Math 52H Homework 7 Solutions

Xiannan Li

February 22, 2011

1. a. The integral is

$$\begin{aligned}\int_0^1\int_{2y}^{y+1} x - y dx dy &= \int_0^1 \frac{(y+1)^2}{2} - (y+1)y - \frac{(2y)^2}{2} + 2y^2 dy \\ &= \int_0^1 \frac{-y^2+1}{2} dy \\ &= 1/2 - 1/6 \\ &= 1/3.\end{aligned}$$

For a, b, c either 0 or 1, the rest of the questions all integrate $\int_B x^a y^b z^c dV = \int_0^1 \int_0^1 \int_0^{xy} x^a y^b z^c dz dx dy = \int_0^1 \int_0^1 x^a y^b (xy)^{c+1}/(c+1) dx dy = \frac{1}{(c+1)(a+c+2)(b+c+2)}$. The numerical answers follow from the above formula, and are (in order) $1/6, 1/6, 1/18$, and $1/9$.

2. a. Using polar coordinates, the given bound translates to $r^2 \leq 2r\cos\theta$, whence $0 \leq r \leq 2\cos\theta$. Note that this implies that $\cos\theta \geq 0$, so that $-\pi/2 \leq \theta \leq \pi/2$. (Alternatively, you can complete the square to see that the equation of the circle is $(x-1)^2 + y^2 = 1$, which is on the right side of the y-axis.) We get that the integral is

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3 dr d\theta &= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta \\ &= 4\int_{-\pi/2}^{\pi/2} \cos^2\theta(1-\sin^2\theta) d\theta \\ &= 4\int_{-\pi/2}^{\pi/2} \frac{\cos 2\theta + 1}{2} d\theta - \int_{-\pi/2}^{\pi/2} \sin^2 2\theta d\theta \\ &= 2\pi - \int_{-\pi/2}^{\pi/2} \frac{1-\cos 4\theta}{2} d\theta \\ &= \frac{3\pi}{2}.\end{aligned}$$

b. Change coordinates to $u = x/a$ and $v = y/b$ which has Jacobian ab. The integral then becomes $ab\int_D \sqrt{1-u^2-v^2}dV$ where $D = \{u^2+v^2 \leq 1\}$. Now change to polar coordinates to get

$$\begin{aligned} ab\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}rdrd\theta &= ab\int_0^{2\pi} 1d\theta[-1/3(1-r^2)^{3/2}]_0^1 \\ &= 2ab\pi/3. \end{aligned}$$

3. Let the region in question be D. We calculate $\frac{\partial u}{\partial x} = \frac{2x}{y}$, $\frac{\partial u}{\partial y} = -\frac{x^2}{y^2}$, $\frac{\partial v}{\partial x} = -\frac{y^2}{x^2}$, and $\frac{\partial v}{\partial y} = \frac{2y}{x}$. The Jacobian determinant is $\frac{\partial(u,v)}{\partial(x,y)} = 3$. Thus we have $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}$. The bounds given in the question translate to $a \leq u \leq b$, and $\alpha \leq v \leq \beta$.

Thus, the area of D is

$$\int_D 1dV = \frac{1}{3}\int_a^b\int_\alpha^\beta 1dvdu = \frac{(b-a)(\beta-\alpha)}{3}.$$

4. Use cylindrical coordinates, so that the hyperboloid becomes $r^2 - z^2 = a^2$, and the region bounded by the sphere is described by $r^2 + z^2 \leq 3a^2$. We calculate the volume of the region given by $r^2 - z^2 \geq a^2$. These bounds imply that $a \leq r \leq \sqrt{3}a$. We divide this into two regions. Let A_1 denote the area of region given by $a \leq r \leq \sqrt{2}a$, and A_2 denote the area of the region given by $\sqrt{2}a \leq r \leq \sqrt{3}a$. Then, by symmetry

$$\begin{aligned} \frac{A_1}{2} &= \int_0^{2\pi}\int_a^{\sqrt{2}a}\int_0^{\sqrt{r^2-a^2}} rdzdrd\theta \\ &= 2\pi\int_a^{\sqrt{2}a}\sqrt{r^2-a^2}rdr \\ &= 2\pi[(r^2-a^2)^{3/2}/3]_a^{\sqrt{2}a} \\ &= \frac{2\pi a^3}{3}. \end{aligned}$$

Similarly,

$$\begin{aligned}\frac{A_2}{2} &= \int_0^{2\pi}\int_{\sqrt{2}a}^{\sqrt{3}a}\int_0^{\sqrt{3a^2-r^2}} rdzdrd\theta \\ &= 2\pi\int_{\sqrt{2}a}^{\sqrt{3}a}\sqrt{3a^2-r^2}rdr \\ &= 2\pi[-(3a^2-r^2)^{3/2}/3]_{\sqrt{2}a}^{\sqrt{3}a} \\ &= \frac{2\pi a^3}{3}.\end{aligned}$$

As a fraction of the total volume of the ball, this is

$$\frac{A_1+A_2}{4/3\pi(\sqrt{3}a)^3} = \frac{2}{3\sqrt{3}}.$$

The ratio is therefore $2 : (3\sqrt{3}-2)$.