Math 52H Homework 5 Solutions

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1. a. We have $\Delta = \partial^2 + \partial d + d\partial + d^2$. Now

$$\partial^2 = *^{-1} d * *^{-1} d* = *^{-1} dd* = 0$$

so $\Delta = \partial d + d\partial$ and this is a map $\Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$.

b. We compute $\Delta(\alpha)$ for an arbitrary 1-form $\alpha(x) = \alpha_1(x)dx_1 + ... + \alpha_n(x)dx_n$. We have

$$\begin{aligned}
\partial d\alpha &= \partial \left(\sum_{i \neq j} D_j \alpha_i dx_j \wedge dx_i \right) \\
&= \partial \left[\sum_{i<j} \left(D_i \alpha_j - D_j \alpha_i \right) dx_i \wedge dx_j \right] \\
&= (-1)^{3n+1} * d \left[\sum_{i<j} \left(D_i \alpha_j - D_j \alpha_i \right) * \left(dx_i \wedge dx_j \right) \right] \\
&= (-1)^{n+1} * \left[\sum_{i<j} \left\{ \left(D_i^2 \alpha_j - D_i D_j \alpha_i \right) dx_i \wedge *(dx_i \wedge dx_j) + \left(D_i D_j \alpha_j - D_j^2 \alpha_i \right) dx_j \wedge *(dx_i \wedge dx_j) \right\} \right]
\end{aligned}$$

Now $dx_i \wedge *(dx_i \wedge dx_j) = \pm * dx_j$. We can determine the sign by noting (ω is the volume form)

$$dx_i \wedge dx_j \wedge *(dx_i \wedge dx_j) = \omega = dx_j \wedge *dx_j.$$

Hence $dx_j \wedge dx_i \wedge *(dx_i \wedge dx_j) = -\omega$ so $dx_i \wedge *(dx_i \wedge dx_j) = - * dx_j$. Similarly we find

$dx_j \wedge *(dx_i \wedge dx_j) = *dx_i$. Plugging this in,

$$\begin{aligned}
\partial d\alpha &= (-1)^{n+1} * \left[\sum_{i<j} \left\{ \left(D_i D_j \alpha_i - D_i^2 \alpha_j \right) * dx_j + \left(D_i D_j \alpha_j - D_j^2 \alpha_i \right) * dx_i \right\} \right] \\
&= (-1)^{n+1+n-1} \sum_{i<j} \left\{ \left(D_i D_j \alpha_i - D_i^2 \alpha_j \right) dx_j + \left(D_i D_j \alpha_j - D_j^2 \alpha_i \right) dx_i \right\} \\
&= \sum_{i \neq j} (D_i D_j \alpha_j - D_j^2 \alpha_i) dx_i .
\end{aligned}$$

We also have

$$\begin{aligned}
d\partial\alpha &= (-1)^{2n+1} d * d \left(\sum_{i=1}^{n} \alpha_i * dx_i \right) = -d * \left(\sum_{i=1}^{n} D_i \alpha_i dx_i \wedge *dx_i \right) \\
&= -d * \left(\sum_{i=1}^{n} D_i \alpha_i \omega \right) = -d \left(\sum_{i=1}^{n} D_i \alpha_i \right) \\
&= - \sum_{i,j=1}^{n} D_i D_j \alpha_i dx_j
\end{aligned}$$

Hence

$$\Delta\alpha = (d\partial + \partial d)\alpha = \sum_{i \neq j} (D_i D_j \alpha_j - D_j^2 \alpha_i) dx_i - \sum_{i,j} D_i D_j \alpha_j dx_i = - \sum_{i,j} D_j^2 \alpha_i dx_i .$$

It follows that

$$\Delta(f\alpha) = - \sum_{i,j} D_i D_j (f \alpha_j) dx_i = - \sum_{i,j} (\alpha_j D_i D_j f + D_i f D_j \alpha_j + D_j f D_i \alpha_j + f D_i D_j \alpha_j) dx_i .$$

c. Let's agree to write ∇^2 for the second order differential operator $\nabla^2 = \sum_{j=1}^{n} D_j^2$. Thus our calculation from part b shows that for 1-form $\alpha = \alpha_1 dx_1 + \alpha_2 dx_2 + \alpha_3 dx_3$ we have

$$\Delta\alpha = -(\nabla^2 \alpha_1 dx_1 + \nabla^2 \alpha_2 dx_2 + \nabla^2 \alpha_3 dx_3).$$

Now we claim that Δ commutes with the dual operator $*$, i.e. $*\Delta\alpha = \Delta * \alpha$. Assuming this for the moment, we get by dualizing that for α the 2-form

$$\alpha = \alpha_1 dx_2 \wedge dx_3 + \alpha_2 dx_3 \wedge dx_1 + \alpha_3 dx_1 \wedge dx_2,$$

$$\Delta\alpha = -(\nabla^2 \alpha_1 dx_2 \wedge dx_3 + \nabla^2 \alpha_2 dx_3 \wedge dx_1 + \nabla^2 \alpha_3 dx_1 \wedge dx_2).$$

For α a zero form, $\partial\alpha = \pm * d * \alpha = 0$ since $*\alpha$ is an n-form. Hence

$$\begin{aligned}\Delta\alpha = (d\partial + \partial d)\alpha = \partial d\alpha &= (-1)^{2n+1} * d(D_1\alpha dx_2 \wedge dx_3 + D_2\alpha dx_3 \wedge dx_1 + D_3\alpha dx_1 \wedge dx_2)\\ &= -(D_1^2\alpha + D_2^2\alpha + D_3^2\alpha) = -\nabla^2\alpha.\end{aligned}$$

Dualizing, $\Delta\alpha(x)dx_1 \wedge dx_2 \wedge dx_3 = -\nabla^2\alpha dx_1 \wedge dx_2 \wedge dx_3$.

It remains to show that $*\Delta = \Delta*$. Indeed, $*\Delta\alpha = *(d\partial+\partial d) = -(*d*^{-1}d*+d*d)$ while $\Delta*\alpha = (d\partial+\partial d)* = -(d*^{-1}d**+*^{-1}d*d*)$. The two are equal since $*^{-1} = (-1)^{k(n-k)}*$, and $** = (-1)^{k(n-k)}\mathrm{Id}$.

d. We note that $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial\theta}$ give an orthonormal basis for each tangent plane at every point $x \in \mathbb{R}^2$. The dual forms are dr and $rd\theta$, from which we infer that $*dr = rd\theta$ and $*d\theta = -\frac{1}{r}dr$. Say that $k = 0$ and fix a 0-form f. Then $d\partial f = 0$ and

$$\begin{aligned}\partial df &= \partial\left(D_r f dr + D_\theta f d\theta\right)\\ &= - * d\left(D_r f r d\theta - D_\theta f \frac{1}{r} dr\right)\\ &= - * \left(D_r f dr \wedge d\theta + r D_r D_r f dr \wedge d\theta - D_\theta D_\theta f \frac{1}{r} dr \wedge dr\right)\\ &= -\left(\frac{1}{r}D_r f + D_r D_r f + \frac{1}{r^2}D_\theta D_\theta f\right).\end{aligned}$$

For the case $k = 2$, we may write

$$\begin{aligned}\Delta f dr \wedge d\theta = \Delta * \frac{f}{r} &= *\Delta\frac{f}{r}\\ &= - * \left(\frac{1}{r}D_r(f/r) + D_r D_r(f/r) + \frac{1}{r^3}D_\theta D_\theta f\right)\\ &= -\left(\frac{1}{r}D_r(f/r) + D_r D_r(f/r) + \frac{1}{r^3}D_\theta D_\theta f\right) r dr \wedge d\theta\\ &= -\left(D_r D_r f - \frac{D_r f}{r} + \frac{f}{r^2} + \frac{1}{r^2}D_\theta D_\theta f\right) dr \wedge d\theta.\end{aligned}$$

When $k = 1$ write the arbitrary 1-form $\alpha = a_r dr + a_\theta d_\theta$. We have $\partial d a_r dr = \partial D_\theta a_r d\theta \wedge dr = *d\frac{D_\theta a_r}{r} = -\left(\frac{D_\theta^2 a_r}{r^2}dr + rD_r D_\theta\left(\frac{a_r}{r}\right)d\theta\right)$. Also $d\partial a_r dr = -d*d(ra_r d\theta) = -d*(a_r + rD_r a_r)dr \wedge d\theta = -d(a_r/r + D_r a_r) = -\left(\frac{D_\theta a_r}{r}d\theta + D_\theta D_r a_r d\theta + D_r\left(\frac{a_r}{r}\right)dr + D_r^2 a_r dr\right)$. Adding the two gives

$$\Delta a_r dr = -\left(\frac{D_\theta^2 a_r}{r^2} + D_r\left(\frac{a_r}{r}\right) + D_r^2 a_r\right) dr - 2D_\theta D_r a_r d\theta.$$

Similarly, $\partial d a_\theta d\theta = \partial D_\theta a_r d\theta \wedge dr = -*d\frac{D_r a_\theta}{r} = -*\left(\frac{D_r^2 a_\theta}{r}dr - \frac{D_r a_\theta}{r^2}dr + \frac{D_r D_\theta a_\theta}{r}d\theta\right) = -\left(D_r^2 a_\theta d\theta - \frac{D_r a_\theta}{r}d\theta - \frac{D_\theta D_r a_\theta}{r^2}\right)$. Also $d\partial a_\theta d\theta = -d*d(a_\theta/rdr) = d*\frac{D_\theta a_\theta}{r}dr\wedge d\theta = d\frac{D_\theta a_\theta}{r^2} = \frac{D_\theta^2 a_\theta}{r^2}d\theta + \left(\frac{D_\theta D_r a_\theta}{r^2} - 2D_\theta\frac{a_\theta}{r^3}\right)dr$. Adding the two gives

$$\Delta a_\theta d\theta = -\left(-\frac{D_\theta^2 a_\theta}{r^2} - D_r\left(\frac{a_\theta}{r}\right) + D_r^2 a_\theta\right)dr - 2\left(\frac{D_\theta a_\theta}{r^3} - \frac{D_\theta D_r a_\theta}{r^2}\right)dr.$$

Adding the two gives the final answer.

2. We calculate

$$\operatorname{div}(\nabla f) = \operatorname{div}(\sum_{j=1}^n D_j f dx_j) = \sum_{j=1}^n D_j^2 f$$

$$\operatorname{div}(\operatorname{curl} v) = \Lambda^{-1}(d(\rfloor\rfloor^{-1}d(\mathcal{D}v))) = \Lambda^{-1}(dd(\mathcal{D}v)) = 0$$

$$\operatorname{curl}(\nabla f) = \rfloor^{-1}d(\mathcal{D}\mathcal{D}^{-1}(df)) = \rfloor^{-1}(ddf) = 0$$

$$\begin{aligned}\operatorname{curl}(fv) &= (D_2(fv_3) - D_3(fv_2), D_3(fv_1) - D_1(fv_3), D_1(fv_2) - D_2(fv_1))\\ &= f\operatorname{curl}(v) + (v_3 D_2 f - v_2 D_3 f, v_1 D_3 f - v_3 D_1 f, v_2 D_1 f - v_1 D_2 f)\\ &= f\operatorname{curl}(v) + \nabla f \times v\\ \operatorname{div}(fv) &= D_1(fv_1) + D_2(fv_2) + D_3(fv_3) = f\operatorname{div}(v) + v_1 D_1 f + v_2 D_2 f + v_3 D_3 f\\ &= f\operatorname{div}(v) + v\cdot\nabla f.\end{aligned}$$

3. By definition of div, it suffices to show that $d(\lrcorner\nabla f \times \nabla g) = 0$. Following the hint, since we know that $\mathcal{D}(\nabla f) = df$ and similarly $\mathcal{D}(\nabla g) = dg$, this is the same as

$$\begin{aligned}d\lrcorner\mathcal{D}^{-1}*(df\wedge dg) &= d*\mathcal{D}\mathcal{D}^{-1}*(df\wedge dg)\\ &= d(df\wedge dg)\\ &= d^2(fdg)\\ &= 0.\end{aligned}$$

In the above, we have used that $\lrcorner = *\mathcal{D}$, $*^2 = \mathrm{Id}$, and $d^2 = 0$.