Math 52H Homework 1 Solutions

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1. Let $e_1, ..., e_n, f_1, ..., f_n$ be the standard basis on $\mathbb{R}^{2n}$ with $x_1, ..., x_n, y_1, ..., y_n$ the dual basis. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be given by $Ae_i = \sum_{j=1}^n A_{ji} e_j$.

a. We have

$$(L_A)^*\omega = \sum_{1 \le i < j \le n} (L_A)^*\omega(e_i, e_j) x_i \wedge x_j.$$

Now

$$\begin{aligned}
(L_A)^*\omega(e_i, e_j) &= \omega(L_A e_i, L_A e_j) \\
&= \sum_{l=1}^n (x_l \otimes y_l - y_l \otimes x_l)((e_i + \sum_{k=1}^n A_{ki} f_k), (e_j + \sum_{k=1}^n A_{kj} f_k)) \\
&= \sum_{l=1}^n \left[x_l(e_i + \sum_{k=1}^n A_{ki} f_k) y_l(e_j + \sum_{k=1}^n A_{kj} f_k) - y_l(e_i + \sum_{k=1}^n A_{ki} f_k) x_l(e_j + \sum_{k=1}^n A_{kj} f_k) \right] \\
&= y_i(e_j + \sum_{k=1}^n A_{kj} f_k) - y_j(e_i + \sum_{k=1}^n A_{ki} f_k) \\
&= A_{ij} - A_{ji}.
\end{aligned}$$

So $(L_A)^*\omega = \sum_{1 \le i < j \le n} (A_{ij} - A_{ji}) x_i \wedge x_j$ and $(L_A)^*\omega = 0$ if and only if A is symmetric.

b. Write $C_\omega(e_k) = \sum_j (\alpha_j x_j + \beta_j y_j)$. We have that $\alpha_j = C_\omega(e_k)(e_j) = \omega(e_k, e_j) = \sum_j x_i \wedge y_i(e_k, e_j) = 0$, and $\beta_j = C_\omega(e_k)(f_j) = \omega(e_k, f_j) = \sum_j x_i \wedge y_i(e_k, f_j) = \delta_{k,j}$.

Similarly, write $C_\omega(f_k) = \sum_j (\alpha_j x_j + \beta_j y_j)$. We then have $\alpha_j = C_\omega(f_k)(e_j) = \omega(f_k, e_j) = \sum_j x_i \wedge y_i(f_k, e_j) = -\delta_{k,j}$, and $\beta_j = 0$ similar to before.

The vector $(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n)$ corresponding to (e_k, f_k) is the kth column of the required matrix. Thus $C = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, where 0 denotes the $n \times n$ zero matrix, and I denotes the $n \times n$ identity matrix.

2. Consider the linear operator $L = \frac{1}{2}(* + I) : \wedge^2((\mathbb{R}^4)^*) \to \wedge^2((\mathbb{R}^4)^*)$; the two-form β is self-dual if and only if $L\beta = \beta$.

Since $*^2 = I$ is the identity, $L^2 = \frac{1}{4}(*^2 + 2* + I) = \frac{1}{2}(* + I) = L$ so L is a projection, that is,

$$\{v : Lv = v\} = \text{im}(L).$$

We have

$$L(x_1 \wedge x_2) = \frac{1}{2}(x_1 \wedge x_2 + x_3 \wedge x_4) = L(x_3 \wedge x_4), \quad L(x_1 \wedge x_3) = \frac{1}{2}(x_1 \wedge x_3 - x_2 \wedge x_4) = -L(x_2 \wedge x_4)$$

$$L(x_1 \wedge x_4) = \frac{1}{2}(x_1 \wedge x_4 + x_2 \wedge x_3) = L(x_2 \wedge x_3),$$

so the space of self-dual forms is 3-dimensional, with basis

$$\{\frac{1}{2}(x_1 \wedge x_2 + x_3 \wedge x_4), \frac{1}{2}(x_1 \wedge x_3 - x_2 \wedge x_4), \frac{1}{2}(x_1 \wedge x_4 + x_2 \wedge x_3)\}.$$

3. We let A be the matrix representation of $\mathcal{A}$, and let $e_1, ..., e_n$ be the basis of $\mathbb{R}^n$ correponding to $x_1, ..., x_n$. Then apply the identity given in the question to $(e_1, e_2, ..., e_n)$. The left hand side is $\det A$. The right hand side is, by definition of exterior product

$$\sum_{i_1 < i_2 < ... < i_k, i_{k+1} < ... < i_n} (-1)^{\text{inv}(i_1, ..., i_n)} \mathcal{A}^*(x_1 \wedge ... \wedge x_k)(e_{i_1}, ..., e_{i_k}) \mathcal{A}^*(x_{k+1} \wedge ... \wedge x_n)(e_{i_{k+1}}, ..., e_{i_n}),$$

and this is precisely the right hand side by Proposition 1.9.5 in the notes.

4. Let $Y = \alpha X + \beta Z$ where $< X, Z >= 0$. Then $X \times Y = \beta X \times Z$, since $X \times X = 0$ whereas $\mathcal{D}^{-1}(*(\mathcal{D}(X) \wedge \mathcal{D}(Y))) = \beta \mathcal{D}^{-1}(*(\mathcal{D}(X) \wedge \mathcal{D}(Z)))$ since $\mathcal{D}(X) \wedge \mathcal{D}(X) = 0$. It thus suffices to check that

$$X \times Z = \mathcal{D}^{-1}(*(\mathcal{D}(X) \wedge \mathcal{D}(Z))).$$

We may assume that $X, Z \neq 0$, and by dividing the above by $||X||||Z||$, we may further assume that X and Z are orthonormal. Let X, Z, W be an orthonormal basis for $\mathbb{R}^3$, where $W = X \times Z$, so that the basis defines the standard orientation for $\mathbb{R}^3$. Note that $\mathcal{D}(X), \mathcal{D}(Z), \mathcal{D}(W)$ forms the dual basis. By §1.16 in the notes [1], we have that $*(\mathcal{D}(X) \wedge \mathcal{D}(Z)) = \mathcal{D}(W)$, and so $\mathcal{D}^{-1}(*(\mathcal{D}(X) \wedge \mathcal{D}(Z))) = W = X \times Z$ as desired.

5. We first prove the following lemma.

Lemma 1. *For any two $n \times n$ matrices A and B such that $AB = BA$, we have that*

$$\exp(A + B) = \exp(A) \exp(B).$$

[1] See, for instance, the second example given in that section.

Proof.

$$\begin{aligned}\exp(A+B) &= \sum_{k\geq 0}\frac{(A+B)^k}{k!}\\ &= \sum_{j\geq 0}\sum_{m=0}^{k}\frac{A^m B^{j-m}}{m!(j-m)!}\\ &= \sum_{m,n}\frac{A^m}{m!}\frac{B^n}{n!},\end{aligned}$$

as desired. In the above, we have used that $AB = BA$ in our binomial expansion. □

Next note that for any $n\times n$ matrix M that $\exp(M)^T = \left(\sum_{k\geq 0}\frac{M^k}{k!}\right)^T = \sum_{k\geq 0}\frac{(M^T)^k}{k!} = \exp(M^T)$. Now let A be skew-symmetric so $A^T = -A$. Then $A^TA = AA^T$ so applying the above and the Lemma gives

$$\exp(A)\exp(A)^T = \exp(A)\exp(A^T) = \exp(A+A^T) = \exp(0) = I,$$

so $\exp(A)$ is orthogonal.

Conversely, say that $\exp(tA)$ is orthogonal for all t. Then

$$\begin{aligned}I &= \exp(tA)\exp(tA^T)\\ &= \left(\sum_{k\geq 0}\frac{t^kA^k}{k!}\right)\left(\sum_{k\geq 0}\frac{t^k(A^T)^k}{k!}\right)\\ &= I + t(A+A^T) + \frac{t^2}{2}(2AA^T + A^2 + (A^T)^2) + ...\end{aligned}$$

Apply $\frac{d}{dt}\Big|_{t=0}$ to the equation above. The left hand side is 0, whereas the right hand side is $(A + A^T + 2t(2AA^T + A^2 + (A^T)^2) + ...)\Big|_{t=0} = A + A^T$. Hence $A = -A^T$, as desired.