Math 52H Homework 4 Solutions

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1. We have

$$
\begin{aligned}
dx &= \sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta \\
dy &= \sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta \\
dz &= \cos\phi d\rho - \rho\sin\phi d\phi.
\end{aligned}
$$

Hence the 1-form is equal to

$$
\begin{aligned}
&\cos\phi d\rho - \rho\sin\phi d\phi + \frac{1}{2}\Big(\rho\sin\phi\cos\theta\left[\sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta\right] \\
&\qquad - \rho\sin\phi\sin\theta\left[\sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta\right]\Big) \\
&= \cos\theta d\rho - \rho\sin\phi d\phi + \rho^2\sin^2\phi d\theta.
\end{aligned}
$$

2. There was some confusion as to whether the image of f was to be taken in Cartesian or polar coordinates, so we'll show the solution in both cases.

Suppose first that the image is in Cartesian coordinates, i.e.

$$f(r,\phi) = re_1 + (\phi + r^2\sin\phi)e_2$$

where e_1 and e_2 are the dual basis to x and y. Then

$$f^*(x\wedge y) = dr\wedge(2r\sin\phi dr + (1 + r^2\cos\phi)d\phi) = (1 + r^2\cos\phi)dr\wedge d\phi.$$

If instead we take

$$f(r,\phi) = r\hat{r} + (\phi + r^2\sin\phi)\hat{\phi}$$

with $\hat{r},\hat{\phi}$ dual to r and ϕ, then recall that $dx\wedge dy = rdr\wedge d\phi$, so that we find

$$f^*(rdr\wedge d\phi) = r(dr\wedge(2r\sin\phi dr + (1 + r^2\cos\phi)d\phi)) = (r + r^3\cos\phi)dr\wedge d\phi.$$

3. Since $\beta \wedge d\beta = xdy \wedge (dx \wedge dy) = 0$ we have for any smooth $f : \mathbb{R}^3 \to \mathbb{R}^3$,

$$0 = f^*(\beta \wedge d\beta) = f^*(\beta) \wedge f^*(d\beta) = f^*(\beta) \wedge df^*(\beta).$$

Since $\alpha \wedge d\alpha = (dx + ydz) \wedge (dy \wedge dz) = dx \wedge dy \wedge dz \neq 0$ it is not possible that there is smooth f with $f^*(\beta) = \alpha$.

4. Since ω is closed,

$$d\omega = (D_1F_2 - D_2F_1)dx \wedge dy + (D_3F_1 - D_1F_3)dz \wedge dx + (D_2F_3 - D_3F_2)dy \wedge dz = 0$$

so we have the three equalities

$$D_1F_2 = D_2F_1, \quad D_3F_1 = D_1F_3, \quad D_2F_3 = D_3F_2. \tag{1}$$

Differentiating the homogeneity equation with respect to t, and then setting $t = 1$ we obtain

$$F_k(x, y, z) = xD_1F_k(x, y, z) + yD_2F_k(x, y, z) + zD_3F_k(x, y, z). \tag{2}$$

Hence

$$\begin{aligned} df &= \frac{1}{2}\Big((F_1 + xD_1F_1 + yD_1F_2 + zD_1F_3)dx + (xD_2F_1 + F_2 + yD_2F_2 + zD_2F_3)dy \\ &\qquad + (xD_3F_1 + yD_3F_2 + F_3 + zD_3F_3)dz\Big) \\ &= \frac{1}{2}\omega + \frac{1}{2}\Big((xD_1F_1 + yD_2F_1 + zD_3F_1)dx + (xD_1F_2 + yD_2F_2 + zD_3F_2)dy \\ &\qquad + (xD_1F_3 + yD_2F_3 + zD_3F_3)dz\Big) \\ &= \omega. \end{aligned}$$

5. Let $f = \sum_{i=1}^n x_i x_{i+n}$ and let $\mathcal{V} = x_1 \wedge ... \wedge x_{2n}$ denote the volume form. Then

$$\begin{aligned} d\Omega &= \frac{d(\omega \wedge \theta)}{f^n} - n\sum_1^n x_{i+n}dx_i \wedge \frac{\omega \wedge \theta}{f^{n+1}} \\ &= \frac{n}{f^n}\left(\mathcal{V} - \sum_1^n x_{i+n}dx_i \wedge \frac{\omega \wedge \theta}{f}\right) \\ &= \frac{n}{f^n}\left(\mathcal{V} - \sum_1^n x_i x_{i+n}\frac{\mathcal{V}}{f}\right) \\ &= 0, \end{aligned}$$

as desired.