Math 52H Homework 6 Solutions

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1. Let $C_0 = [0,1]$, and for $n \geq 1$, let C_n denote the set obtained from C_{n-1} by removing the central open subinterval of every interval in C_{n-1}. We have that $C = \cap_n C_n$, and the volume of C_n is $(2/3)^n$. Thus C is covered by Riemann measurable sets C_n of arbitrarily small volume, hence C is measurable with volume 0 by Prop 4.4.4.

2.a. Write $\alpha = \frac{xdx+ydy+zdz}{(x^2+y^2+z^2)^{3/2}}$ and let $G : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be defined by $G(x,y,z) = -(x^2+y^2+z^2)^{\frac{-1}{2}}$. Then

$$dG = -(x^2+y^2+z^2)^{\frac{-3}{2}}(\frac{-1}{2})(2xdx + 2ydy + 2zdz) = \alpha$$

so α is exact on $\mathbb{R}^3 \setminus \{0\}$, which contains the curve γ. It follows that

$$\int_\gamma \alpha = G(\gamma(1)) - G(\gamma(0)) = (\frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{3}}).$$

b. Write $\alpha = \frac{xdy-ydx}{x^2+y^2}$. On the set $\{(x,y) : x > 0\}$ we have $\alpha = d\arctan(\frac{y}{x})$, so γ is exact on this domain. Since $\gamma_x(t) = 1 + \frac{1}{2}\sin\frac{t^2}{\pi} > 0$ for all t, the curve γ is contained in $\{(x,y) : x > 0\}$. Hence

$$\int_\gamma \alpha = \arctan(\frac{\gamma_y(pi)}{\gamma_x(\pi)}) - \arctan(\frac{\gamma_y(0)}{\gamma_x(0)}) = 0 - \frac{\pi}{4} = \frac{-\pi}{4}.$$

3. a. Write $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$. Assume that there is a primitive F. If we integrate α_1 with respect to x, we get

$$F = x^3/3 - x^2/2 + xy + xz - xyz + f(y,z),$$

for some function $f(y,z)$ depending only on y and z. Differentiate this w.r.t. y and set it equal to α_2 to get

$$x - xz + \frac{\partial f}{\partial y} = y^2 + x - y + z - xz,$$

which is equivalent to $f = y^3/3 - y^2/2 + yz + g(z)$ for some function g depending only on z. Now differentiate the expression we have for F w.r.t. z and set equal to α_3 to get

$$x - xy + y + g'(z) = z^2 + x + y - z - xy,$$

whence $g(z) = z^3/3 - z^2/2 + C$ for some constant C. We thus have

$$F(x,y,z) = \frac{x^3 + y^3 + z^3}{3} - \frac{x^2 + y^2 + z^2}{2} + xy + xz + yz - xyz + C.$$

0.1 Remark:

It is important to note that the fact that we were able to find a solution to the equations above implies that α is exact with F as its primitive. In fact, the equations we solved above imply that $dF = \alpha$. Also, note that finding a primitive for an exact 1-form is the same as finding a potential function for a vector field. Finally, we remark here that one can guess the form of F very easily by symmetry.

b. Since $\alpha = dF$, we have that

$$\int_\gamma \alpha = F(\gamma(\pi)) - F(\gamma(0)) = 0,$$

since $\gamma(\pi) = \gamma(0)$. In other words, γ is a closed path, and so the integral of the exact form α along γ is 0.