+ \left\{\int_0^1 (1-t)\left[P((1-t)(x,y,z)+tr)(r_y-y) - Q((1-t)(x,y,z)+tr)(r_x-x)\right]dt\right\}dz. \end{aligned}$$

3. Suppose for contradiction that we have diffeomorphism f from the sphere

$$\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 = 1\}$$

to the torus

$$\{(x_1,x_2,y_1,y_2)\in\mathbb{R}^4 : x_1^2+x_2^2 = y_1^2+y_2^2 = 1\}.$$

Parametrize the torus by $F(\theta,\phi) = (\cos\theta, \sin\theta, \cos\phi, \sin\phi)$, $\theta,\phi \in [0,2\pi]$. Let Γ be the closed path $(\cos t, \sin t, 1, 0)$, $t \in [0,2\pi]$ on the torus. We have

$$\int_\Gamma \frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2} = \int_\Gamma d\theta = 2\pi.$$

Hence $\int_{f^{-1}(\Gamma)} f^* d\theta = 2\pi$. But observe that $d\theta$ is a closed 1-form on the torus so $f^* d\theta$ is closed on the sphere. Moreover, $f^{-1}(\Gamma)$ is not the entire sphere because Γ is not the entire torus, and f^{-1} is a bijection. Hence, after deleting a point not on $f^{-1}(\Gamma)$, $f^* d\theta$ is a closed 1-form on the punctured sphere and so it's integral along any closed path is zero because the punctured sphere is contractible. This gives a contradiction.

4. An orthonormal basis for the plane $x + y + z = 0$ is given by

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \end{bmatrix}.$$

Hence a parametrization of C is given by

$$\gamma(t) = \cos t v_1 + \sin t v_2 = \begin{bmatrix} \frac{\cos t}{\sqrt{2}} + \frac{\sin t}{\sqrt{6}} \\ \frac{-\cos t}{\sqrt{2}} + \frac{\sin t}{\sqrt{6}} \\ \frac{-2\sin t}{\sqrt{6}} \end{bmatrix}, \qquad t \in [0,2\pi].$$

We can check that the projection $P(\gamma)$ of γ into the xy-plane revolves in a counter-clockwise direction by noting that projection of v_1 and v_2 onto the xy-plane endow $\mathbb{R}$ with the same orientation as the usual one. Thus γ assigns the correct orientation to C.

The desired integral is now given by

$$\int_0^{2\pi} \frac{-8\sin^3 t}{6\sqrt{6}} \left(\frac{-\sin t}{\sqrt{2}} + \frac{\cos t}{\sqrt{6}} \right) dt = \int_0^{2\pi} \frac{2\sin^4 t}{3\sqrt{3}} dt = \frac{\pi}{2\sqrt{3}}$$

5. We may assume that M is connected after possibly restricting to a single connected component. Since M is closed we have $\int_{\partial M} \omega = \int_M d\omega = 0$. Write $d\omega = f\sigma$ where σ is the volume form on M. Since M is compact, f achieves it's minimum f_- and its maximum f_+ on M. Note that

$$f_- vol(M) = \int_M f_- d\sigma \leq \int_M f d\sigma = 0 \leq \int_M f_+ d\sigma = f_+ vol(M).$$

Since $vol(M) > 0$ we conclude $f_- \leq 0$ and $f_+ \geq 0$. Let x_- and x_+ be points where f attains the values f_- and f_+. Let $\gamma(t), t \in [0,1]$ be a path in M such that $\gamma(0) = x_-$ and $\gamma(1) = x_+$.

Then $f \circ \gamma$ is continuous on $[0,1]$ with $f(0) \le 0$ and $f(1) \ge 0$. It follows that there exists $c \in [0,1]$ with $f(c) = 0$, that is, $\gamma(c) \in M$ satisfies $d\omega(\gamma(c)) = 0$.

6. Assume for the sake of eventual contradiction that there exists a smooth map $f : S^{2k} \to S^{2k}$ such that $f(x) \neq x, -x$ for any x. Then for each x, there exists a unique arc of length less than π connecting x and $f(x)$. Let $\gamma_x(t)$ be a parametrization of the half circle connecting x and $-x$ and passing through $f(x)$. That is, $\gamma_x(0) = x$ and $\gamma_x(t) = -x$ with $\gamma_x(t_0) = f(x)$ for some $t_0 \in (0,1)$. Define a homotopy $f_t(x) = \gamma_x(t)$ and note that $f_0 = I$ while $f_1 = -I$, where I denotes the identity map.

Let σ denote the volume form on S^{2k}. By Lemma 6.2.3, $\int_{S^{2k}} f_0^*\sigma = \int_{S^{2k}} f_1^*\sigma$. On the other hand, $f_0^*\sigma = \sigma$ while $f_1^*\sigma = -\sigma$, so that the first integral is $\mathrm{Vol}(S^{2k})$, while the second is $-\mathrm{Vol}(S^{2k})$, a contradiction. Note that $f_1^*\sigma = -\sigma$ only on spheres of even dimension (one way to prove this is by noting that $\sigma = n \lrcorner \Omega$ where $n = (x_1, \dots, x_{2k+1})$ is the normal vector and $\Omega = dx_1 \wedge \dots \wedge dx_{2k+1}$ is the usual volume form for $\mathbb{R}^{2k+1}$).

7. a. We have $\omega(\frac{X}{\|X\|}, Y) = \frac{1}{\|X\|}\omega(X,Y)$ and $P(\frac{X}{\|X\|}, Y) = \frac{1}{\|X\|}P(X,Y)$ so we may assume that X is a unit vector. Write $Y = Y_1 + Y_2$ with Y_1 a scalar times X and $Y_2 \in X^\perp$. Then $\omega(X,Y) = \omega(X,Y_2)$ and $P(X,Y) = P(X,Y_2)$. In particular, if X and Y are linearly dependent then $\omega(X,Y) = P(X,Y) = 0$, so it suffices to consider the case X and Y orthogonal unit vectors.

Let $e_1, \dots, e_n$ dual to $x_1, \dots, x_n$ and $f_1, \dots, f_n$ dual to $y_1, \dots, y_n$. Let $X = \sum_{i=1}^n a_i e_i + b_i f_i$ and $Y = \sum_{i=1}^n c_i e_i + d_i f_i$. Then

$$\omega(X,Y) = [a_1, \dots, a_n, b_1, \dots, b_n] \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ d_n \end{bmatrix} = X^t A Y = (A^t X)^t Y = (A^t X) \cdot Y \le \|AX\| \|Y\|.$$

The matrix A is orthogonal so we deduce $\omega(X,Y) \le \|X\|\|Y\| = 1$ with equality if and only if $Y = A^t X$, that is

$$Y = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ b_n \end{bmatrix} \quad \Leftrightarrow \quad \forall i,\ c_i = -b_i,\ d_i = a_i \quad \Leftrightarrow \quad Y = iX.$$

b. By part a, ω is equal to the area form on A. Hence

$$area(A) = \int_A \omega = \int_{\partial A} d\omega = \int_{\partial B} d\omega = \int_B \omega \le \int_B \sigma_B = area(B).$$