Math 52H Homework 3 Solutions

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1. a. This is question 6 on Homework 1.

b. Let $e_1, ..., e_{2n}$ be basis vectors for $\mathbb{R}^{2n}$, where e_{2k-1} corresponds to x_k and e_{2k} corresponds to y_k for $1 \leq k \leq n$. We first find the matrix corresponding to the skew-symmetric bilinear form ω. Say that $i \leq j$. Then, we have that $\omega(e_i, e_j) = 0$ unless $i = 2k-1$ and $j = 2k$ for some k in which case $\omega(e_i, e_j) = 1$. By skew-symmetry of ω, the matrix corresponding to ω is $-J$. Now $\mathcal{A}^*\omega = \omega$ implies that for $u, v \in \mathbb{R}^{2n}$ that

$$u^t(-J)v = u^t A^t(-J)Av,$$

which immediately gives $J = A^t JA$, as desired.

c. Let $\mathcal{U}$ be an orthogonal operator with matrix U. Then $\mathcal{U}$ is unitary $\Leftrightarrow UJ = JU \Leftrightarrow J = U^{-1}JU = U^tJU \Leftrightarrow \mathcal{U}$ is symplectic.

d. L is Lagrangian is equivalent to $0 = \omega(u, v) = -u^tJv$ for all $u, v \in L \Leftrightarrow u$ is orthogonal to Jv for all $u, v \in L \Leftrightarrow J(L) = L^\perp$. The last part follows since J is one to one, and L is dimension n so $J(L)$ is a dimension n subspace orthogonal to L.

e. First, say that $L = \mathcal{U}(L_0)$ for some unitary $\mathcal{U}$. Then, for all $u, v \in L$, let $u = \mathcal{U}(u_0)$ and $v = \mathcal{U}(v_0)$ for $u_0, v_0 \in L_0$. Then $\omega(u, v) = -< u, Jv >= -< \mathcal{U}u_0, J\mathcal{U}v_0 >= -< u_0, Jv_0 >= 0$. Thus, L is Lagragian.

Now, say that L is Lagrangian, and let $g_1, g_2, ..., g_n$ be an orthonormal basis for L. L being Lagrangian implies that $g_1, Jg_1, g_2, Jg_2, ..., g_n, Jg_n$ forms an orthonormal basis. With notation as in a., note that L_0 is spanned by e_{2k-1}. Define an orthogonal operator $\mathcal{U}$ by letting $\mathcal{U}(e_{2k-1}) = g_k$ and $\mathcal{U}(e_{2k}) = Jg_k$ for $1 \leq k \leq n$. Since $\mathcal{U} \circ \mathcal{J}(e_i) = \mathcal{J} \circ \mathcal{U}(e_i)$, $\mathcal{U} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{U}$ on all of $\mathbb{R}^{2n}$ whence $\mathcal{U}$ is an unitary operator such that $L = \mathcal{U}(L_0)$.

2. We have $\nabla f(y) = \mathcal{D}^{-1}(d_y f)$. Hence for any $h = h_1 v_1 + h_2 v_2 \in V$

$$h_1 \frac{\partial f}{\partial y_1}(y) + h_2 \frac{\partial f}{\partial y_2}(y) = \langle \nabla f(y), h \rangle = \langle g_1(y) v_1 + g_2(y) v_2, h_1 v_1 + h_2 v_2 \rangle,$$

or compactly,

$$\left[\frac{\partial f}{\partial y_1}(y), \frac{\partial f}{\partial y_2}(y) \right] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = [g_1(y), g_2(y)] \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix},$$

so, since h was arbitrary,

$$[g_1(y), g_2(y)] = \left[\frac{\partial f}{\partial y_1}(y), \frac{\partial f}{\partial y_2}(y) \right] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \left[\frac{2\partial f}{\partial y_1}(y) - \frac{\partial f}{\partial y_2}(y), \frac{-\partial f}{\partial y_1}(y) + \frac{\partial f}{\partial y_2}(y) \right].$$

3. As usual, write $e_1, ..., e_n$ for the standard basis on $\mathbb{R}^n$ so that $X = \sum_{i=1}^n f_i e_i$ and $Y = \sum_{i=1}^n g_i e_i$.

Let h be an arbitrary smooth function $\mathbb{R}^n \to \mathbb{R}$. Then

$$L_X L_Y(h) = L_X(\sum_{i=1}^n g_i \frac{\partial h}{\partial x_i}) = \sum_{j=1}^n f_j \frac{\partial}{\partial x_j}(\sum_{i=1}^n g_i \frac{\partial h}{\partial x_i}).$$

By linearity of the derivative and the Leibniz rule, we obtain

$$L_X L_Y(h) = \sum_{i,j=1}^n f_j \left(\frac{\partial g_i}{\partial x_j} \frac{\partial h}{\partial x_i} + g_i \frac{\partial^2 h}{\partial x_j \partial x_i} \right) = \sum_{i,j=1}^n f_j \frac{\partial g_i}{\partial x_j} \frac{\partial h}{\partial x_i} + \sum_{i,j=1}^n f_j g_i \frac{\partial^2 h}{\partial x_j \partial x_i}.$$

Similarly,

$$L_Y L_X = \sum_{i,j=1}^n g_i \frac{\partial f_j}{\partial x_i} \frac{\partial h}{\partial x_j} + \sum_{i,j=1}^n f_j g_i \frac{\partial^2 h}{\partial x_i \partial x_j}.$$

Hence, using that the mixed partials of h commute (h is smooth) we obtain

$$L_X L_Y(h) - L_Y L_X(h) = \sum_{i,j=1}^n \left[f_j \frac{\partial g_i}{\partial x_j} \frac{\partial h}{\partial x_i} - g_i \frac{\partial f_j}{\partial x_i} \frac{\partial h}{\partial x_j} \right] = \sum_{i=1}^n \left[\sum_{j=1}^n f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \right] \frac{\partial h}{x_i} = L_Z(h)$$

where Z is the vector field

$$Z = \sum_{i=1}^n \left[\sum_{j=1}^n f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \right] e_i.$$

This holds for arbitrary smooth h, so $L_X L_Y - L_Y L_X = L_Z$ as an equality of first order differential operators (on smooth functions).