

Math 52H Homework 8 Solutions

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1. a. Let $A = (0, 1)$, $B = (1, \infty)$ be 1-dimensional submanifolds of \mathbb{R} and let $f : A \rightarrow B$ be given by $f(t) = \frac{1}{t}$. This is a diffeomorphism.

b. Let X denote the 1-dimensional connected closed submanifold. Since X is compact, we have a finite collection of coordinate charts $(U_1, \phi_1), \dots, (U_k, \phi_k)$ so that $X = \bigcup_j U_j$. Without loss of generality, we also assume that $G_j := \phi_j(U_j) = (0, 1)$, and that $U_i \not\subseteq U_j$ for any $i \neq j$. Denote by ψ_j the parameterization maps $\phi_j^{-1} : (0, 1) \rightarrow U_j$. There exists $\epsilon > 0$ such that $X = \bigcup_j \psi_j((\epsilon, 1 - \epsilon))$. Denote $V_j = \psi_j((\epsilon, 1 - \epsilon))$ and $\bar{V}_j = \psi_j([\epsilon, 1 - \epsilon])$. We will also denote by a_j and b_j the end points $a_j = \psi_j(\epsilon)$ and $v_j = \psi_j(1 - \epsilon)$ of \bar{V}_j . By choosing an appropriate $\epsilon > 0$ we can also arrange that all points a_j, b_j , $j = 1, \dots, k$ are disjoint.

Using the connectivity of X we can inductively order V_j in such a way that $V_j \cap V_{j-1} \neq \emptyset$ for $j = 2, \dots, k$. We will assume that this reordering is done.

Now we will prove by induction that for each $s = 1, \dots, k$ the union $\bar{X}_j = \bigcup_1^s \bar{V}_j$ is diffeomorphic either to a closed interval, or to the circle S^1 . This will imply that X is diffeomorphic to S^1 , because $X = \bar{X}_k = \bigcup_1^k V_j$ is by our assumption is compact.

For $s = 1$ the statement is true. Assuming that the claim holds for \bar{X}_{s-1} we will prove it for \bar{X}_s . If \bar{X}_{s-1} is diffeomorphic to S^1 then we must have $\bar{X}_s \subset S^1$. Indeed, take any point $a \in \bar{V}_s \setminus \bar{X}_{s-1}$ and any point $b \in \bar{X}_{s-1}$. In view of connectivity of X there exists a path $\gamma : [0, 1] \rightarrow X$ connecting the points $a = \gamma(0)$ and $b = \gamma(1)$. Let $c = \inf\{t \in [0, 1]; \gamma(t) \in \bar{X}_{s-1}\}$. Then $\gamma(c) \in \bar{X}_{s-1}$ because the X_{s-1} is compact (because it is diffeomorphic to the circle S^1 which is compact). This proves that in this case V_s is a subset of \bar{X}_s , and hence $\bar{X}_s = \bar{X}_{s-1} \cup \bar{V}_s = \bar{X}_s$ is diffeomorphic to the circle.

Consider now the case when \bar{X}_{s-1} is diffeomorphic to a closed interval and consider a diffeomorphism $\Psi_{s-1} : [0, 1] \rightarrow \bar{X}_{s-1}$. Note that the boundary points $A_{s-1} = \Psi_{s-1}(0)$ and $B_{s-1} = \Psi_{s-1}(1)$ coincide with some of the end points a_j, b_j , $j = 1, \dots, s-1$, and hence by our assumption that all points a_j, b_j are disjoint we conclude that all four points A_{s-1} , B_{s-1} , a_s and b_s are disjoint.

We can assume that $\overline{V}_s \not\subseteq \overline{X}_{s-1}$. It then follows that at least one of the points a_s, b_s must belong to \overline{X}_{s-1} . There could be two cases:

Case 1 Both points a_s, b_s belong to \overline{X}_{s-1} ;

Case 2 Only one of the points a_s, b_s belongs to \overline{X}_{s-1} .

We claim that in Case 1 the manifold $\overline{X}_s = \overline{X}_{s-1} \cup \overline{V}_s$ is diffeomorphic to an interval, and in Case 2 \overline{X}_s is diffeomorphic to the circle. We will consider here only Case 2. An argument in Case 1 is similar. We have in this case $A_{s-1}, B_{s-1} \in V_s$. Denote $\alpha_{s-1} = \phi_s(A_{s-1})$, $\beta_{s-1} = \phi_s(B_{s-1})$. We can assume that $0 < \beta_{s-1} < \alpha_{s-1} < 1$. (If $\alpha_{s-1} < \beta_{s-1}$ then one can change the orientation of the diffeomorphism ϕ_s by composing it with a diffeomorphism $x \mapsto 1 - x$.) Denote $\overline{W}_s = \psi_s([\beta_{s-1}, \alpha_{s-1}])$. Note that $\overline{W}_s = \overline{X}_s \setminus X_{s-1}$. Consider the unit circle $S^1 \subset \mathbb{R}^2$. It is parameterized by the polar coordinate θ . Let us define a diffeomorphism g_+ of \overline{X}_{s-1} onto the upper semicircle $S_+ = \{0 \leq \theta \leq \phi\}$ by the formula $x \mapsto \theta = \pi \Phi_{s-1}(x)$, $x \in \overline{X}_{s-1}$, where we denoted $\Phi_{s-1} := \Psi_{s-1}^{-1}$. Let us note that one can construct a diffeomorphism $h : [\beta_{s-1}, \alpha_{s-1}] \rightarrow [\pi, 2\pi]$ with any prescribed positive values of its derivative at the end points. Given such h , we define a diffeomorphism g_- of \overline{W}_s onto the lower semi-circle $S_- = \{\pi \leq \theta \leq 2\pi\}$ by the formula $x \mapsto \phi = h \circ \phi_s$, where we choose the diffeomorphism h to arrange that the map $g : \overline{X}_s \rightarrow S^1$ which is equal to g_- on \overline{W}_s and to g_+ on \overline{X}_{s-1} is C^1 -smooth at the points A_{s-1} and B_{s-1} . Then g will automatically be a C^1 -diffeomorphism.

By choosing h with more care one can make the diffeomorphism g C^∞ -smooth, see Section 4.11 in the Online Text.

c. Let $D = [0, 2\pi) \times [0, 2\pi)$, and define parametrizations of T and U respectively by $P_T(\theta, \phi) = (\cos \theta, \sin \theta, \cos \phi, \sin \phi)$ and $P_U(\theta, \phi) = (2 \cos \theta, 2 \sin \theta, 0) + (\cos \theta \cos \phi, \sin \theta \cos \phi, \sin \phi)$. The first parametrization P_T is obvious. For P_U , view U as a union of circles C_θ centered at $(2 \cos \theta, 2 \sin \theta, 0)$ with the projection of C_θ onto the xy plane being a line segment parallel to $(\cos \theta, \sin \theta, 0)$. Hence a diffeomorphism from T to U is given by $g(x_1, x_2, x_3, x_4) = (x_1(2 + x_3), x_2(2 + x_3), x_4)$. This map is well-defined and bijective since $g = P_U \circ P_T^{-1}$. It is also clearly smooth. For $h(x, y, z) = (x^2 + y^2 + z^2 - 5)/4$, we also have $g^{-1}(x, y, z) = (\frac{x}{2+h(z)}, \frac{y}{2+h(z)}, h(z), z)$ is smooth.

2. Write F for the parametrization $F(x_1, \dots, x_n) = (x_1, \dots, x_n, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ and $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$. Then

$$\int_{L_f} \omega^{\wedge k} = \int_{F^{-1}(L_f)} F^*(\omega^{\wedge k}) = \int_{F^{-1}(L_f)} (F^*\omega)^{\wedge k}.$$

But we calculate

$$F^*\omega = \sum_{i=1}^n dx_i \wedge \left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \right) = \sum_{i \neq j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j = \sum_{i < j} \frac{\partial^2 f}{\partial x_i \partial x_j} (dx_i \wedge dx_j + dx_j \wedge dx_i) = 0.$$

Hence the integral vanishes.

3. Recall $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$. Hence, since ∂A is empty,

$$0 = \int_{\partial A} \omega \wedge \eta = \int_A d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

It follows

$$\int_A \omega \wedge d\eta = (-1)^{k+1} \int_A d\omega \wedge \eta = (-1)^{k+1+(k+1)l} \int_A \eta \wedge d\omega.$$

The constant C may thus be taken to be $(-1)^{(k+1)(l+1)}$.

4. We parametrize X by $F : [0, 2\pi) \times [0, 2\pi) \rightarrow X$,

$$F(\theta, \phi) = (\cos \theta, \sin \theta, \cos \phi, \sin \phi).$$

The point $x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is covered by $\theta = \phi = 0$ and at $\theta = \phi = 0$ we have

$$\frac{dF}{d\theta} = \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{dF}{d\phi} = \begin{bmatrix} 0 \\ 0 \\ -\sin \phi \\ \cos \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the orientation $\{\hat{\theta}, \hat{\phi}\}$ of the parametrization space induces an orientation of the

tangent space of X given at the point a by $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Since

$$x_2 \wedge x_4 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) = 1 > 0$$

the parametrization orientation agrees with the specified orientation of X .

Note

$$F^*(x_1) = \cos \theta, F^*(x_2) = \sin \theta, F^*(x_3) = \cos \phi, F^*(x_4) = \sin \phi$$

$$F^*(dx_1) = -\sin \theta d\theta, F^*(dx_2) = \cos \theta d\theta, F^*(dx_3) = -\sin \phi d\phi, F^*(dx_4) = \cos \phi d\phi.$$

a.

$$\int_X dx_1 \wedge dx_2 + dx_3 \wedge dx_4 = \int_0^{2\pi} \int_0^{2\pi} -\sin \theta \cos \theta - \sin \phi \cos \phi d\phi d\theta = 0.$$

b.

$$\int_X dx_1 \wedge dx_3 + dx_2 \wedge dx_4 = \int_0^{2\pi} \int_0^{2\pi} \sin \theta \sin \phi + \cos \theta \cos \phi d\phi d\theta = 0.$$

c.

$$\int_X x_2 x_4 dx_1 \wedge dx_3 = \int_0^{2\pi} \int_0^{2\pi} \sin^2 \theta \sin^2 \phi d\phi d\theta = \pi^2.$$

5. Let V denote the submanifold given by $x^2 + y^2 + z^2 \leq 1$ with $z \geq 0$, and let D denote the disc of radius 1 centered at the origin and lying on the xy plane oriented with upwards pointing normal. Then $\partial V = S \cup D$. Let $\omega = dx \wedge dy + zdz \wedge dx$. Note that $d\omega = 0$, and so by Stoke's Theorem

$$= \int_V d\omega = \int_S \omega - \int_D \omega.$$

On D , $z = 0$ so $\int_S \omega = \int_D dx \wedge dy = \pi$, the latter being just the area of the disc D .

To compute directly, parametrize S by $P(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$ for $(x, y) \in D$. Note that the tangent plane is spanned by $T_x = (1, 0, \text{something})$ and $T_y = (0, 1, \text{something})$ so $T_x \times T_y$ always has positive z component.

Now

$$dz = \frac{2}{\sqrt{1 - x^2 - y^2}}(-2xdx - 2ydy),$$

so

$$zdz \wedge dx = z \frac{y}{\sqrt{1 - x^2 - y^2}} dx \wedge dy = ydx \wedge dy.$$

Thus $\int_S \omega = \int_D (1 + y)dx \wedge dy = \pi$ since the first term gives the area of the the disc, and the second is odd in y while D is symmetric in y .

6. Let $\lambda = \omega - d\alpha$. We want to show that $\lambda = 0$. We claim that $\lambda = \beta \wedge \alpha$ for some 1-form β . Fix any point a , and if necesary, change the basis of \mathbb{R}_a^5 to v_1, \dots, v_5 so that $\alpha_a = x_5$, where x_1, \dots, x_5 denote the coordinate functions. Then let $\lambda_a = \sum_{i < j} a_{i,j} x_i \wedge x_j$. Since $\lambda_a(x_i, x_j) = 0$ for $1 \leq i < j \leq 4$, we conclude that $\lambda_a = (\sum_{i < 5} a_{i,5} x_i) \wedge \alpha_a$ (recall that $\alpha_a = x_5$). Thus, there exists some 1-form β such that $\lambda = \beta \wedge \alpha$.¹

¹Note that $\alpha_a = x_5$ is only a local condition at a . For instance, it is not possible to assume that $\alpha = dx_5$ globally.

Now $0 = d\omega = d(a\alpha + \lambda) = d\lambda = d\beta \wedge \alpha - \beta \wedge d\alpha$. Let \mathcal{H} denote the hyperplane field defined by $\alpha = 0$. Then the above gives $\beta \wedge d\alpha|_{\mathcal{H}} = 0$. It now suffices to show that $\beta|_{\mathcal{H}} = 0$, since this implies that $\beta = c\alpha$ for some constant c , which in turn implies that $\lambda = 0$.

Before proceeding, let us note that the last condition $\alpha \wedge \omega \wedge \omega \neq 0$ is equivalent to $\alpha \wedge d\alpha \wedge d\alpha \neq 0$, upon substituting $\omega = \beta \wedge \alpha + d\alpha$. We thus conclude that $d\alpha \wedge d\alpha \neq 0$.

Lemma 1. *Fix any exterior 2-form γ on \mathbb{R}^4 such that $\gamma \wedge \gamma \neq 0$. We claim that the map $f : \Lambda^1((\mathbb{R}^4)^*) \rightarrow \Lambda^1((\mathbb{R}^4)^*)$ defined by $f(\delta) = \delta \wedge \gamma$ is an isomorphism.*

Proof. Again, fix a basis v_1, \dots, v_4 with dual basis x_1, \dots, x_4 . Let $\gamma = \sum_{i < j} a_{i,j} x_i \wedge x_j$. The matrix for γ is a skew-symmetric matrix consisting of the $a_{i,j}$ on the upper half and $-a_{i,j}$ on the lower half with 0 on the diagonal. Since all skew-symmetric matrices of even dimension can be block diagonalized, we may change basis so that

$$\gamma = l_1 x_1 \wedge x_2 + l_2 x_3 \wedge x_4.$$

Then the condition that $\gamma \wedge \gamma \neq 0$ is equivalent to $l_1 l_2 \neq 0$.²

Now, examine the linear map f on the basis x_1, \dots, x_4 of $\Lambda^1((\mathbb{R}^4)^*)$. We have that $f(x_1) = l_2 x_1 \wedge x_3 \wedge x_4$, $f(x_2) = l_2 x_2 \wedge x_3 \wedge x_4$, $f(x_3) = l_1 x_3 \wedge x_1 \wedge x_2 = l_1 x_1 \wedge x_2 \wedge x_3$, and $f(x_4) = l_1 x_1 \wedge x_2 \wedge x_4$. Since f sends the basis elements of $\Lambda^1((\mathbb{R}^4)^*)$ to basis elements of $\Lambda^3((\mathbb{R}^4)^*)$, it is an isomorphism. \square

Now, we are nearly done. Apply the above lemma with $\gamma = d\alpha_a|_{\mathcal{H}}$ to see that $(\beta \wedge d\alpha)|_{\mathcal{H}} = 0 \Rightarrow \beta|_{\mathcal{H}} = 0$, as desired.

²Incidentally, $\pm il_1, \pm il_2$ are the (totally imaginery) eigenvalues of the matrix.