Math 52H Homework 1 Solutions

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1. Let $T = x_1 \otimes x_1 \otimes x_2$, $u = (1,0,0)$ and $v = (0,1,0)$. Then $T(u,u,v) = 1$ and $T(u,v,u) = 0$. Say that $T = A + B$ where A is symmetric and B is anti-symmetric. Then $A(u,u,v) = A(u,v,u)$, whereas $B(u,u,v) = 0 = B(u,v,u)$, a contradiction.

This shows that not all 3-linear functions on $\mathbb{R}^3$ can be expressed as a sum of a symmetric function and a skew-symmetric function.

2. To verify that $< \cdot, \cdot >$ is an inner product we check the axioms. First, $< X, Y >= \mathrm{Tr}(XY) = \mathrm{Tr}(YX) =< Y, X >$. Also $< \alpha X + Y, Z >= \mathrm{Tr}((\alpha X + Y)Z) = \alpha \mathrm{Tr}(XZ) + \mathrm{Tr}(YZ) = \alpha < X, Z > + < Y, Z >$. Finally, for X symmetric, X is diagonalizable over $\mathbb{R}$ with real eigenvalues λ_1 and λ_2. Thus $< X, X >= \mathrm{Tr} < X^2 >= \lambda_1^2 + \lambda_2^2 \geq 0$. Equality holds if and only if $\lambda_1 = \lambda_2 = 0$ which is equivalent to $X = 0$.

Let $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $A_3 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It is easy to check that $A_i A_j$ has zeros on the diagonal unless $i = j$. Moreover $\mathrm{Tr}(A_i^2) = 1$ if $i = 1, 3$ and $\mathrm{Tr}(A_2^2) = 2$. Thus the matrix for the biliear form is $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

3. Let $\{e_1, ..., e_n\}$ be the usual basis for $\mathbb{R}^n$, dual to $x_1, ..., x_n$. We have

$$f \otimes g = \sum_{1 \leq i_1, i_2, i_3, i_4 \leq n} f \otimes g(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}) x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4}$$

[Proof: Let

$$H = \sum_{1 \leq i_1, i_2, i_3, i_4 \leq n} f \otimes g(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}) x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4}.$$

Then $H(e_{j_1}, e_{j_2}, e_{j_3}, e_{j_4}) = f \otimes g(e_{j_1}, e_{j_2}, e_{j_3}, e_{j_4})$. Equality at general v_1, v_2, v_3, v_4 then follows by multilinearity of both H and $f \otimes g$.]

But

$$f \otimes g(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}) = f(e_{i_1}, e_{i_2}) g(e_{i_3}, e_{i_4}) = a_{i_1 i_2} b_{i_3 i_4}$$

so

$$f \otimes g = \sum_{1 \le i_1, i_2, i_3, i_4 \le n} a_{i_1 i_2} b_{i_3 i_4} x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4}.$$

4. (Assume $\beta \neq 0$.) Let $\alpha = a_1 x_2 \wedge x_3 + a_2 x_3 \wedge x_1 + a_3 x_1 \wedge x_2$ and $\beta = b_1 x_1 + b_2 x_2 + b_3 x_3$. Then $\alpha \wedge \beta = (a_1 b_1 + a_2 b_2 + a_3 b_3) x_1 \wedge x_2 \wedge x_3$ so

$$\alpha \wedge \beta = 0 \quad \Leftrightarrow \quad \langle a, b \rangle = 0.$$

This is equivalent to $a = b \times c$ for some $c \in \mathbb{R}^3$, i.e.

$$a_1 = b_2 c_3 - b_3 c_2, \quad a_2 = b_3 c_1 - b_1 c_3, \quad a_3 = b_1 c_2 - b_2 c_1$$

which says exactly that

$$\alpha = (b_1 x_1 + b_2 x_2 + b_3 x_3) \wedge (c_1 x_1 + c_2 x_2 + c_3 x_3).$$

5. We have $\theta = \sum_{i=1}^{n-1} x_i \otimes x_{i+1} - \sum_{i=1}^{n-1} x_{i+1} \otimes x_i$ so

$$\begin{aligned} \theta(A, B) &= \sum_{i=1}^{n-1} A_i B_{i+1} - \sum_{i=1}^{n-1} A_{i+1} B_i \\ &= \sum_{i=1}^{n-1} (-1)^{i+1} - \sum_{i=1}^{n-1} (-1)^i = \begin{cases} 2 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}. \end{aligned}$$

6. a. The n-fold wedge product of ω with itself is given by

$$(\omega)^{\wedge n} = \underbrace{\omega \wedge \ldots \wedge \omega}_{n \text{ times}} = n!(x_1 \wedge y_1) \wedge (x_2 \wedge y_2) \wedge \ldots \wedge (x_n \wedge y_n).$$

b. Now we have

$$F^*(\omega^{\wedge n}) = (F^* \omega)^{\wedge n} \tag{1}$$

so

$$n! F^*((x_1 \wedge y_1) \wedge \ldots \wedge (x_n \wedge y_n)) = n!(x_1 \wedge y_1) \wedge \ldots \wedge (x_n \wedge y_n).$$

Since $(x_1 \wedge y_1) \wedge \ldots \wedge (x_n \wedge y_n)$ is (up to sign/orientation) the volume form on $\mathbb{R}^{2n}$

$$F^*((x_1 \wedge y_1) \wedge \ldots \wedge (x_n \wedge y_n) = \det F (x_1 \wedge y_1) \wedge \ldots \wedge (x_n \wedge y_n) \tag{2}$$

and $\det F = 1$.

[Proof of (1): This follows by repeatedly applying (i.e. with induction) the identity

$$F^*(\omega_1 \wedge \omega_2) = (F^*\omega_1) \wedge (F^*\omega_2) \tag{3}$$

valid for any $\omega_1 \in \wedge^k(V^*)$, $\omega_2 \in \wedge^l(V^*)$. To prove (3), write

$$\begin{aligned}
F^*(\omega_1 \wedge \omega_2)&(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}) \\
&= \omega_1 \wedge \omega_2(Fv_1, ..., Fv_k, Fv_{k+1}, ..., Fv_{k+l}) \\
&= \sum_{i_1<...<i_k, i_{k+1}<...<i_{k+l}} (-1)^{inv(i_1,...,i_{k+l})} \omega_1(Fv_{i_1}, ..., Fv_{i_k}) \omega_2(Fv_{i_{k+1}}, ..., Fv_{i_{k+l}}) \\
&= \sum_{i_1<...<i_k, i_{k+1}<...<i_{k+l}} (-1)^{inv(i_1,...,i_{k+l})} F^*\omega_1(v_{i_1}, ..., v_{i_k}) F^*\omega_2(v_{i_{k+1}}, ..., v_{i_{k+l}}) \\
&= (F^*\omega_1) \wedge (F^*\omega_2)(v_1, ..., v_k, v_{k+1}, ..., v_{k+l}). \qquad]
\end{aligned}$$

[Proof of (2): Let $F : \mathbb{R}^n \to \mathbb{R}^n$. Then $F^*(x_1 \wedge ... \wedge x_n) = cx_1 \wedge ... \wedge x_n$ since the space of exterior n-forms on $\mathbb{R}^n$ is one-dimensional. To determine c, we check

$$c = F^*(x_1 \wedge ... \wedge x_n)(e_1, ..., e_n) = x_1 \wedge ... \wedge x_n(Fe_1, ..., Fe_n) = \begin{vmatrix} | & & | \\ Fe_1 & ... & Fe_n \\ | & & | \end{vmatrix} = |Mat_n(F)|. \qquad]$$