# Math 52H: Multilinear algebra, differential forms and Stokes' theorem

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March 8, 2011

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# 1 Linear and multilinear functions

## 1.1 Dual space

Let $V$ be a finite-dimensional real vector space. The set of all linear functions on $V$ will be denoted by $V^*$.

**Proposition 1.1.1.** *$V^*$ is a vector space of the same dimension as $V$.*

*Proof.* One can add linear functions and multiply them by real numbers:

$$\begin{aligned}(l_1 + l_2)(x) &= l_1(x) + l_2(x)\\ (\lambda l)(x) &= \lambda l(x) \quad \text{for } l, l_1, l_2 \in V^*,\ x \in V,\ \lambda \in \mathbb{R}\end{aligned}$$

It is straightforward to check that all axioms of a vector space are satisfied for $V^*$. Let us now check that $\dim V = \dim V^*$.

Choose a basis $v_1 \dots v_n$ of $V$. For any $x \in V$ let $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be its coordinates in the basis $v_1 \dots v_n$. Notice that each coordinate $x_1, \dots, x_n$ can be viewed as a linear function on $V$. Indeed,

1) the coordinates of the sum of two vectors are equal to the sum of the corresponding coordinates;

2) when a vector is multiplied by a number, its coordinates are being multiplied by the same number.

Thus $x_1, \dots, x_n$ are vectors from the space $V^*$. Let us show now that they form a basis of $V^*$. Indeed, any linear function $l \in V^*$ can be written in the form $l(x) = a_1x_1 + \ldots + a_nx_n$

which means that $l$ is a linear combination of $x_1 \dots x_n$ with coefficients $a_1, \dots, a_n$. Thus $x_1, \dots, x_n$ generate $V^*$. On the other hand, if $a_1 x + \dots + a_n x_n$ is the 0-function, then all the coefficients must be equal to 0; i.e. functions $x_1, \dots, x_n$ are linearly independent. Hence $x_1, \dots, x_n$ form a basis of $V$ and therefore $\dim V^* = n = \dim V$. ■

The space $V^*$ is called *dual* to $V$ and the basis $x_1, \dots, x_n$ *dual* to $v_1 \dots v_n$.[1]

**Exercise 1.1.2.** *Prove the converse: given any basis $l_1, \dots, l_n$ of $V^*$ we can construct a dual basis $w_1, \dots, w_n$ of $V$ so that the functions $l_1, \dots, l_n$ serve as coordinate functions for this basis.*

Recall that vector spaces of the same dimension are isomorphic. For instance, if we fix bases in both spaces, we can map vectors of the first basis into the corresponding vectors of the second basis, and extend this map by linearity to an isomorphism between the spaces. In particular, sending a basis $S = \{v_1, \dots, v_n\}$ of a space $V$ into the dual basis $x_1, \dots, x_n$ of the dual space $V^*$ we can establish an isomorphism $i_S : V \to V^*$. However, this isomorphism *is not canonical*, i.e. it depends on the choice of the basis $v_1, \dots, v_n$.

If $V$ is a Euclidean space, i.e. a space with a scalar product $\langle x, y \rangle$, then this allows us to define another isomorphism $V \to V^*$, different from the one described above. This isomorphism associates with a vector $v \in V$ a linear function $l_v(x) = \langle v, x \rangle$. We will denote the corresponding map $V \to V^*$ by $\mathcal{D}$. Thus we have $\mathcal{D}(v) = l_v$ for any vector $v \in V$.

**Exercise 1.1.3.** *Prove that $\mathcal{D} : V \to V^*$ is an isomorphism. Show that $\mathcal{D} = i_S$ for any* orthonormal *basis $S$.*

The isomorphism $\mathcal{D}$ is independent of a choice of an orthonormal basis, but is still not completely canonical: it depends on a choice of a scalar product.

[1] It is sometimes customary to denote dual bases in $V$ and $V^*$ by the same letters but using lower indices for $V$ and upper indices for $V^*$, e.g. $v_1, \dots, v_n$ and $v^1, \dots, v^n$. However, in these notes we do not follow this convention.

**Remark 1.1.4.** The definition of the dual space $V^*$ also works in the infinite-dimensional case.

**Exercise 1.1.5.** *Show that both maps $i_S$ and $\mathcal{D}$ are injective in the infinite case as well. However, neither one is surjective if $V$ is infinite-dimensional.*

## 1.2 Canonical isomorphism between $(V^*)^*$ and $V$

The space $(V^*)^*$, dual to the dual space $V$, is *canonically isomorphic* in the finite-dimensional case to $V$. The word *canonically* means that the isomorphism is "god-given", i.e. it is independent of any additional choices.

When we write $f(x)$ we usually mean that the function $f$ is fixed but the argument $x$ can vary. However, we can also take the opposite point of view, that $x$ is fixed but $f$ can vary.

If $x \in V$ and $f \in V^*$ then the above argument allows us to consider vectors of the space $V$ as linear functions on the dual space $V^*$. Thus we can define a map $I : V \to V^{**}$ by the formula

$$x \mapsto I(x) \in (V^*)^*, \quad \text{where} \quad I(x)(l) = l(x) \quad \text{for any} \quad l \in V^* .$$

**Exercise 1.2.1.** *Prove that if $V$ is finite-dimensional then $I$ is an isomorphism. What can go wrong in the infinite-dimensional case?*

## 1.3 The map $\mathcal{A}^*$

Given a map $\mathcal{A} : V \to W$ one can define a *dual map* $\mathcal{A}^* : W^* \to V^*$ as follows. For any linear function $l \in W^*$ we define the function $\mathcal{A}^*(l) \in V^*$ by the formula $\mathcal{A}^*(l)(x) = l(A(x))$, $x \in V$. In other words, $\mathcal{A}^*(l) = l \circ \mathcal{A}$.[2]

[2] In fact, the above formula makes sense in much more general situation. Given any map $\Phi : X \to Y$ between two sets $X$ and $Y$ the formula $\Phi^*(h) = h \circ \Phi$ defines a map $\Phi^* : F(Y) \to F(X)$ between the spaces of functions on $Y$ and $X$. Notice that this map goes in the opposite direction as the map $\Phi$.

Given bases $\mathcal{B}_v = \{v_1, \dots, v_n\}$ and $\mathcal{B}_w = \{w_1, \dots, w_k\}$ in the vector spaces $V$ and $W$ one can associate with the map $\mathcal{A}$ a matrix $A = M_{\mathcal{B}_v \mathcal{B}_w}(\mathcal{A})$. Its columns are coordinates of the vectors $\mathcal{A}(v_j), j = 1, \dots, n$, in the basis $\mathcal{B}_w$. Dual spaces $V^*$ and $W^*$ have dual bases $X = \{x_1, \dots x_n\}$ and $Y = \{y_1, \dots, y_k\}$ which consist of coordinate functions corresponding to the basis $\mathcal{B}_v$ and $\mathcal{B}_w$. Let us denote by $A^*$ the matrix of the dual map $\mathcal{A}^*$ with respect to the bases $Y$ and $X$, i. e. $A^* = M_{YX}(\mathcal{A}^*)$.

**Proposition 1.3.1.** *The matrices $A$ and $A^*$ are transpose to each other, i.e. $A^* = A^T$.*

**Proof.** By the definition of the matrix of a linear map we should take vectors of the basis $Y = \{y_1, \dots, y_k\}$, apply to them the map $\mathcal{A}^*$, expand the images in the basis $X = \{x_1, \dots x_n\}$ and write the components of these vectors as *columns* of the matrix $\mathcal{A}^*$. Set $\tilde{y}_i = \mathcal{A}^*(y_i)$, $i = 1, \dots, k$. For any vector $u = \sum\limits_{j=i}^{n} u_j v_j \in V$, we have $\tilde{y}_i(u) = y_i(\mathcal{A}(u))$. The coordinates of the vector $\mathcal{A}(u)$ in the basis $w_1, \dots, w_k$ may be obtained by multiplying the matrix $A$ by the column $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$. Hence,

$$\tilde{y}_i(u) = y_i(\mathcal{A}(u)) = \sum_{j=1}^{n} a_{ij} u_j.$$

But we also have

$$\sum_{j=1}^{n} a_{ij} x_j(u) = \sum_{j=1}^{n} a_{ij} u_j.$$

Hence, the linear function $\tilde{y}_i \in V^*$ has an expansion $\sum\limits_{j=1}^{n} a_{ij} x_j$ in the basis $X = \{x_1, \dots x_n\}$ of the space $V^*$. Hence the $i$-th column of the matrix $A^*$ equals $\begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix}$, so that the whole matrix $A^*$ has the form

$$A^* = \begin{pmatrix} a_{11} & \cdots & a_{k1} \\ \cdots & \dots & \cdots \\ a_{1n} & \cdots & a_{kn} \end{pmatrix} = A^T .$$

■

**Exercise 1.3.2.** *Given a linear map $\mathcal{A} : V \to W$ with a matrix $A$, find $\mathcal{A}^*(y_i)$.*

*Answer.* The map $\mathcal{A}^*$ sends the coordinate function $y_i$ on $W$ to the function $\sum_{j=1}^{n} a_{ij} x_j$ on $V$.

**Proposition 1.3.3.** *Consider linear maps*

$$U \xrightarrow{\mathcal{A}} V \xrightarrow{\mathcal{B}} W\ .$$

*Then $(\mathcal{B} \circ \mathcal{A})^* = \mathcal{A}^* \circ \mathcal{B}^*$.*

**Proof.** For any linear function $l \in W^*$ we have

$$(\mathcal{B} \circ \mathcal{A})^*(l)(x) = l(\mathcal{B}(\mathcal{A}(x)) = \mathcal{A}^*(\mathcal{B}^*(l))(x)$$

for any $x \in U$. ■

**Exercise 1.3.4.** *Suppose that $V$ is a Euclidean space and $\mathcal{A}$ is a linear map $V \to V$. Prove that for any 2 vectors $X, Y \in V$ we have $\langle \mathcal{A}(X), Y \rangle = \langle X, \mathcal{D}^{-1} \circ \mathcal{A}^* \circ \mathcal{D}(Y) \rangle$.*

Let us recall that if $V$ is an Euclidean space, then operator $\mathcal{B} : V \to V$ is called *adjoint* to $\mathcal{A} : V \to V$ if for any two vectors $X, Y \in V$ one has

$$\langle \mathcal{A}(X), Y \rangle = \langle X, \mathcal{B}(Y) \rangle.$$

The adjoint operator always exist and unique. It is denoted by $\mathcal{A}^\star$. Clearly, $(\mathcal{A}^\star)^\star = \mathcal{A}$. In any *orthonormal* basis the matrices of an operaor and it adjoint are transpose to each other. An operator $\mathcal{A} : V \to V$ is called *self-adjoint* if $\mathcal{A}^* = \mathcal{A}$, or equivalently, if for any two vectors $X, Y \in V$ one has

$$\langle \mathcal{A}(X), Y \rangle = \langle X, \mathcal{A}(Y) \rangle.$$

The statement of Exercise 1.3.4 can be otherwise expressed by saying that the operator $\mathcal{D}^{-1} \circ \mathcal{A}^* \circ \mathcal{D} : V \to V$ is *adjoint* to $\mathcal{A}$.

**Remark 1.3.5.** Recall that a linear transformation $\mathcal{A} : V \to V$ of a Euclidean space $V$ is called *orthogonal* if $\langle \mathcal{A}(X), \mathcal{A}(Y) \rangle = \langle X, Y \rangle$ for any vectors $X, Y \in V$. Equivalently, one can say that $\mathcal{A}$ is orthogonal if it maps an orthonormal basis to an orthonormal basis. Exercise 1.1.3 can be equivalently reformulated as follows: For any orthogonal transformation $\mathcal{A} : V \to V$ the operator $\mathcal{A}^*$ *commutes* with $\mathcal{D}$, i.e. $\mathcal{A}^* \circ \mathcal{D} = \mathcal{D} \circ \mathcal{A}^*$.

## 1.4 Multilinear functions

A function $l(X_1, X_2, \dots, X_k)$ of $k$ vector arguments $X_1, \dots, X_k \in V$ is called *$k$-linear* (or multilinear) if it is linear with respect to each argument when all other arguments are fixed. We say *bilinear* instead of 2-linear. Multilinear functions are also called *tensors*. Sometimes, one may also say a "$k$-linear *form*", or simply $k$-form instead of a "$k$-linear functions". However, we will reserve the term $k$-form for a skew-symmetric tensors which will be defined in Section 1.7 below.

If one fixes a basis $v_1 \dots v_n$ in the space $V$ then with each bilinear function $f(X, Y)$ one can associate a square $n \times n$ matrix as follows. Set $a_{ij} = f(v_i, v_j)$. Then $A = (a_{ij})_{i,j=1,\dots,n}$ is called *the matrix of the function $f$ in the basis* $v_1, \dots, v_n$. For any 2 vectors

$$X = \sum_1^n x_i v_i,\ Y = \sum_1^n y_j v_j$$

we have

$$f(X, Y) = f(\sum_{i=1}^n x_i v_i, \sum_{j=1}^n y_j v_j) = \sum_{i,j=1}^n x_i y_j f(v_i, v_j) = \sum_{i,j=1}^n a_{ij} x_i y_j = X^T A Y\,.$$

**Exercise 1.4.1.** *How does the matrix of a bilinear function depend on the choice of basis?*

**Answer.** The matrices $A$ and $\tilde{A}$ of the bilinear form $f(x, y)$ in the bases $v_1, \dots, v_n$ and $\tilde{v}_1, \dots, \tilde{v}_n$ are related by the formula $\tilde{A} = C^T A C$, where $C$ is the matrix of transition from the basis $v_1 \dots v_n$ to the basis $\tilde{v}_1 \dots \tilde{v}_n$, i.e the matrix whose columns are the coordinates of the basis $\tilde{v}_1 \dots \tilde{v}_n$ in the basis $v_1 \dots v_n$.

Similarly, with a $k$-linear function $f(X_1, \dots, X_k)$ on $V$ and a basis $v_1, \dots, v_n$ one can associate a "$k$-dimensional" matrix

$$A = \{a_{i_1 i_2 \dots i_k};\ \ 1 \le i_1, \dots, i_k \le n\},$$

where

$$a_{i_1 i_2 \dots i_k} = f(v_{i_1}, \dots, v_{i_k})\,.$$

If $X_i = \sum\limits_{j=1}^{n} x_{ij} v_j,\ \ i = 1, \dots, k$, then

$$f(X_1, \dots, X_k) = \sum_{i_1, i_2, \dots i_k = 1}^{n} a_{i_1 i_2 \dots i_k} x_{1i_1} x_{2i_2} \dots x_{ki_k}\ ,$$

see Proposition 1.6.1 below.

## 1.5 Tensor product

Given a $k$-linear function $\phi$ and a $l$-linear function $\psi$, one can form a $(k+l)$-linear function, which will be denoted by $\phi \otimes \psi$ and called *the tensor product* of the functions $\phi$ and $\psi$. By definition

$$\phi \otimes \psi(X_1, \dots, X_k, X_{k+1}, \dots, X_{k+l}) := \phi(X_1, \dots, X_k) \cdot \psi(X_{k+1}, \dots, X_{k+l}).$$

For instance, the tensor product two linear functions $l_1$ and $l_2$ is a bilinear function $l_1 \otimes l_2$ defined by the formula

$$l_1 \otimes l_2(U, V) = l_1(U) l_2(V).$$

Let $v_1 \dots v_n$ be a basis in $V$ and $x_1, \dots, x_n$ a dual basis in $V^*$, i.e. $x_1, \dots, x_n$ are coordinates of a vector with respect to the basis $v_1, \dots, v_n$.

The tensor product $x_i \otimes x_j$ is a bilinear function $x_i \otimes x_j(Y, Z) = y_i z_j$. Thus a bilinear function $f$ with a matrix $A$ can be written as a linear combination of the functions $x_i \otimes x_j$ as follows:

$$f = \sum_{i,j=1}^{n} a_{ij}\, x_i \otimes x_j,$$

where $a_{ij}$ is the matrix of the form $f$ in the basis $v_1 \dots v_n$. Similarly any $k$-linear function $f$ with a "$k$-dimensional" matrix $A = \{a_{i_1 i_2 \dots i_k}\}$ can be written (see 1.6.1 below) as a linear combination of functions

$$x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_k},\ 1 \le i_1, i_2, \dots, i_k \le n\,.$$

Namely, we have

$$f = \sum_{i_1, i_2, \dots i_k = 1}^{n} a_{i_1 i_2 \dots i_k} x_{i_1} \otimes x_{i_2} \otimes \dots \otimes x_{i_k}\,.$$

## 1.6 Spaces of multilinear functions

All $k$-linear functions, or *$k$-tensors*, on a given $n$-dimensional vector space $V$ themselves form a vector space, which will be denoted by $V^{*\otimes k}$. The space $V^{*\otimes 1}$ is, of course, just the dual space $V^*$.

**Proposition 1.6.1.** *Let $v_1, \dots v_n$ be a basis of $V$, and $x_1, \dots, x_k$ be the dual basis of $V^*$ formed by coordinate functions with respect to the basis $V$. Then $n^k$ $k$-linear functions $x_{i_1} \otimes \dots \otimes x_{i_k}$, $1 \le i_1, \dots, i_k \le n$, form a basis of the space $V^{*\otimes k}$.*

**Proof.** Take a $k$-linear function $F$ from $V^{*\otimes k}$ and evaluate it on vectors $v_{i_1}, \dots, v_{i_k}$:

$$F(v_{i_1}, \dots, v_{i_k}) = a_{i_1 \dots i_k}\,.$$

We claim that we have

$$F = \sum_{1 \le i_1, \dots, i_k \le n} a_{i_1 \dots i_k} x_{i_1} \otimes \dots \otimes x_{i_k}\,.$$

Indeed, the functions on the both sides of this equality being evaluated on any set of $k$ basic vectors $v_{i_1}, \dots, v_{i_k}$, give the same value $a_{i_1 \dots i_k}$. The same argument shows that if $\sum\limits_{1 \le i_1, \dots, i_k \le n} a_{i_1 \dots i_k} x_{i_1} \otimes \dots \otimes x_{i_k} = 0$, then all coefficients $a_{i_1 \dots i_k}$ should be equal to 0. Hence the functions $x_{i_1} \otimes \dots \otimes x_{i_k}$, $1 \le i_1, \dots, i_k \le n$, are linearly independent, and therefore form a basis of the space $V^{*\otimes k}$ ■

Similar to the case of spaces of linear functions, a linear map $\mathcal{A}: V \to W$ induces a linear map $\mathcal{A}^*: W^{*\otimes k} \to V^{*\otimes k}$, which sends a $k$-linear function $F \in W^{*\otimes k}$ to a $k$-linear function $\mathcal{A}^*(F) \in V^{*\otimes k}$, defined by the formula

$$\mathcal{A}^*(F)(X_1, \dots, X_k) = F(\mathcal{A}(X_1), \dots, \mathcal{A}(X_k)) \quad \text{for any vectors} \quad X_1, \dots X_k \in V\,.$$

## 1.7 Symmetric and skew-symmetric tensors

A multilinear function (tensor) is called *symmetric* if it remains unchanged under the transposition of any two of its arguments:

$$f(X_1, \dots, X_i, \dots, X_j, \dots, X_k) = f(X_1, \dots, X_j, \dots, X_i, \dots, X_k)$$

Equivalently, one can say that a k-tensor $f$ is symmetric if

$$f(X_{i_1}, \dots, X_{i_k}) = f(X_1, \dots, X_k)$$

for any permutation $i_1, \dots, i_k$ of indices $1, \dots, k$.

**Exercise 1.7.1.** *Show that a bilinear function $f(X, Y)$ is symmetric if and only if its matrix (in any basis) is symmetric.*

Notice that the tensor product of two symmetric tensors usually *is not* symmetric.

**Example 1.7.2.** *Any linear function is (trivially) symmetric. However, the tensor product of two functions $l_1 \otimes l_2$ is not a symmetric bilinear function unless $l_1 = l_2$. On the other hand, the function $l_1 \otimes l_2 + l_2 \otimes l_1$, is symmetric.*

A tensor is called *skew-symmetric (or anti-symmetric)* if it changes its sign when one transposes any two of its arguments:

$$f(X_1, \dots, X_i, \dots, X_j, \dots, X_k) = -f(X_1, \dots, X_j, \dots, X_i, \dots, X_k).$$

Equivalently, one can say that a k-tensor $f$ is anti-symmetric if

$$f(X_{i_1},\dots,X_{i_k}) = (-1)^{\mathrm{inv}(i_1\dots i_k)} f(X_1,\dots,X_k)$$

for any permutation $i_1,\dots,i_k$ of indices $1,\dots,k$, where $\mathrm{inv}(i_1\dots i_k)$ is the number of inversions in the permutation $i_1,\dots,i_k$. Recall that two indices $i_k, i_l$ form an *inversion* if $k < l$ but $i_k > i_l$.

The matrix $A$ of a bilinear skew-symmetric function is skew-symmetric, i.e.

$$A^T = -A.$$

**Example 1.7.3.** *The determinant* $\det(X_1,\dots,X_n)$ *(considered as a function of columns* $X_1,\dots,X_n$ *of a matrix) is a skew-symmetric* $n$*-linear function.*

**Exercise 1.7.4.** *Prove that any* $n$*-linear skew-symmetric function on* $\mathbb{R}^n$ *is proportional to the determinant.*

Linear functions are trivially anti-symmetric (as well as symmetric).

As in the symmetric case, the tensor product of two skew-symmetric functions is not skew-symmetric. We will define below in Section 1.9 a new product, called an *exterior product* of skew-symmetric functions, which will again be a skew-symmetric function.

## 1.8 Symmetrization and anti-symmetrization

The following constructions allow us to create symmetric or anti-symmetric tensors from arbitrary tensors. Let $f(X_1,\dots,X_k)$ be a $k$-tensor. Set

$$f^{\mathrm{sym}}(X_1,\dots,X_k) := \sum_{(i_1\dots i_k)} f(X_{i_1},\dots,X_{i_k})$$

and

$$f^{\mathrm{asym}}(X_1,\dots,X_k) := \sum_{(i_1\dots i_k)} (-1)^{\mathrm{inv}(i_1,\dots,i_k)} f(X_{i_1},\dots,X_{i_k})$$

where the sums are taken over all permutations $i_1, \dots, i_k$ of indices $1, \dots, k$. The tensors $f^{\text{sym}}$ and $f^{\text{assym}}$ are called, respectively, *symmetrization* and *anti-symmetrization* of the tensor $f$. It is now easy to see that

**Proposition 1.8.1.** *The function $f^{sym}$ is symmetric. The function $f^{asym}$ is skew-symmetric. If $f$ is symmetric then $f^{sym} = k!f$ and $f^{asym} = 0$. Similarly, if $f$ is anti-symmetric then $f^{asym} = k!f$, $f^{sym} = 0$.*

**Exercise 1.8.2.** *Let $x_1, \dots, x_n$ be coordinate function in $\mathbb{R}^n$. Find $(x_1 \otimes x_2 \otimes \dots \otimes x_n)^{asym}$.*

**Answer.** The determinant.

## 1.9 Exterior product

For our purposes skew-symmetric functions will be more important. Thus we will concentrate on operations on them.

Skew-symmetric $k$-linear functions are also called *exterior $k$-forms*. Let $\phi$ be an exterior $k$-form and $\psi$ an exterior $l$-form. We define an exterior $(k+l)$-form $\psi \wedge \psi$, *the exterior product of $\phi$ and $\psi$* as

$$\phi \wedge \psi := \frac{1}{k!l!}(\phi \otimes \psi)^{\text{asym}}.$$

In other words,

$$\phi \wedge \psi(X_1, \dots, X_k, X_{k+1}, \dots, X_{k+l}) = \frac{1}{k!l!} \sum_{i_1, \dots i_{k+l}} (-1)^{\text{inv}(i_1, \dots, i_{k+l})} \phi(X_{i_1} \dots, X_{i_k}) \psi(X_{i_{k+1}}, \dots, X_{i_{k+l}}),$$

where the sum is taken over all permutations of indices $1, \dots, k+l$.

Note that because the anti-symmetrization of an anti-symmetric $k$-tensor amounts to its multiplication by $k!$, we can also write

$$\phi \wedge \psi(X_1, \dots, X_{k+l}) = \sum_{i_1 < \dots < i_k, i_{k+1} < \dots < i_{k+l}} (-1)^{\text{inv}(i_1, \dots, i_{k+l})} \phi(X_{i_1}, \dots, X_{i_k}) \psi(X_{i_{k+1}}, \dots, X_{i_{k+l}}),$$

where the sum is taken over all permutations $i_1, \dots, i_{k+l}$ of indices $1, \dots, k+l$.

**Exercise 1.9.1.** *The exterior product operation has the following properties:*

- *For any exterior $k$-form $\phi$ and exterior $l$-form $\psi$ we have $\phi \wedge \psi = (-1)^{kl}\psi \wedge \phi$.*
- *Exterior product is linear with respect to each factor:*

$$\begin{aligned}(\phi_1 + \phi_2) \wedge \psi &= \phi_1 \wedge \psi + \phi_2 \wedge \psi \\ (\lambda\phi) \wedge \psi &= \lambda(\phi \wedge \psi)\end{aligned}$$

  *for $k$-forms $\phi, \phi_1, \phi_2$, $l$-form $\psi$ and a real number $\lambda$.*

- *Exterior product is associative: $(\phi \wedge \psi) \wedge \omega = \phi \wedge (\psi \wedge \omega)$.*

First two properties are fairly obvious. To prove associativity one can check that both sides of the equality $(\phi \wedge \psi) \wedge \omega = \phi \wedge (\psi \wedge \omega)$ are equal to

$$\frac{1}{k!l!m!}(\phi \otimes \psi \otimes \omega)^{\text{asym}}.$$

In particular, if $\phi, \psi$ and $\omega$ are 1-forms, i.e. if $k = l = m = 1$ then

$$\psi \wedge \phi \wedge \omega = (\phi \otimes \psi \otimes \omega)^{\text{asym}}.$$

This formula can be generalized for computing the exterior product of any number of 1-forms:

$$\phi_1 \wedge \cdots \wedge \phi_k = (\phi_1 \otimes \cdots \otimes \phi_k)^{\text{asym}}. \tag{1.1}$$

**Example 1.9.2.** $x_1 \wedge x_2 = x_1 \otimes x_2 - x_2 \otimes x_1$. For 2 vectors, $U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, V = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, we have

$$x_1 \wedge x_2(U, V) = u_1 v_2 - u_2 v_1 = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}.$$

For 3 vectors $U, V, W$ we have

$$x_1 \wedge x_2 \wedge x_3(U, V, W) =$$

$$= x_1 \wedge x_2(U, V)x_3(W) + x_1 \wedge x_2(V, W)x_3(U) + x_1 \wedge x_2(W, U)x_3(V) =$$

$$(u_1v_2 - u_2v_1)w_3 + (v_1w_2 - v_2w_1)u_3 + (w_1u_2 - w_2u_1)v_3 = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}.$$

The last equality is just the expansion formula of the determinant according to the last row.

**Proposition 1.9.3.** *Any exterior 2-form $f$ can be written as*

$$f = \sum_{1 \le i < j \le n} a_{ij}\, x_i \wedge x_j$$

*Proof.* We had seen above that any bilinear form can be written as $f = \sum_{ij} a_{i,j}\, x_i \otimes x_j$. If $f$ is skew-symmetric then the matrix $A = (a_{ij})$ is skew-symmetric, i.e. $a_{ii} = 0$, $a_{ij} = -a_{ji}$ for $i \neq j$.

Thus, $f = \sum_{1 \le i < j \le n} a_{ij}(x_i \otimes x_j - x_j \otimes x_i) = \sum_{1 \le i < j \le n} a_{ij}\, x_i \wedge x_j$. ∎

**Exercise 1.9.4.** *Prove that any exterior $k$-form $f$ can be written as*

$$f = \sum_{1 \le i, < \ldots < i_k \le n} a_{i_1 \ldots i_k}\, x_{i_1} \wedge x_{i_2} \wedge \ldots x_{i_k}.$$

The following proposition can be proven by induction over $k$, similar to what has been done in Example 1.9.2 for the case $k = 3$.

**Proposition 1.9.5.** *For any $k$ 1-forms $l_1, \ldots, l_k$ and $k$ vectors $X_1, \ldots, X_k$ we have*

$$l_1 \wedge \cdots \wedge l_k(X_1, \ldots, X_k) = \begin{vmatrix} l_1(X_1) & \ldots & l_1(X_k) \\ \ldots & \ldots & \ldots \\ l_k(X_1) & \ldots & l_k(X_k) \end{vmatrix}. \qquad (1.2)$$

**Corollary 1.9.6.** *The* 1*-forms* $l_1, \dots, l_k$ *are linearly dependent as vectors of* $V^*$ *if and only if* $l_1 \wedge \dots \wedge l_k = 0$*. In particular,* $l_1 \wedge \dots \wedge l_k = 0$ *if* $k > n = \dim V$*.*

**Proof.** If $l_1, \dots, l_k$ are dependent then for any vectors $X_1, \dots, X_k \in V$ the rows of the determinant in the equation (1.9.5) are linearly dependent. Therefore, this determinant is equal to 0, and hence $l_1 \wedge \dots \wedge l_k = 0$. In particular, when $k > n$ then the forms $l_1, \dots, l_k$ are dependent (because $\dim V^* = \dim \mathrm{V} = n$).

On the other hand, if $l_1, \dots, l_k$ are linearly independent, then the vectors $l_1, \dots, l_k \in V^*$ can be completed to form a basis $l_1, \dots, l_k, l_{k+1}, \dots, l_n$ of $V^*$. According to Exercise 1.1.2 there exists a basis $w_1, \dots, w_n$ of $V$ that is dual to the basis $l_1, \dots, l_n$ of $V^*$. In other words, $l_1, \dots, l_n$ can be viewed as coordinate functions with respect to the basis $w_1, \dots, w_n$. In particular, we have $l_i(w_j) = 0$ if $i \neq j$ and $l_i(w_i) = 1$ for all $i, j = 1, \dots, n$. Hence we have

$$l_1 \wedge \dots \wedge l_k(w_1, \dots, w_k) = \begin{vmatrix} l_1(w_1) & \dots & l_1(w_k) \\ \dots & \dots & \dots \\ l_k(w_1) & \dots & l_k(w_k) \end{vmatrix} = \begin{vmatrix} 1 & \dots & 0 \\ \dots & 1 & \dots \\ 0 & \dots & 1 \end{vmatrix} = 1 \,,$$

i.e. $l_1 \wedge \dots \wedge l_k \neq 0$. ■

Proposition 1.9.5 can be also deduced from formula (1.1).

Corollary 1.9.6 and Exercise 1.9.4 imply that there are no non-zero $k$-forms on an $n$-dimensional space for $k > n$.

## 1.10 Operator $\mathcal{A}^*$ on spaces of tensors

For any linear operator $\mathcal{A} : V \to W$ we introduced above in Section 1.3 the notion of a dual linear operator $\mathcal{A}^* : W^* \to V^*$. Namely $\mathcal{A}^*(l) = l \circ \mathcal{A}$ for any element $l \in V^*$, which is just a linear function on $V$. In this section we extend this construction to $k$-tensors for $k \geq 1$, i.e. we will define a map $\mathcal{A}^* : W^{*\otimes k} \to V^{*\otimes k}$.

Given a $k$-tensor $\phi \in W^{*\otimes k}$ and $k$ vectors $X_1, \dots, X_k \in V$ we define

$$\mathcal{A}^*(\phi)(X_1, \dots, X_k) = \phi\left(\mathcal{A}(X_1), \dots, \mathcal{A}(X_k)\right).$$

Note that if $\phi$ is symmetric, or anti-symmetric, so is $\mathcal{A}^*(\phi)$. Hence, the map $\mathcal{A}^*$ also induces the maps $S^k(W^*) \to S^k(V^*)$ and $\Lambda^k(W^*) \to \Lambda^k(V^*)$. We will keep the same notation $\mathcal{A}^*$ for both of these maps as well.

**Proposition 1.10.1.** *Let $\mathcal{A} : V \to W$ be a linear map. Then*

1. $\mathcal{A}^*(\phi \otimes \psi) = \mathcal{A}^*(\phi) \otimes \mathcal{A}^*(\psi)$ *for any* $\phi \in W^{*\otimes k}, \psi \in W^{*\otimes l}$;
2. $\mathcal{A}^*(\phi^{\mathrm{asym}}) = (\mathcal{A}^*(\phi))^{\mathrm{asym}}$, $\mathcal{A}^*(\phi^{\mathrm{sym}}) = (\mathcal{A}^*(\phi))^{\mathrm{sym}}$;
3. $\mathcal{A}^*(\phi \wedge \psi) = \mathcal{A}^*(\phi) \wedge \mathcal{A}^*(\psi)$ *for any* $\phi \in \Lambda^k(W^*), \psi \in \Lambda^l(W^*)$.

*If $\mathcal{B} : W \to U$ is another linear map then $(\mathcal{B} \circ \mathcal{A})^* = \mathcal{A}^* \circ \mathcal{B}^*$.*

**Proof.**

1. Take any $k + l$ vectors $X_1, \dots, X_{k+l} \in V$. Then by definition of the operator $\mathcal{A}^*$ we have

$$\begin{aligned}\mathcal{A}^*(\phi \otimes \psi)(X_1, \dots, X_{k+l}) =& \phi \otimes \psi(\mathcal{A}(X_1), \dots, \mathcal{A}(X_{n+k}) = \\ & \phi(\mathcal{A}(X_1), \dots, \mathcal{A}(X_k)\psi(\mathcal{A}(X_{k+1}), \dots, \mathcal{A}(X_{n+k})) = \\ & \mathcal{A}^*(\phi)(X_1, \dots, X_k)\mathcal{A}^*(\psi)(X_{k+1}, \dots, X_{k+n}) = \\ & \mathcal{A}^*(\phi) \otimes \mathcal{A}^*(\psi)(X_1, \dots, X_{k+l}).\end{aligned}$$

2. Given $k$ vectors $X_1, \dots, X_k \in V$ we get

$$\mathcal{A}^*(\phi^{asym})(X_1, \dots, X_k) = \phi^{asym}(\mathcal{A}(X_1), \dots, \mathcal{A}(X_k)) = \sum_{(i_1 \dots i_k)} \phi(\mathcal{A}(X_{i_1}), \dots, \mathcal{A}(X_{i_k})) =$$
$$\sum_{(i_1 \dots i_k)} \mathcal{A}^*(\phi)(X_{i_1}, \dots, X_{i_k}) = (\mathcal{A}^*(\phi))^{asym}(X_1, \dots, X_k),$$

where the sum is taken over all permutations $i_1, \dots, i_k$ of indices $1, \dots, k$. Similarly one proves that $\mathcal{A}^*(\phi^{\mathrm{sym}}) = (\mathcal{A}^*(\phi))^{\mathrm{sym}}$.

3. $\mathcal{A}^*(\phi \wedge \psi) = \mathcal{A}^*\left((\phi \otimes \psi)^{asym})\right) = (\mathcal{A}^*(\phi \otimes \psi))^{asym} = \mathcal{A}^*(\phi) \wedge \mathcal{A}^*(\psi)$.

The last statement of Proposition 1.10.1 is straightforward and its proof is left to the reader.

Let us now discuss how to compute $\mathcal{A}^*(\phi)$ in coordinates. Let us fix bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$ in spaces $V$ and $W$. Let $x_1, \dots, x_m$ and $y_1, \dots, y_n$ be coordinates and

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

be the matrix of a linear map $\mathcal{A} : V \to W$ in these bases. Note that the map $\mathcal{A}$ in these coordinates is given by $n$ linear coordinate functions:

$$\begin{aligned} y_1 &= l_1(x_1, \dots, x_m) = a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m \\ y_2 &= l_2(x_1, \dots, x_m) = a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m \\ &\dots \\ y_n &= l_n(x_1, \dots, x_k) = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_n \end{aligned}$$

We have already computed in Section 1.3 that $\mathcal{A}^*(y_k) = l_k = \sum\limits_{j=1}^{m} a_{kj}x_j$, $k = 1, \dots, n$. Indeed, the coefficients of the function $l_k = \mathcal{A}^*(x_k)$ form the $k$-th column of the transpose matrix $A^T$. Hence, using Proposition 1.10.1 we compute:

$$\mathcal{A}^*(y_{j_1} \otimes \dots \otimes y_{j_k}) = l_{j_1} \otimes \dots \otimes l_{j_k}$$

and

$$\mathcal{A}^*(y_{j_1} \wedge \dots \wedge y_{j_k}) = l_{j_1} \wedge \dots \wedge l_{j_k}.$$

Consider now the case when $V = W$, $n = m$, and we use the same basis $v_1, \dots, v_n$ in the source and target spaces.

**Proposition 1.10.2.**

$$\mathcal{A}^*(x_1 \wedge \dots \wedge x_n) = \det A \, x_1 \wedge \dots \wedge x_n.$$

Note that the determinant $\det A$ is independent of the choice of the basis. Indeed, the matrix of a linear map changes to a similar matrix $C^{-1}AC$ in a different basis, and $\det C^{-1}AC = \det A$. Hence, we can write $\det \mathcal{A}$ instead of $\det A$, i.e. attribute the determinant to the linear operator $\mathcal{A}$ rather than to its matrix $A$.

**Proof.** We have

$$\mathcal{A}^*(x_1 \wedge \dots \wedge x_n) = l_1 \wedge \dots \wedge l_n = \sum_{i_1=1}^{n} a_{1i_1} x_{i_1} \wedge \dots \wedge \sum_{i_n=1}^{n} a_{ni_n} x_{i_n} =$$
$$\sum_{i_1,\dots,i_n=1}^{n} a_{1i_1} \dots a_{ni_n} x_{i_1} \wedge \dots \wedge x_{i_n}.$$

Note that the in the latter sum all terms with repeating indices vanish, and hence we can replace this sum by a sum over all permutations of indices $1,\dots,n$. Thus, we can continue

$$\mathcal{A}^*(x_1 \wedge \dots \wedge x_n) = \sum_{i_1,\dots,i_n} a_{1i_1} \dots a_{ni_n} x_{i_1} \wedge \dots \wedge x_{i_n} =$$
$$\left( \sum_{i_1,\dots,i_n} (-1)^{\mathrm{inv}(i_1,\dots,i_n)} a_{1i_1} \dots a_{ni_n} \right) x_1 \wedge \dots \wedge x_n = \det A \, x_1 \wedge \dots \wedge x_n.$$

**Exercise 1.10.3.** *Apply the equality*

$$\mathcal{A}^*(x_1 \wedge \dots \wedge x_k \wedge x_{k+1} \wedge \dots \wedge x_n) = \mathcal{A}^*(x_1 \wedge \dots \wedge x_k) \wedge \mathcal{A}^*(x_{k+1} \wedge \dots \wedge x_n)$$

*for a map $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ to deduce the formula for expansion of a determinant according to its first $k$ rows:*

$$\det A = \sum_{i_1<\dots<i_k,\ j_1<\dots<j_{n-k};\ i_m \neq j_l} (-1)^{\mathrm{inv}(i_1,\dots,j_{n-k})} \begin{vmatrix} a_{1,i_1} & \dots & a_{1,i_k} \\ \vdots & \vdots & \vdots \\ a_{k,i_1} & \dots & a_{k,i_k} \end{vmatrix} \begin{vmatrix} a_{k+1,j_1} & \dots & a_{k+1,j_{n-k}} \\ \vdots & \vdots & \vdots \\ a_{n,j_1} & \dots & a_{n,j_{n-k}} \end{vmatrix}.$$

## 1.11 Spaces of symmetric and skew-symmetric tensors

As was mentioned above, $k$-tensors on a vector space $V$ form a vector space under the operation of addition of functions and multiplication by real numbers. We denoted this space by $V^{*\otimes k}$. Symmetric and skew-symmetric tensors form subspaces of this space $V^{*\otimes k}$, which we denote, respectively, by $S^k(V^*)$ and $\Lambda^k(V^*)$. In particular, we have

$$V^* = S^1(V^*) = \Lambda^1(V^*)$$

.

**Exercise 1.11.1.** *What is the dimension of the spaces $S^k(V^*)$ and $\Lambda^k(V^*)$?*

**Answer.**

$$\dim \Lambda^k(V^*) = \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$$

$$\dim S^k(V^*) = \frac{(n+k-1)!}{k!(n-1)!}.$$

The basis of $\Lambda^k(V^*)$ is formed by exterior $k$-forms $x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_k}, 1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

## 1.12 Orientation

We say that two bases $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ of a vector space $V$ define *the same orientation* of $V$ if the matrix of transition from one of these bases to the other has a positive determinant. Clearly, if we have 3 bases, and the first and the second define the same orientation, and the second and the third define the same orientation then the first and the third also define the same orientation. Thus, one can subdivide the set of all bases of $V$ into the two classes. All bases in each of these classes define the same orientation; two bases chosen from different classes define opposite orientation of the space. To choose an orientation of the space simply means to choose one of these two classes of bases.

There is no way to say which orientation is "positive" or which is "negative"—it is a question of convention. For instance, the so-called counter-clockwise orientation of the plane depends from which side we look at the plane. The positive orientation of our physical 3-space is a physical, not mathematical, notion.

Suppose we are given two oriented spaces $V, W$ of the same dimension. An invertible linear map (= isomorphism) $\mathcal{A} : V \to W$ is called *orientation preserving* if it maps a basis which defines the given orientation of $V$ to a basis which defines the given orientation of $W$.

Any non-zero exterior $n$-form $\eta$ on $V$ induces an orientation of the space $V$. Indeed, the preferred set of bases is characterized by the property $\eta(v_1, \dots, v_n) > 0$.

## 1.13 Orthogonal transformations

Let $V$ be a Euclidean vector space. A linear operator $\mathcal{U} : V \to V$ is called *orthogonal* if it preserves the scalar product, i.e. if

$$\langle \mathcal{U}(X), \mathcal{U}(Y) \rangle = \langle X, Y \rangle, \tag{1.3}$$

for any vectors $X, Y \in V$. Recall that we have

$$\langle \mathcal{U}(X), \mathcal{U}(Y) \rangle = \langle X, \mathcal{U}^* (\mathcal{U}(Y)) \rangle,$$

where $\mathcal{U}^* : V \to V$ is the adjoint operator to $\mathcal{U}$, see Section 1.3 above. Hence, the orthogonality of an operator $\mathcal{U}$ is equivalent to the identity $\mathcal{U}^* \circ \mathcal{U} = \mathrm{Id}$, or $\mathcal{U}^* = \mathcal{U}^{-1}$. Here we denoted by Id the identity operator, i.e. $\mathrm{Id}(X) = X$ for any $X \in V$.

Let $v_1, \dots, v_n$ be an orthonormal basis in $V$ and $U$ be the matrix of $\mathcal{U}$ in this basis. The matrix of the adjoint operator in an orthonormal basis is the transpose of the matrix of this operator. Hence, the equation $\mathcal{U}^* \circ \mathcal{U} = \mathrm{Id}$ translates into the equation $U^T U = E$, or equivalently $UU^T = E$, or $U^{-1} = U^T$ for its matrix. Matrices, which satisfy this equation are called *orthogonal*. If we write

$$U = \begin{pmatrix} u_{11} & \dots & u_{1n} \\ \dots & \dots & \dots \\ u_{n1} & \dots & u_{nn} \end{pmatrix},$$

then the equation $U^T U = E$ can be rewritten as

$$\sum_i u_{ki} u_{ji} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j, \end{cases} .$$

Similarly, the equation $UU^T = E$ can be rewritten as

$$\sum_i u_{ik} u_{ij} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j, \end{cases} .$$

In particular, we have

$$1 = \det(U^T U) = \det(U^T) \det U = (\det U)^2 ,$$

and hence $\det U = \pm 1$. In other words, the determinant of any orthogonal matrix is equal $\pm 1$. We can also say that the determinant of an orthogonal *operator* is equal to $\pm 1$ because the determinant of the matrix of an operator is independent of the choice of a basis. Orthogonal transformations with $\det = 1$ preserve the orientation of the space, while those with $\det = -1$ reverse it.

Composition of two orthogonal transformations, or the inverse of an orthogonal transformation is again an orthogonal transformation. The set of all orthogonal transformations of an $n$-dimensional Euclidean space is denoted by $O(n)$. Orientation preserving orthogonal transformations sometimes called *special* orthogonal transformations, and the set of special orthogonal transformations is denoted by $SO(n)$. For instance $O(1)$ consists of two elements and $SO(1)$ of one: $O(1) = \{1, -1\}$, $SO(1) = \{1\}$. $SO(2)$ consists of rotations of the plane, while $O(2)$ consists of rotations an reflections with respect to lines.

## 1.14 Determinant and Volume

We begin by recalling some facts from Linear Algebra. Let $V$ be an $n$-dimensional Euclidean space with an inner product $\langle \ , \ \rangle$. Given a linear subspace $L \subset V$ and a point $x \in V$, the *projection* $\text{proj}_L(x)$ is a vector $y \in L$ which is uniquely characterized by the property $x - y \perp L$, i.e. $\langle x - y, z \rangle = 0$ for any $z \in L$. The length $||x - \text{proj}_L(x)||$ is called the *distance* from $x$ to $L$; we denote it by $\text{dist}(x, L)$.

Let $U_1, \dots, U_k \in V$ be linearly independent vectors. The $k$-dimensional *parallelepiped spanned by vectors* $U_1, \dots, U_k$ is, by definition, the set

$$P(U_1, \dots, U_k) = \left\{ \sum_1^k \lambda_j U_j;\ 0 \le \lambda_1, \dots, \lambda_k \le 1 \right\} \subset \text{Span}(U_1, \dots, U_k).$$

Given a $k$-dimensional parallelepiped $P = P(U_1, \dots, U_k)$ we will define its $k$-dimensional *volume* by the formula

$$\text{Vol} P = ||U_1|| \text{dist}(U_2, \text{Span}(U_1)) \text{dist}(U_3, \text{Span}(U_1, U_2)) \dots \text{dist}(U_k, \text{Span}(U_1, \dots, U_{k-1})). \tag{1.4}$$

Of course we can write $\text{dist}(U_1, 0)$ instead of $||U_1||$. This definition agrees with the definition of the area of a parallelogram, or the volume of a 3-dimensional parallelepiped in the elementary geometry.

**Proposition 1.14.1.** *Let $v_1, \dots, v_n$ be an orthonormal basis in $V$. Given $n$ vectors $U_1, \dots, U_n$ let us denote by $U$ the matrix whose columns are coordinates of these vectors in the basis $v_1, \dots, v_n$:*

$$U := \begin{pmatrix} u_{11} & \dots & u_{1n} \\ & \vdots & \\ u_{n1} & \dots & u_{nn} \end{pmatrix}$$

*Then*

$$\text{Vol}\, P(U_1, \dots, U_n) = |\det U|.$$

**Proof.** If the vectors $U_1, \dots, U_n$ are linearly dependent then $\text{Vol}\, P(U_1, \dots, U_n) = \det U = 0$. Suppose now that the vectors $U_1, \dots, U_n$ are linearly independent, i.e. form a basis. Consider first the case where this basis is orthonormal. Then the matrix $U$ is orthogonal. i.e. $UU^T = E$, and hence $\det U = \pm 1$. But in this case $\text{Vol}\, P(U_1, \dots, U_n) = 1$, and hence $\text{Vol}\, P(U_1, \dots, U_n) = |\det U|$.

Now let the basis $U_1, \dots, U_n$ be arbitrary. Let us apply to it the Gram-Schmidt orthonormalization process. Recall that this process consists of the following steps. First, we normalize the vector $U_1$, then subtract from $U_2$ its projection to $\mathrm{Span}(U_1)$, Next, we normalize the new vector $U_2$, then subtract from $U_3$ its projection to $\mathrm{Span}(U_1, U_2)$, and so on. At the end of this process we obtain an orthonormal basis. It remains to notice that each of these steps affected $\mathrm{Vol}\, P(U_1, \dots, U_n)$ and $|\det U|$ in a similar way. Indeed, when we multiplied the vectors by a positive number, both the volume and the determinant were multiplied by the same number. When we subtracted from a vector $U_k$ its projection to $\mathrm{Span}(U_1, \dots, U_{k-1})$, this affected neither the volume nor the determinant. ■

**Corollary 1.14.2.** *1. Let $x_1, \dots, x_n$ be a Cartesian coordinate system. Then*

$$\mathrm{Vol}\, P(U_1, \dots, U_n) = |x_1 \wedge \dots x_n(U_1, \dots, U_n)|.$$

*2. Let $\mathcal{A} : V \to V$ be a linear map. Then*

$$\mathrm{Vol}\, P(\mathcal{A}(U_1), \dots, \mathcal{A}(U_n)) = |\det \mathcal{A}| \mathrm{Vol}\, P(U_1, \dots, U_n).$$

**Proof.**

1. According to 1.9.5, $x_1 \wedge \dots x_n(U_1, \dots, U_n) = \det U$.

2. $x_1 \wedge \dots x_n(\mathcal{A}(U_1), \dots, \mathcal{A}(U_n)) = \mathcal{A}^*(x_1 \wedge \dots \wedge x_n)(U_1, \dots, U_n) = \det \mathcal{A} x_1 \wedge \dots x_n(U_1, \dots, U_n)$.

■

In view of Proposition 1.14.1 and the first part of Corollary 1.14.2 the value

$$x_1 \wedge \dots x_n(U_1, \dots, U_n) = \det U$$

is called sometimes the *signed volume* of the parallelepiped $P(U_1, \dots, U_n)$. It is positive when the basis $U_1, \dots, U_n$ defines the given orientation of the space $V$, and it is negative otherwise.

Note that $x_1 \wedge \dots x_k(U_1, \dots, U_k)$ for $0 \le k \le n$ is the signed $k$-dimensional volume of the orthogonal projection of the parallelepiped $P(U_1, \dots, U_k)$ to the coordinate subspace $\{x_{k+1} = \dots = x_n = 0\}$.

For instance, let $\omega$ be the 2-form $x_1 \wedge x_2 + x_3 \wedge x_4$ on $\mathbb{R}^4$. Then for any two vectors $U_1, U_2 \in \mathbb{R}^4$ the value $\omega(U_1, U_2)$ is the sum of signed areas of projections of the parallelogram $P(U_1, U_2)$ to the coordinate planes spanned by the two first and two last basic vectors.

## 1.15 Volume and Gram matrix

In this section we will compute the $\mathrm{Vol}_k P(v_1, \ldots, v_k)$ in the case when the number $k$ of vectors is less than the dimension $n$ of the space.

Let $V$ be an Euclidean space. Given vectors $v_1, \ldots, v_k \in V$ we can form a $k \times k$-matrix

$$G(v_1, \ldots, v_k) = \begin{pmatrix} \langle v_1, v_1 \rangle & \ldots & \langle v_1, v_k \rangle \\ \ldots & \ldots & \ldots \\ \langle v_k, v_1 \rangle & \ldots & \langle v_k, v_k \rangle \end{pmatrix}, \tag{1.5}$$

which is called the *Gram matrix* of vectors $v_1, \ldots, v_k$.

Suppose we are given Cartesian coordinate system in $V$ and let us form a matrix $C$ whose columns are coordinates of vectors $v_1, \ldots, v_k$. Thus the matrix $C$ has $n$ rows and $k$ columns. Then

$$G(v_1, \ldots, v_k) = C^T C,$$

because in Cartesian coordinates the scalar product looks like the dot-product.

We also point out that if $k = n$ and vectors $v_1, \ldots, v_n$ form a basis of $V$, then $G(v_1, \ldots, v_k)$ is just the matrix of the bilinear function $\langle X, Y \rangle$ in the basis $v_1, \ldots, v_n$.

**Proposition 1.15.1.** *Given any $k$ vectors $v_1, \ldots, v_k$ in an Euclidean space $V$ the volume $\mathrm{Vol}_k P(v_1, \ldots, v_k)$ can be computed by the formula*

$$\mathrm{Vol}_k P(v_1, \ldots, v_k)^2 = \det G(v_1, \ldots, v_k) = \det C^T C, \tag{1.6}$$

*where $G(v_1, \ldots, v_k)$ is the Gram matrix and $C$ is the matrix whose columns are coordinates of vectors $v_1, \ldots, v_k$ in some orthonormal basis.*

**Proof.** Suppose first that $k = n$. Then according to Proposition 1.14.1 we have $\mathrm{Vol}_k P(v_1, \ldots, v_k) = |\det C|$. But $\det C^T C = \det C^2$, and the claim follows.

Let us denote vectors of our orthonormal basis by $w_1, \ldots, w_n$. Consider now the case when

$$\mathrm{Span}(v_1, \ldots, v_k) \subset \mathrm{Span}(w_1, \ldots, w_k). \tag{1.7}$$

In this case the elements in the $j$-th row of the matrix $C$ are zero if $j > k$. Hence, if we denote by $\widetilde{C}$ the square $k \times k$ matrix formed by the first $k$ rows of the matrix $C$, then $C^T C = \widetilde{C}^T \widetilde{C}$ and thus $\det C^T C = \det \widetilde{C}^T \widetilde{C}$. But $\det \widetilde{C}^T \widetilde{C} = \mathrm{Vol}_k P(v_1, \ldots, v_k)$ in view of our above argument in the equi-dimensional case applied to the subspace $\mathrm{Span}(w_1, \ldots, w_k) \subset V$, and hence

$$\mathrm{Vol}_k P(v_1, \ldots, v_k) = \det C^T C = \det G(v_1, \ldots, v_k).$$

But neither $\mathrm{Vol}_k P(v_1, \ldots, v_k)$, nor the Gram matrix $G(v_1, \ldots, v_k)$ depends on the choice of an orthonormal basis. On the other hand, using Gram-Schmidt process one can always find an orthonormal basis which satisfies condition (1.7).

**Remark 1.15.2.** *Note that* $\det G(v_1, \ldots, v_k) \geq 0$ *and* $\det G(v_1, \ldots, v_k) = 0$ *if an and only if the vectors* $v_1, \ldots, v_k$ *are linearly dependent.*

## 1.16 Duality between $k$-forms and $(n-k)$-forms on a $n$-dimensional Euclidean space $V$

Let $V$ be an $n$-dimensional vector space. As we have seen above, the space $\Lambda^k(V^*)$ of $k$-forms, and the space $\Lambda^{n-k}(V^*)$ of $(n-k)$-forms have the same dimension $\frac{n!}{k!(n-k)!}$; these spaces are therefore isomorphic. Suppose that $V$ is an *oriented Euclidean* space, i.e. it is supplied with an orientation and an inner product $\langle\ ,\ \rangle$. It turns out that in this case there is a canonical way to establish this isomorphism which will be denoted by

$$\star : \Lambda^k(V^*) \to \Lambda^{n-k}(V^*)\,.$$

**Definition 1.16.1.** *Let $\alpha$ be a $k$-form. Then given any vectors $U_1,\dots,U_{n-k}$, the value $\star\alpha(U_1,\dots,U_{n-k})$ can be computed as follows. If $U_1,\dots,U_{n-k}$ are linearly dependent then $\star\alpha(U_1,\dots,U_{n-k}) = 0$. Otherwise, let $S^\perp$ denote the orthogonal complement to the space $S = \mathrm{Span}(U_1,\dots,U_{n-k})$. Choose a basis $Z_1,\dots,Z_k$ of $S^\perp$ such that*

$$\mathrm{Vol}_k(Z_1,\dots,Z_k) = \mathrm{Vol}_{n-k}(U_1,\dots,U_{n-k})$$

*and the basis $Z_1,\dots,Z_k,U_1,\dots,U_{n-k}) > 0$. defines the given orientation of the space $V$. Then*

$$\star\,\alpha(U_1,\dots,U_{n-k}) = \alpha(Z_1,\dots,Z_k). \tag{1.8}$$

Let us first show that

**Lemma 1.16.2.** *$\star\alpha$ is a $(n-k)$-form, i.e. $\star\alpha$ is skew-symmetric and multilinear.*

**Proof.** To verify that $\star\alpha$ is skew-symmetric we note that for any $1 \le i < j \le n-q$ the bases $Z_1,Z_2,\dots,Z_k,U_1,\dots,U_i,\dots,U_j,\dots,U_{n-k},\ -Z_1,Z_2,\dots,Z_k,U_1,\dots,U_j,\dots,U_i,\dots,U_{n-k}$ define the same orientation of the space $V$, and hence

$$\star\,\alpha(U_1,\dots,U_j,\dots,U_i,\dots,U_{n-k}) = \alpha(-Z_1,Z_2,\dots,Z_k)$$
$$= -\,\alpha(Z_1,Z_2,\dots,Z_k) = \star\alpha(U_1,\dots,U_i,\dots,U_j,\dots,U_{n-k}).$$

Hence, in order to check the *multi*-linearity it is sufficient to prove the linearity of $\alpha$ with respect to the first argument only. It is also clear that

$$\star\,\alpha(\lambda U_1,\dots,U_{n-k}) = \star\lambda\alpha(U_1,\dots,U_{n-k}). \tag{1.9}$$

Indeed, multiplication by $\lambda \neq 0$ does not change the span of the vectors $U_1,\dots,U_{n-q}$ , and hence if $\star\alpha(U_1,\dots,U_{n-k}) = \alpha(Z_1,\dots,Z_k)$ then $\star\alpha(\lambda U_1,\dots,U_{n-k}) = \alpha(\lambda Z_1,\dots,Z_k) = \lambda\alpha(Z_1,\dots,Z_k)$.

Thus it remains to check that

$$\alpha(U_1+\widetilde{U}_1,U_2,\dots,U_{n-k}) = \alpha(U_1,U_2,\dots,U_{n-k}) + \alpha(\widetilde{U}_1,U_2,\dots,U_{n-k})).$$

Let us denote $L := \mathrm{Span}(U_2, \dots, U_{n-k})$ and observe that $\mathrm{proj}_L(U_1 + \widetilde{U}_1) = \mathrm{proj}_L(U_1) + \mathrm{proj}_L(\widetilde{U}_1)$. Denote $N := U_1 - \mathrm{proj}_L(U_1)$ and $\widetilde{N} := \widetilde{U}_1 - \mathrm{proj}_L(\widetilde{U}_1)$. The vectors $N$ and $\widetilde{N}$ are normal components of $U_1$ and $\widetilde{U}_1$ with respect to the subspace $L$, and the vector $N + \widetilde{N}$ is the normal component of $U_1 + \widetilde{U}_1$ with respect to $L$. Hence, we have

$$\alpha(U_1, \dots, U_{n-k}) = \alpha(N, \dots, U_{n-k}), \quad \alpha(\widetilde{U}_1, \dots, U_{n-k}) = \alpha(\widetilde{N}, \dots, U_{n-k}),$$

and

$$\alpha(U_1 + \widetilde{U}_1, \dots, U_{n-k}) = \alpha(N + \widetilde{N}, \dots, U_{n-k}).$$

Indeed, in each of these three cases,

- vectors on both side of the equality span the same space;
- the parallelepiped which the generate have the same volume, and
- the orientation which these vectors define together with a basis of the complementary space remains unchanged.

Hence, it is sufficient to prove that

$$\alpha(N + \widetilde{N}, U_2, \dots, U_{n-k}) = \alpha(N, U_2, \dots, U_{n-k}) + \alpha(\widetilde{N}, U_2, \dots, U_{n-k})). \tag{1.10}$$

If the vectors $N$ and $\widetilde{N}$ are linearly dependent, i.e. one of them is a multiple of the other, then (1.10) follows from (1.9).

Suppose now that $N$ and $\widetilde{N}$ are linearly independent. Let $L^\perp$ denote the orthogonal complement of $L = \mathrm{Span}(U_2, \dots, U_{n-k})$. Then $\dim L^\perp = k + 1$ and we have $N, \widetilde{N} \in L^\perp$. Let us denote by $M$ the plane in $L^\perp$ spanned by the vectors $N$ and $\widetilde{N}$, and by $N^\perp$ its orthogonal complement in $L^\perp$. Choose any orientation of $M$ so that we can talk about counter-clockwise rotation of this plane. Let $Y, \widetilde{Y} \in M$ be vectors obtained by rotating $N$ and $\widetilde{N}$ in $M$ counter-clockwise by the angle $\frac{\pi}{2}$. Then $Y + \widetilde{Y}$ can be obtained by rotating $N + \widetilde{N}$ in $M$ counter-clockwise by the angle $\frac{\pi}{2}$. Let us choose in $N^\perp$ a basis $Z_2, \dots, Z_k$ such that

$$\mathrm{Vol}_{k-1} P(Z_2, \dots, Z_k) = \mathrm{Vol}_{n-k-1} P(U_2, \dots, U_{n-k}).$$

Note that the orthogonal complements to $\mathrm{Span}(N, U_2, \ldots, U_{n-k})$, $\mathrm{Span}(\widetilde{N}, U_2, \ldots, U_{n-k})$, and to $\mathrm{Span}(N + \widetilde{N}, U_2, \ldots, U_{n-k})$ in $V$ coincide, respectively, with the orthogonal complements to the the vectors $N, \widetilde{N}$ and to $N + \widetilde{N}$ in $L^\perp$. In other words, we have

$$(\mathrm{Span}(N, U_2, \ldots, U_{n-k}))_V^\perp = \mathrm{Span}(Y, Z_2, \ldots, Z_k),$$

$$\Big(\mathrm{Span}(\widetilde{N}, U_2, \ldots, U_{n-k})\Big)_V^\perp = \mathrm{Span}(\widetilde{Y}, Z_2, \ldots, Z_k) \text{ and}$$

$$\Big(\mathrm{Span}(N + \widetilde{N}, U_2, \ldots, U_{n-k})\Big)_V^\perp = \mathrm{Span}(Y + \widetilde{Y}, Z_2, \ldots, Z_k).$$

Next, we observe that

$$\mathrm{Vol}_{n-k} P(N, U_2, \ldots, U_{n-k}) = \mathrm{Vol}_k P(Y, Z_2, \ldots, Z_k),$$

$$\mathrm{Vol}_{n-k} P(\widetilde{N}, U_2, \ldots, U_{n-k}) = \mathrm{Vol}_k P(\widetilde{Y} Z_2, \ldots, Z_k) \text{ and}$$

$$\mathrm{Vol}_{n-k} P(N + \widetilde{N}, U_2, \ldots, U_{n-k}) = \mathrm{Vol}_k P(Y + \widetilde{Y}, Z_2, \ldots, Z_k).$$

Consider the following 3 bases of $V$:

$$Y, Z_2, \ldots, Z_k, N, U_2, \ldots, U_{n-k},$$

$$\widetilde{Y}, Z_2, \ldots, Z_k, \widetilde{N}, U_2, \ldots, U_{n-k},$$

$$Y + \widetilde{Y}, Z_2, \ldots, Z_k, N + \widetilde{N}, U_2, \ldots, U_{n-k},$$

and observe that all 3 of them induce the same orientation of $V$. Let us assume that this the orientation given by the volume form $x_1 \wedge \cdots \wedge x_n$. Then we have, by definition of the operator $\star$:

$$\begin{aligned}
&\star\alpha(N + \widetilde{N}, U_2, \ldots, U_{n-k}) = \alpha(Y + \widetilde{Y}, Z_2, \ldots, Z_k) = \\
&\alpha(Y, Z_2, \ldots, Z_k) + \alpha(\widetilde{Y}, Z_2, \ldots, Z_k) = \\
&\star\alpha(N, U_2, \ldots, U_{n-k}) + \star\alpha(\widetilde{N}, U_2, \ldots, U_{n-k}).
\end{aligned}$$

This completes the proof that $\star\alpha$ is an $(n-k)$-form. ■

Thus the map $\alpha \mapsto \star\alpha$ defines a map $\star : \Lambda^k(V^*) \to \Lambda^{n-k}(V^*)$. Clearly, this map is linear. In order to check that $\star$ is an isomorphism let us choose an orthonormal basis in $V$ and consider the coordinates $x_1, \dots, x_n \in V^*$ corresponding to that basis.

Let us recall that the forms $x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_k}$, $1 \le i_1 < i_2 < \dots < i_k \le n$, form a basis of the space $\Lambda^k(V^*)$.

**Lemma 1.16.3.**

$$\star\, x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_k} = (-1)^{\mathrm{inv}(i_1,\dots,i_k,j_1,\dots,j_{n-k})} x_{j_1} \wedge x_{j_2} \wedge \dots \wedge x_{j_{n-k}}\,, \tag{1.11}$$

*where $j_1 < \dots < j_{n-k}$ is the set of indices, complementary to $i_1, \dots, i_k$. In other words, $i_1, \dots, i_k, j_1, \dots, j_{n-k}$ is a permutation of indices $1, \dots, n$.*

**Proof.** Evaluating $\star(x_{i_1} \wedge \dots \wedge x_{i_k})$ on basic vectors $v_{j_1}, \dots, v_{j_{n-k}}$, $1 \le j_1 < \dots < j_{n-q} \le n$, we get 0 unless all the indices $j_1, \dots, j_{n-k}$ are all different from $i_1, \dots, i_k$, while in the latter case we get

$$\star(x_{i_1} \wedge \dots \wedge x_{i_k})(v_{j_1}, \dots, v_{j_{n-k}}) = (-1)^{\mathrm{inv}(i_1,\dots,i_k,j_1,\dots,j_{n-k})}.$$

Hence,

$$\star x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_k} = (-1)^{\mathrm{inv}(i_1,\dots,i_k,j_1,\dots,j_{n-k})} x_{j_1} \wedge x_{j_2} \wedge \dots \wedge x_{j_{n-k}}.$$

■

Thus $\star$ establishes a 1 to 1 correspondence between the bases of the spaces $\Lambda^k(V^*)$ and the space $\Lambda^{n-k}(V^*)$, and hence it is an isomorphism. Note that by linearity for any form $\alpha = \sum\limits_{1 \le i_1 < \dots < i_k \le n} a_{i_1 \dots i_k} x_{i_1} \wedge \dots \wedge x_{i_k})$ we have

$$\star\alpha = \sum_{1 \le i_1 < \dots < i_k \le n} a_{i_1 \dots i_k} \star (x_{i_1} \wedge \dots \wedge x_{i_k})\,.$$

**Examples**.

1. $\star C = C x_1 \wedge \dots \wedge x_n$ ; in other words the isomorphism $\star$ acts on constants (= 0-forms) by multiplying them by the volume form.

2. In $\mathbb{R}^3$ we have

$$\star x_1 = x_2 \wedge x_3 \,, \star x_2 = -x_1 \wedge x_3 = x_3 \wedge x_1 \,, \star x_3 = x_1 \wedge x_2 \,,$$

$$\star(x_1 \wedge x_2) = x_3 \,, \star(x_3 \wedge x_1) = x_2 \,, \star(x_2 \wedge x_3) = x_1 \,.$$

3. More generally, given a 1-form $l = a_1 x_1 + \cdots + a_n x_n$ we have

$$\star l = a_1 x_2 \wedge \cdots \wedge x_n - a_2 x_1 \wedge x_3 \wedge \cdots \wedge x_n + \cdots + (-1)^{n-1} a_n x_1 \wedge \cdots \wedge x_{n-1} \,.$$

In particular for $n = 3$ we have

$$\star(a_1 x_1 + a_2 x_2 + a_3 x_3) = a_1 x_2 \wedge x_3 + a_2 x_3 \wedge x_1 + a_3 x_1 \wedge x_2 \,.$$

**Proposition 1.16.4.**

$$\star^2 = (-1)^{k(n-k)} \mathrm{Id}, \quad \textit{i.e.} \quad \star(\star\omega) = (-1)^{k(n-k)}\omega \quad \textit{for any } k\textit{-form} \quad \omega \,.$$

*In particular, if dimension $n = \dim V$ is odd then $*^2 = \mathrm{Id}$. If $n$ is even and $\omega$ is a $k$-form then $\star(\star\omega) = \omega$ if $k$ is even, and $\star(\star\omega) = -\omega$ if $k$ is odd.*

**Proof.** It is sufficient to verify the equality

$$\star(\star\omega) = (-1)^{k(n-k)}\omega$$

for the case when $\omega$ is a basic form, i.e.

$$\omega = x_{i_1} \wedge \cdots \wedge x_{i_k} \,, \quad 1 \le i_1 < \cdots < i_k \le n \,.$$

We have

$$\star(x_{i_1} \wedge \cdots \wedge x_{i_k}) = (-1)^{\mathrm{inv}(i_1,\ldots,i_k,j_1,\ldots,j_{n-k})} x_{j_1} \wedge x_{j_2} \wedge \cdots \wedge x_{j_{n-k}}$$

and

$$\star(x_{j_1} \wedge x_{j_2} \wedge \cdots \wedge x_{j_{n-k}}) = (-1)^{\mathrm{inv}(j_1,\ldots,j_{n-k},i_1,\ldots,i_k)} * x_{i_1} \wedge \cdots \wedge x_{i_k} \,.$$

But the permutations $i_1 \dots i_k j_1 \dots j_{n-k}$ and $j_1 \dots j_{n-k} i_1 \dots i_k$ differ by $k(n-k)$ transpositions of pairs of its elements. Hence, we get

$$(-1)^{\mathrm{inv}(i_1,\dots,i_k,j_1,\dots,j_{n-k})} = (-1)^{k(n-k)}(-1)^{\mathrm{inv}(j_1,\dots,j_{n-k},i_1,\dots,i_k)},$$

and, therefore,

$$\begin{aligned}
\star\big(\star(x_{i_1} \wedge \dots \wedge x_{i_k})\big) &= \star\big((-1)^{\mathrm{inv}(i_1,\dots,i_k,j_1,\dots,j_{n-k})} x_{j_1} \wedge x_{j_2} \wedge \dots \wedge x_{j_{n-k}}\big) \\
&= (-1)^{\mathrm{inv}(i_1,\dots,i_k,j_1,\dots,j_{n-k})} \star (x_{j_1} \wedge x_{j_2} \wedge \dots \wedge x_{j_{n-k}}) \\
&= (-1)^{\mathrm{inv}(i_1,\dots,i_k,j_1,\dots,j_{n-k})+\mathrm{inv}(j_1,\dots,j_{n-k},i_1,\dots,i_k)} x_{i_1} \wedge \dots \wedge x_{i_k} \\
&= (-1)^{k(n-k)} x_{i_1} \wedge \dots \wedge x_{i_k}.
\end{aligned}$$

■

**Exercise 1.16.5.** *(a) For any special orthogonal operator $\mathcal{A}$ the operators $\mathcal{A}^*$ and $*$ commute, i.e.*

$$\mathcal{A}^* \circ * = * \circ \mathcal{A}^*.$$

*(b) Let $A$ be an orthogonal matrix with $\det A = 1$. Prove that the absolute value of each $k$-minor $M$ of $A$ is equal to the absolute value of its complementary minor of order $(n-k)$. (Hint: Apply (a) to the form $x_{i_1} \wedge \dots \wedge x_{i_k}$).*

*(c) Let $V$ be an oriented 3-dimensional Euclidean space. Prove that for any two vectors $X, Y \in V$, their cross-product can be written in the form*

$$X \times Y = \mathcal{D}^{-1}\left(*(\mathcal{D}(X) \wedge \mathcal{D}(Y))\right).$$

## 1.17 Euclidean structure on the space of exterior forms

Suppose that the space $V$ is oriented and Euclidean, i.e. it is endowed with an inner product $\langle\,,\,\rangle$ and an orientation.

Given two forms $\alpha, \beta \in \Lambda^k(V^*)$, $k = 0, \dots, n$, let us define

$$\langle\langle \alpha, \beta \rangle\rangle = \star(\alpha \wedge \star\beta).$$

Note that $\alpha \wedge \star\beta$ is an $n$-form for every $k$, and hence, $\langle\langle \alpha, \beta \rangle\rangle$ is a 0-form, i.e. a real number.

**Proposition 1.17.1.** *1. The operation $\langle\langle\, , \,\rangle\rangle$ defines an inner product on $\Lambda^k(V^*)$ for each $k = 0, \dots, n$.*

*2. If $\mathcal{A} : V \to V$ is a special orthogonal operator then the operator $\mathcal{A}^* : \Lambda^k(V^*) \to \Lambda^k(V^*)$ is orthogonal with respect to the inner product $\langle\langle\, , \,\rangle\rangle$.*

**Proof.**

1. We need to check that $\langle\langle \alpha, \beta \rangle\rangle$ is a symmetric bilinear function on $\Lambda^k(V^*)$ and $\langle\langle \alpha, \alpha \rangle\rangle > 0$ unless $\alpha = 0$. Bilinearity is straightforward. Hence, it is sufficient to verify the remaining properties for basic vectors $\alpha = x_{i_1} \wedge \dots \wedge x_{i_k}, \beta = x_{j_1} \wedge \dots \wedge x_{j_k}$, where $1 \le i_1 < \dots < i_k \le n$, $1 \le j_1 < \dots < j_k \le n$. Here $(x_1, \dots, x_n)$ is any Cartersian coordinates in $V$ which define its given orientation.

Note that $\langle\langle \alpha, \beta \rangle\rangle = 0 = \langle\langle \beta, \alpha \rangle\rangle$ unless $i_m = j_m$ for all $m = 1, \dots, k$, and in the the latter case we have $\alpha = \beta$. Furthermore, we have

$$\langle\langle \alpha, \alpha \rangle\rangle = \star(\alpha \wedge \star\alpha) = \star(x_1 \wedge \dots \wedge x_n) = 1 > 0.$$

2. The inner product $\langle\langle\, , \,\rangle\rangle$ is defined only in terms of the Euclidean structure and the orientation of $V$. Hence, for any special orthogonal operator $\mathcal{A}$ (which preserves these structures) the induced operator $\mathcal{A}^* : \Lambda^k(V^*) \to \Lambda^k(V^*)$ preserves the inner product $\langle\langle\, , \,\rangle\rangle$.

■

Note that we also proved that the basis of $k$-forms $x_{i_1} \wedge \dots \wedge x_{i_k}$, $1 \le i_1 < \dots < i_k \le n$, is orthonormal with respect to the scalar product $\langle\langle, \rangle\rangle$. Hence, we get

**Corollary 1.17.2.** *Suppose that a $k$-form $\alpha$ can be written in Cartesian coordinates as*

$$\alpha = \sum_{1 \le i_1 < \dots < i_k \le n} a_{i_1 \dots i_k} x_{i_1} \wedge \dots \wedge x_{i_k}.$$

*Then*

$$\| \alpha \|^2 = \langle\langle \alpha, \alpha \rangle\rangle = \sum_{1 \le i_1 < \dots < i_k \le n} a^2_{i_1 \dots i_k}.$$

**Corollary 1.17.3.** *Let $V$ be a Euclidean $n$-dimensional space. Choose an orthonormal basis $e_1, \dots, e_n$ in $V$. Then for any vectors $Z_1 = (z_{11}, \dots, z_{n1}), \dots, Z_k = (z_{1k}, \dots, z_{nk}) \in V$ we have*

$$(\mathrm{Vol}_k P(Z_1, \dots, Z_k))^2 = \sum_{1 \le i_1 < \dots < i_k \le n} Z^2_{i_1, \dots, i_k}, \tag{1.12}$$

*where*

$$Z_{i_1, \dots, i_k} = \begin{vmatrix} z_{i_1 1} & \dots & z_{i_1 k} \\ \dots & \dots & \dots \\ z_{i_k 1} & \dots & z_{i_k k} \end{vmatrix}.$$

**Proof.** Consider linear functions $l_j = \mathcal{D}(Z_j) = \sum\limits_{i=1}^{n} z_{ij} x_i \in V^*$, $j = 1, \dots, k$. Then

$$l_1 \wedge \dots \wedge l_k = \sum_{i_1=1}^{n} z_{i_1 j} x_{i_1} \wedge \dots \wedge \sum_{i_k=1}^{n} z_{i_k j} x_{i_k} =$$
$$\sum_{i_1, \dots, i_k} z_{i_1} \dots z_{i_k} x_{i_1} \wedge x_{i_k} = \sum_{1 \le i_1 < \dots < i_k \le n} Z_{i_1, \dots, i_k} x_{i_1} \wedge \dots x_{i_k}. \tag{1.13}$$

In particular, if one has $Z_1, \dots Z_k \in \mathrm{Span}(e_1, \dots, e_k)$ then $Z_{1 \dots k} = \mathrm{Vol}\, P(Z_1, \dots, Z_k)$ and hence

$$l_1 \wedge \dots \wedge l_k = Z_{1 \dots k}\, x_1 \wedge \dots \wedge x_k = \mathrm{Vol}\, P(Z_1, \dots, Z_k) x_1 \wedge \dots \wedge x_k,$$

which yields the claim in this case.

In the general case, according to Proposition 1.17.2 we have

$$\| l_1 \wedge \dots \wedge l_k \|^2 = \sum_{1 \le i_1 < \dots < i_k \le n} Z^2_{i_1, \dots, i_k}, \tag{1.14}$$

which coincides with the right-hand side of (1.12). Thus it remains to check to that

$$\| l_1 \wedge \dots \wedge l_k \|^2 = (\mathrm{Vol}_k P(Z_1, \dots, Z_k))^2.$$

Given any orthogonal transformation $\mathcal{A} : V \to V$ we have, according to Proposition 1.17.1, the equality

$$\| l_1 \wedge \cdots \wedge l_k \| = \| \mathcal{A}^* l_1 \wedge \cdots \wedge \mathcal{A}^* l_k \|, \tag{1.15}$$

and applying Corollary 1.14.2 we have

$$|\mathrm{Vol}_k P(Z_1, \dots, Z_k)| = |\mathrm{Vol}_k P(\mathcal{A}(Z_1), \dots, \mathcal{A}(Z_k))|. \tag{1.16}$$

On the other hand, there exists an orthogonal transformation $\mathcal{A} : V \to V$ such that $\mathcal{A}(Z_1), \dots, \mathcal{A}(Z_k) \in \mathrm{Span}(e_1, \dots, e_k)$. As was pointed out above we then have

$$|\mathrm{Vol}_k P(\mathcal{A}(Z_1), \dots, \mathcal{A}(Z_k))| = \| \mathcal{A}^* l_1 \wedge \cdots \wedge \mathcal{A}^* l_k \|, \tag{1.17}$$

and the claim follows from (1.15), (1.16) and (1.17). ■

We recall that an alternative formula for computing $\mathrm{Vol}_k P(Z_1, \dots, Z_k)$ was given earlier in Proposition 1.15.1.

## 1.18 Contraction

Let $V$ be a vector space and $\phi \in \Lambda^k(V^*)$ a $k$-form. Define a $(k-1)$-form $\psi = v \lrcorner \phi$ by the formula

$$\psi(X_1, \dots, X_{k-1}) = \phi(v, X_1, \dots, X_{k-1})$$

for any vectors $X_1, \dots, X_{k-1} \in V$.

**Proposition 1.18.1.** $\lrcorner$ *is a bilinear operation, i.e.*

$$\begin{aligned}
(v_1 + v_2) \lrcorner \phi &= v_1 \lrcorner \phi + v_2 \lrcorner \phi \\
(\lambda v) \lrcorner \phi &= \lambda (v \lrcorner \phi) \\
v \lrcorner (\phi_1 + \phi_2) &= v \lrcorner \phi_1 + v \lrcorner \phi_2 \\
v \lrcorner (\lambda \phi) &= \lambda (v \lrcorner \phi).
\end{aligned}$$

*Here* $v, v_1, v_2 \in V$; $\phi, \phi_1, \phi_2 \in \Lambda^k(V^*)$; $\lambda \in \mathbb{R}$.

The proof is straightforward.

Let $\phi$ be a non-zero $n$-form. Then we have

**Proposition 1.18.2.** *The map* $\lrcorner : V \to \Lambda^{n-1}(V^*)$, *defined by the formula* $\lrcorner(v) = v \lrcorner \phi$ *is an isomorphism between the vector spaces* $V$ *and* $\Lambda^{n-1}(V^*)$.

*Proof.* Take a basis $v_1, \dots, v_n$. Let $x_1, \dots, x_n \in V^*$ be the dual basis, i.e. the corresponding coordinate system. Then $\phi = a x_1 \wedge \dots \wedge x_n$, where $a \neq 0$. To simplify the notation let us assume that $a = 1$, so that

$$\phi = x_1 \wedge \dots \wedge x_n.$$

Then

$$v \lrcorner \phi(U_1, \dots, U_{n-1}) = \det(v, U_1, \dots, U_{n-1}) = \begin{vmatrix} a_1 & u_{1,1} & \dots & u_{1,n-1} \\ \dots & \dots & \dots & \dots \\ a_n & u_{n,1} & \dots & u_{n,n-1} \end{vmatrix},$$

where $\begin{pmatrix} u_{1,i} \\ \vdots \\ u_{n,i} \end{pmatrix}$ are coordinates of the vector $U_i \in V$ in the basis $V_1, \dots, V_n$ and $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ are coordinates of $v$.

Expanding the determinant according to the first column, we get

$$v \lrcorner \phi(U_1, \dots, U_k) = a_1 \begin{vmatrix} u_{2,1} & \dots & u_{2,n-1} \\ u_{3,1} & \dots & u_{3,n-1} \\ \dots & \dots & \dots \\ u_{n,1} & \dots & u_{n,n-1} \end{vmatrix} + \dots + (-1)^{n+1} a_n \begin{vmatrix} u_{1,1} & \dots & u_{1,n-1} \\ u_{2,1} & \dots & u_{2,n-1} \\ \dots & \dots & \dots \\ u_{n-1,1} & \dots & u_{n-1,n-1} \end{vmatrix}.$$

Take the $(k-1)$-form

$$\omega = a_1 x_2 \wedge \dots \wedge x_n - a_2 x_1 \wedge x_3 \wedge \dots \wedge x_n + + \dots + (-1)^{n+1} a_n x_1 \wedge \dots \wedge x_{n-1}. \qquad (1.18)$$

Then we have

$$\omega(u_1, \ldots, u_{n-1}) = v \lrcorner \phi(u_1, \ldots, u_{n-1})$$

for any $(n-1)$ vectors $u_1, \ldots, u_{n-1} \in V$. Hence,

$$\omega = v \lrcorner \phi.$$

But any $(n-1)$-form coefficients $a_1, \ldots, a_n$. Hence, the correspondence $v \mapsto v \lrcorner \phi$ is an isomorphism $V \to \Lambda^{n-1}(V^*)$. ■

Suppose that $\dim V = 3$. Then the formula (1.18) can be rewritten as

$$v \lrcorner (x_1 \wedge x_2 \wedge x_3) = a_1 x_2 \wedge x_3 + a_2 x_3 \wedge x_1 + a_3 x_1 \wedge x_2,$$

where $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ are coordinates of the vector $V$.

**Exercise 1.18.3.** *Let*

$$\alpha = x_{i_1} \wedge \cdots \wedge x_{i_k}, \quad 1 \leq i_1 < \cdots < i_k \leq n$$

*and* $v = (a_1, \ldots, a_n)$. *Show that*

$$v \lrcorner \alpha = \sum_{j=1}^{k} (-1)^{j+1} a_{i_j} x_{i_1} \wedge \ldots x_{i_{j-1}} \wedge x_{i_{j+1}} \wedge \ldots x_{i_k}.$$

Let us describe the geometric meaning of the operation $\lrcorner$. Set $\omega = v \lrcorner (x_1 \wedge x_2 \wedge x_3)$. Then $\omega(U_1, U_2)$ is the volume of the parallelogram defined by the vectors $U_1, U_2$ and $v$. Let $\nu$ be the unit normal vector to the plane $L(U_1, U_2) \subset V$. Then we have

$$\omega(U_1, U_2) = \text{ Area } P(U_1, U_2) \cdot \langle v, \nu \rangle.$$

If we interpret $v$ as the velocity of a fluid flow in the space $V$ then $\omega(U_1, U_2)$ is just an amount of fluid flown through the parallelogram $\Pi$ generated by vectors $U_1$ and $U_2$ for the unit time. It is called *the flux* of $v$ through the parallelogram $\Pi$.

The next proposition establishes a relation between the isomorphisms $\star, \lrcorner$ and $\mathcal{D}$.

**Proposition 1.18.4.** *Let $V$ be a Euclidean space, and $x_1, \ldots, x_n$ be coordinates in an orthonormal basis. Then for any vector $v \in V$ we have*

$$\star\mathcal{D}v = v \lrcorner (x_1 \wedge \cdots \wedge x_n)\,.$$

**Proof.** Let $v = (a_1, \ldots, a_n)$. Then $\mathcal{D}v = a_1 x_1 + \cdots + a_n x_n$ and

$$\star\mathcal{D}v = (a_1 x_2 \wedge \cdots \wedge x_n - a_2 x_1 \wedge x_3 \wedge \cdots \wedge x_n \wedge x_1 + \cdots + (-1)^{n-1} a_n x_1 \wedge \cdots \wedge x_{n-1})\,.$$

But according to Proposition 1.18.2 the $(n-1)$-form $v \lrcorner (x_1 \wedge \cdots \wedge x_n)$ is defined by the same formula. ■

# 2 Vector fields and differential forms

## 2.1 Differential and gradient of a smooth function

Given a vector space $V$ we will denote by $V_x$ the vector space $V$ with the origin translated to the point $x \in V$. One can think of $V_x$ as that tangent space to $V$ at the point $x$. Though the parallel transport allows one to identify spaces $V$ and $V_x$ it will be important for us to think about them as different spaces.

Let $f : U \to \mathbb{R}$ be a function on a domain $U \subset V$ in a vector space $V$. The function $f$ is called *differentiable* at a point $x \in U$ if there exists a linear function $l : V_x \to \mathbb{R}$ such that

$$f(x + h) - f(x) = l(h) + o(||h||)$$

for any sufficiently small vector $h$, where the notation $o(t)$ stands for any function such that $\frac{o(t)}{t} \underset{t\to 0}{\to} 0$. The linear function $l$ is called the *differential* of the function $f$ at the point $x$ and is denoted by $d_x f$. In other words, $f$ is differentiable at $x \in U$ if for any $h \in V_x$ there exists a limit

$$l(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}\,,$$

and the limit $l(h)$ *linearly* depends on $h$. The value $l(h) = d_x f(h)$ is called the *directional derivative* of $f$ at the point $x$ in the direction $h$. The function $f$ is called differentiable on the whole domain $U$ if it is differentiable at each point of $U$.

Simply speaking, the differentiability of a function means that at a small scale near a point $x$ the function behaves approximately like a linear function, the differential of the function at the point $x$. However this linear function varies from point to point, and we call the family $\{d_x f\}_{x \in U}$ of all these linear functions the *differential* of the function $f$, and denote it by $df$ (without a reference to a particular point $x$).

Let us summarize the above discussion. Let $f : U \to \mathbb{R}$ be a differentiable function. Then for each point $x \in U$ there exists a linear function $d_x f : V_x \to \mathbb{R}$, the differential of $f$ at the point $x$ defined by the formula

$$d_x f(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}\,, x \in U, h \in V_x\,.$$

If $v_1, \dots, v_n$ are vectors of a basis of $V$, parallel transported to the point $x$, then we have

$$d_x f(v_i) = \frac{\partial f}{\partial x_i}(x),\ x \in U,\ i = 1, \dots, n\,,$$

where $x_1, \dots, x_n$ are coordinates with respect to the chosen basis $v_1, \dots, v_n$.

Notice that if $f$ is a linear function,

$$f(x) = a_1 x_1 + \dots + a_n x_n\,,$$

then for each $x \in V$ we have

$$d_x f(h) = a_1 h_1 + \dots + a_n h_n\ , h = (h_1, \dots, h_n) \in V_x\,.$$

Thus the differential of a linear function $f$ at any point $x \in V$ coincides with this function, parallel transported to the space $V_x$. This observation, in particular, can be applied to linear coordinate functions $x_1, \dots, x_n$ with respect to a chosen basis of $V$.

In Section 2.6 below we will define the differential for maps $f : U \to W$, where $W$ is a vector space and not just the real line $\mathbb{R}$.

## 2.2 Gradient vector field

If $V$ is an Euclidean space, i.e. a vector space with an inner product $\langle\, ,\, \rangle$, then there exists a canonical isomorphism $\mathcal{D} : V \to V^*$, defined by the formula $\mathcal{D}(v)(x) = \langle v, x\rangle$ for $v, x \in V$. Of course, $\mathcal{D}$ defines an isomorphism $V_x \to V_x^*$ for each $x \in V$. Set

$$\nabla f(x) = \mathcal{D}^{-1}(d_x f).$$

The vector $\nabla f(x)$ is called the *gradient* of the function $f$ at the point $x \in U$. We will also use the notation $\mathrm{grad} f(x)$.

By definition we have

$$\langle \nabla f(x), h\rangle = d_x f(h) \quad \text{for any vector} \quad h \in V\,.$$

If $||h|| = 1$ then $d_x f(h) = ||\nabla f(x)|| \cos\varphi$, where $\varphi$ is the angle between the vectors $\nabla f(x)$ and $h$. In particular, the directional derivative $d_x f(h)$ has its maximal value when $\varphi = 0$. Thus the direction of the gradient is the direction of the maximal growth of the function and the length of the gradient equals this maximal value.

As in the case of a differential, the gradient varies from point to point, and the family of vectors $\{\nabla f(x)\}_{x\in U}$ is called the *gradient vector field* $\nabla f$.

We discuss the general notion of a vector field in the next section.

## 2.3 Vector fields

A *vector field* $v$ on a domain $U \subset V$ is a function which associates to each point $x \in U$ a vector $v(x) \in V_x$, i.e. a vector originated at the point $x$.

A gradient vector field $\nabla f$ of a function $f$ provides us with an example of a vector field, but as we shall see, gradient vector fields form only a small very special class of vector fields.

Let $v$ be a vector field on a domain $U \in V$. If we fix a basis in $V$, and parallel transport this basis to all spaces $V_x, x \in V$, then for any point $x \in V$ the vector $v(x) \in V_x$ is described by its coordinates $(v_1(x), v_2(x), \ldots, v_n(x))$. Thus to define a vector field on $U$ is the same as to define $n$ functions $v_1, \ldots, v_n$ on $U$, i.e. to define a map $(v_1, \ldots, v_n) : U \to \mathbb{R}^n$.

Thus, if a basis of $V$ is fixed, then the difference between the maps $U \to \mathbb{R}^n$ and vector fields on $U$ is just a matter of geometric interpretation. When we speak about a vector field $v$ we view $v(x)$ as a vector in $V_x$, i.e. originated at the point $x \in U$. When we speak about a map $v : U \to \mathbb{R}^n$ we view $v(x)$ as a point of the space $V$, or as a vector, with its origin at $\mathbf{0} \in V$.

Vector fields naturally arise in a context of Physics, Mechanics, Hydrodynamics, etc. as force, velocity and other physical fields.

There is another very important interpretation of vector fields as *first order differential operators.*

Let $C^\infty(U)$ denote the *vector space of infinitely differentiable functions* on a domain $U \subset V$. Let $v$ be a vector field on $V$. Let us associate with $v$ a linear operator

$$D_v : C^\infty(U) \to C^\infty(U),$$

given by the formula

$$D_v(f) = df(v),\ f \in C^\infty(U).$$

In other words, we compute at any point $x \in U$ the directional derivative of $f$ in the direction of the vector $v(x)$. Clearly, the operator $D_v$ is linear: $D_v(af + bg) = aD_v(f) + bD_v(g)$ for any functions $f, g \in C^\infty(U)$ and any real numbers $a, b \in \mathbb{R}$. It also satisfies the *Leibniz rule*:

$$D_v(fg) = D_v(f)g + fD_v(g).$$

In view of the above correspondence between vector fields and first order differential operatopts , it is sometimes convenient just to view a vector field as a differential operator. Hence, when it will not be confusing we may drop the notation $D_v$ and just directly apply the vector $v$ to a function $f$.

Let $v_1, \ldots, v_n$ be a basis of $V$, and $x_1, \ldots, x_n$ be the coordinate functions in this basis. We would like to introduce the notation for the vector field obtained from vectors $v_1, \ldots, v_n$ by parallel transporting them to all points of the domain $U$. To motivate the notation which we are going to introduce, let us temporarily denote these vector fields by $\mathbf{v}_1, \dots, \mathbf{v_n}$. Observe that $D_{\mathbf{v}_i}(f) = \frac{\partial f}{\partial x_i}$, $i = 1, \dots, n$. Thus the operator $D_{\mathbf{v}_i}$ is just the operator $\frac{\partial}{\partial x_i}$ of taking $i$-th partial derivative. Hence, viewing the vector field $\mathbf{v_i}$ as a differential operator we will just use the notation $\frac{\partial}{\partial x_i}$ instead of $\mathbf{v}_i$. Given any vector field $v$ with coordinate functions $a_1, a_2, \dots, a_n : U \to \mathbb{R}$ we have

$$D_v(f)(x) = \sum_{i=1}^{n} a_i(x) \frac{\partial f}{\partial x_i}(x), \quad \text{for any} \quad f \in C^\infty(U),$$

and hence we can write $v = \sum\limits_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$. Note that the coefficients $a_i$ here are *functions* and not constants.

Suppose that $V, \langle\, , \,\rangle$ is a Euclidean vector space. Choose a (not necessarily orthonormal) basis $v_1, \dots, v_n$. Let us find the coordinate description of the gradient vector field $\nabla f$, i.e. find the coefficients $a_j$ in the expansion $\nabla f(x) = \sum\limits_{1}^{n} a_i(x) \frac{\partial}{\partial x_i}$. By definition we have

$$\langle \nabla f(x), h \rangle = d_x f(h) = \sum_{1}^{n} \frac{\partial f}{\partial x_j}(x) h_j \tag{2.1}$$

for any vector $h \in V_x$ with coordinates $(h_1, \dots, h_n)$ in the basis $v_1, \dots, v_n$ parallel transported to $V_x$. Let us denote $g_{ij} = \langle v_i, v_j \rangle$. Thus $G = (g_{ij})$ is a symmetric $n \times n$ matrix, which is called the *Gram matrix* of the basis $v_1, \dots, v_n$. Then the equation (2.1) can be rewritten as

$$\sum_{i,j=1}^{n} g_{ji} a_i h_j = \sum_{1}^{n} \frac{\partial f}{\partial x_j}(x) h_j.$$

Because $h_j$ are arbitrarily numbers it implies that the coefficients with $h_j$ in the right and left sides should coincide for all $j = 1, \dots, n$. Hence we get the following system of linear equations:

$$\sum_{i=1}^{n} g_{ij} a_i = \frac{\partial f}{\partial x_j}(x), \quad j = 1, \dots, n, \tag{2.2}$$

or in matrix form

$$G \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix},$$

and thus

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = G^{-1} \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix}, \tag{2.3}$$

i.e.

$$\nabla f = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x_i}(x) \frac{\partial}{\partial x_j}, \tag{2.4}$$

where we denote by $g^{ij}$ the entries of the inverse matrix $G^{-1} = (g_{ij})^{-1}$

If the basis $v_1, \dots, v_n$ is orthonormal then $G$ is the unit matrix, and thus in this case

$$\nabla f = \sum_{1}^{n} \frac{\partial f}{\partial x_j}(x) \frac{\partial}{\partial x_j}, \tag{2.5}$$

i.e. $\nabla f$ has coordinates $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. However, simple expression (2.5) for the gradient holds **only in the orthonormal basis**. In the general case one has a more complicated expression (2.4).

## 2.4 Differential forms

Similarly to vector fields, we can consider functions on $U \subset V$ which associates to each point a $k$-form from $\Lambda^k(V_x^*)$. These functions are called *differential $k$-forms*.

Thus the relation between $k$-forms and differential $k$-forms is exactly the same as the relation between vectors and vector-fields. For instance, a differential 1-form $\alpha$ associates with each point $x \in U$ a linear function $\alpha(x)$ on the space $V_x$. Sometimes we will write $\alpha_x$ instead of $\alpha(x)$ to leave space for the arguments of the function $\alpha(x)$.

**Example 2.4.1.** 1. Let $f : V \to \mathbb{R}$ be a smooth function. Then the differential $df$ is a differential 1-form. Indeed, with each point $x \in V$ it associates a linear function $d_x f$ on the space $V_x$. As we shall see, most differential 1-form *are not* differentials of functions (just as most vector fields are not gradient vector fields).

2. A differential 0-form $f$ on $U$ associates with each point $x \in U$ a 0-form on $V_x$, i. e. a number $f(x) \in \mathbb{R}$. Thus differential 0-forms on $U$ are just functions $U \to \mathbb{R}$.

## 2.5 Coordinate description of differential forms

Let $x_1, \dots, x_n$ be coordinate linear functions on $V$, which form the basis of $V^*$ dual to a chosen basis $v_1, \dots, v_n$ of $V$. For each $i = 1, \dots, n$ the differential $dx_i$ defines a linear function on each space $V_x, x \in V$. Namely, if $h = (h_1, \dots, h_n) \in V_x$ then $dx_i(h) = h_i$. Indeed

$$d_x x_i(h) = \lim_{t \to 0} \frac{x_i + th_i - x_i}{t} = h_i,$$

independently of the base point $x \in V$. Thus differentials $dx_1, \dots, dx_n$ form a basis of the space $V_x^*$ for each $x \in V$. In particular, any differential 1-form $\alpha$ on $v$ can be written as

$$\alpha = f_1 dx_1 + \dots + f_n dx_n,$$

where $f_1, \dots, f_n$ are *functions* on $V$. In particular,

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n. \tag{2.6}$$

Let us point out that this simple expression of the differential of a function holds in an **arbitrary coordinate system**, while an analogous simple expression (2.5) for the gradient vector field is valid only in the case of Cartesian coordinates. This reflects the fact that while the notion of differential is intrinsic and independent of any extra choices, one needs to have a background inner product to define the gradient.

Similarly, any differential 2-form $w$ on a 3-dimensional space can be written as

$$\omega = b_1(x) dx_2 \wedge dx_3 + b_2(x) dx_3 \wedge dx_1 + b_3(x) dx_1 \wedge dx_2$$

where $b_1, b_2$, and $b_3$ are functions on $V$. Any differential 3-form $\Omega$ on a 3-dimensional space $V$ has the form

$$\Omega = c(x) dx_1 \wedge dx_2 \wedge dx_3$$

for a function $c$ on $V$.

More generally, any differential $k$-form $\alpha$ can be expressed as

$$\alpha = \sum_{1 \le i_1 < i_2 \cdots < i_k \le n} a_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

for some functions $a_{i_1 \dots i_k}$ on $V$.

## 2.6 Smooth maps and their differentials

Let $V, W$ be two vector spaces of arbitrary (not, necessarily, equal) dimensions and $U \subset V$ be an open domain in $V$.

Recall that a map $f : U \to W$ is called *differentiable* if for each $x \in U$ there exists a linear map

$$l : V_x \to W_{f(x)}$$

such that

$$l(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

for any $h \in V_x$. In other words,

$$f(x+th) - f(x) = tl(h) + o(t), \text{ where } \frac{o(t)}{t} \underset{t \to 0}{\to} 0.$$

The map $l$ is denoted by $d_x f$ and is called *the differential* of the map $f$ at the point $x \in U$. Thus, $d_x f$ is a linear map $V_x \to W_{f(x)}$.

When $W = \mathbb{R}$ then we get the notion of the differential of a function, which was introduced earlier in Section 2.1.

Let us pick bases in $V$ and $W$ and let $(x_1, \dots, x_k)$ and $(y_1, \dots, y_n)$ be the corresponding coordinate functions. Then each of the spaces $V_x$ and $W_y$, $x \in V$, $y \in W$ inherits a basis obtained by parallel transport of the bases of $V$ and $W$. In terms of these bases, the differential $d_x f$ is given by the Jacobi matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_k} \\ \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_k} \end{pmatrix}$$

In what follows we will consider only sufficiently smooth maps, i.e. we assume that all maps and their coordinate functions are differentiable as many times as we need it.

## 2.7 Operator $f^*$

Let $U$ be a domain in a vector space $V$ and $f : U \to W$ a smooth map. Then the differential $df$ defines a linear map

$$d_x f : V_x \to W_{f(x)}$$

for each $x \in V$.

Let $\omega$ be a differential $k$-form on $W$. Thus $\omega$ defines an exterior $k$-form on the space $W_y$ for each $y \in W$.

Let us define the differential $k$-form $f^*\omega$ on $U$ by the formula

$$(f^*\omega)|_{V_x} = (d_x f)^*(\omega|_{W_{f(x)}}).$$

Here the notation $\omega|_{W_y}$ stands for the exterior $k$-form defined by the differential form $\omega$ on the space $W_y$.

In other words, for any $k$ vectors, $H_1, \dots, H_k \in V_x$ we have

$$f^*\omega(H_1, \dots, H_k) = \omega(d_x f(H_1), \dots, d_x f(H_k)).$$

We say that the differential form $f^*\omega$ is *induced from* $\omega$ by the map $f$, or that $f^*\omega$ is the *pull-back of* $\omega$ *by* $f$.

**Example 2.7.1.** Let $\Omega = h(x)dx_1 \wedge \dots \wedge dx_n$. Then formula (1.14.2) implies

$$f^*\Omega = h \circ f \det Df dx_1 \wedge \dots \wedge dx_n.$$

Here

$$\det Df = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

is the determinant of the Jacobian matrix of $f = (f_1, \dots, f_n)$.

Similarly to Proposition1.3.3 we get

**Proposition 2.7.2.** *Given 2 maps*

$$U_1 \xrightarrow{f} U_2 \xrightarrow{g} U_3$$

*and a differential $k$ form $\omega$ on $U_3$ we have*

$$(g \circ f)^*(\omega) = f^*(g^*\omega)\,.$$

## 2.8 Coordinate description of the operator $f^*$

Consider first the linear case. Let $\mathcal{A}$ be a linear map $V \to W$ and $\omega \in \Lambda^p(W^*)$. Let us fix coordinate systems $x_1, \dots, x_k$ in $V$ and $y_1, \dots, y_n$ in $W$. If $A$ is the matrix of the map $\mathcal{A}$ then we already have seen in Section 1.10 that

$$\mathcal{A}^* y_j = l_j(x_1, \dots, x_k) = a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jk}x_k, \quad j = 1, \dots, n,$$

and that for any exterior $k$-form

$$\omega = \sum_{1 \le i_1 < \dots < i_p \le n} A_{i_1,\dots,i_p}\, y_{i_1} \wedge \dots \wedge y_{i_p}$$

we have

$$\mathcal{A}^*\omega = \sum_{1 \le i, < \dots < i_p \le n} A_{i_1 \dots i_p} l_{i_1} \wedge \dots \wedge l_{i_p}.$$

Now consider the non-linear situation. Let $\omega$ be a differential $p$-form on $W$. Thus it can be written in the form

$$\omega = \sum A_{i_1 \dots i_p}(y) dy_{i_1} \wedge \dots dy_{i_p}$$

for some functions $A_{i_1 \dots i_p}$ on $W$ .

Let $U$ be a domain in $V$ and $f : U \to W$ a smooth map.

**Proposition 2.8.1.** *$f^*\omega = \sum A_{i_1 \dots, i_p}(f(x)) df_{i_1} \wedge \dots \wedge df_{i_p}$, where $f_1, \dots, f_n$ are coordinate functions of the map $f$.*

*Proof.* For each point $x \in U$ we have, by definition,

$$f^*\omega|_{V_x} = (d_x f)^*(\omega|_{W_{fx}})$$

But the coordinate functions of the linear map $d_x f$ are just the differentials $d_x f_i$ of the coordinate functions of the map $f$. Hence the desired formula follows from the linear case proven in the previous proposition. ■

## 2.9 Examples

1. Consider the domain $U = \{r > 0, 0 \le \varphi < 2\pi\}$ on the plane $V = \mathbb{R}^2$ with cartesian coordinates $(r, \varphi)$. Let $W = \mathbb{R}^2$ be another copy of $\mathbb{R}^2$ with cartesian coordinates $(x, y)$. Consider a map $P : V \to W$ given by the formula

$$P(r, \varphi) = (r \cos \varphi, r \sin \varphi).$$

This map introduces $(r, \varphi)$ as *polar coordinates* on the plane $W$. Set $\omega = dx \wedge dy$. It is called *the area form* on $W$. Then

$$P^*\omega = d(r \cos \varphi) \wedge d(r \sin \varphi) = (\cos \varphi dr + r d(\cos \varphi)) \wedge (\sin \varphi dr + r d(\sin \varphi) =$$

$$(\cos\varphi dr - r\sin\varphi d\varphi) \wedge (\sin\varphi dr + r\cos\varphi d\varphi) =$$

$$\cos\varphi \cdot \sin\varphi dr \wedge dr - r\sin^2\varphi d\varphi \wedge dr + r\cos^2\varphi dr \wedge d\varphi - r^2 \sin\varphi \cos\varphi d\varphi \wedge d\varphi =$$

$$r\cos^2\varphi dr \wedge d\varphi + r\sin^2 dr \wedge d\varphi = r dr \wedge d\varphi.$$

**2.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and the map $F : \mathbb{R}^2 \to \mathbb{R}^3$ be given by the formula

$$F(x, y) = (x, y, f(x, y))$$

Let

$$\omega = P(x, y, z)dy \wedge dz + Q(x, y, z)dz \wedge dx + R(x, y, z)dx \wedge dy$$

be a differential 2-form on $\mathbb{R}^3$. Then

$$\begin{aligned} F^*\omega &= P(x, y, f(x, y))dy \wedge df + \\ &+ Q(x, y, f(x, y))df \wedge dx + R(x, y, f(x, y))dx \wedge dy \\ &= P(x, y, f(x, y))dy \wedge (f_x dx + f_y dy) + \\ &+ Q(x, y, f(x, y))(f_x dx + f_y dy) \wedge dx + \\ &+ R(x, y, f(x, y))dx \wedge dy = \\ &= (R(x, y, f(x, y)) - P(x, y, f(x, y))f_x - Q(x, y, f(x, y))f_y)dx \wedge dy \end{aligned}$$

where $f_x, f_y$ are partial derivatives of $f$.

**3.** If $p > k$ then the pull-back $f^*\omega$ of a $p$-form $\omega$ on $U$ to a $k$-dimensional space $V$ is equal to 0.

# 3 Exterior differential

Let $\Omega^k(U)$ be the space of all differential $k$-forms on $U$. We will define a map

$$d : \Omega^k(U) \to \Omega^{k+1}(U)$$

which is called the *exterior differential.*

We first define it in coordinates and then prove that the result is independent of the choice of the coordinate system. Let us fix a coordinate system $x_1, \dots, x_n$ in $V \supset U$. As a reminder, a differential $k$-form $w \in \Omega^k(U)$ has the form $w = \sum\limits_{i_1 < \ldots < i_k} a_{i_1 \ldots, i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ where $a_{i_1 \ldots i_k}$ are functions on the domain $U$. Define

$$dw := \sum_{i_1 < \ldots < i_k} da_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

**Examples. 1.** Let $w \in \Omega^1(U)$, i.e. $w = \sum\limits_{i=1}^{n} a_i dx_i$. Then

$$dw = \sum_{i=1}^{n} da_i \wedge dx_i = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \frac{\partial a_i}{\partial x_j} dx_j \right) \wedge dx_i = \sum_{1 \le i < j \le n} \left( \frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \right) dx_i \wedge dx_j.$$

For instance, when $n = 2$ we have

$$d(a_1 dx_1 + a_2 dx_2) = \left( \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) dx_1 \wedge dx_2$$

For $n = 3$, we get

$$d(a_1dx_1+a_2dx_2+a_3dx_3) = \left( \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \right) dx_2 \wedge dx_3 + \left( \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} \right) dx_3 \wedge dx_1 + \left( \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) dx_1 \wedge dx_2.$$

**2.** Let $n = 3$ and $w \in \Omega^2(U)$. Then

$$w = a_1 dx_2 \wedge dx_3 + a_2 dx_3 \wedge dx_1 + a_3 dx_1 \wedge dx_2$$

and

$$
\begin{aligned}
dw &= da_1 \wedge dx_2 \wedge dx_3 + da_2 \wedge dx_3 \wedge dx_1 + da_3 \wedge dx_1 \wedge dx_2 \\
&= \left( \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3
\end{aligned}
$$

3. For 0-forms, i.e. functions the exterior differential coincides with the usual differential of a function.

## 3.1 Properties of the operator $d$

**Proposition 3.1.1.** *For any 2 forms, $\alpha \in \Omega^k(U)$, $\beta \in \Omega^l(U)$ we have*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

*Proof.* We have

$$
\begin{aligned}
\alpha &= \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\
\beta &= \sum_{j_1 < \ldots < j_l} b_{j_i \ldots j_l} dx_{j_1} \wedge \ldots \wedge dx_{j_l} \\
\alpha \wedge \beta &= \left( \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k} \right) \wedge \left( \sum_{j_i < \ldots < j_l} b_{j_1 \ldots j_l} dx_{j_1} \wedge \ldots \wedge dx_{j_l} \right) \\
&= \sum_{\substack{i_1 < \ldots < i_k \\ j_1 < \ldots < j_l}} a_{i_1 \ldots i_k} b_{j_1 \ldots j_l} dx_{i_l} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_l} \wedge \ldots \wedge dx_{j_l}
\end{aligned}
$$

$$
\begin{aligned}
d(\alpha \wedge \beta) &= \sum_{\substack{i_1<\ldots<i_k \\ j_1<\ldots<j_l}} (b_{j_1\ldots j_l} da_{i_1\ldots i_k} + a_{i_1\ldots i_k} db_{j_1\ldots j_l}) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l} \\
&= \sum_{\substack{i_1<\ldots<i_k \\ j_1<\ldots<j_l}} b_{j_1\ldots j_l} da_{i_1\ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_l} \wedge \ldots \wedge dx_{j_l} \\
&+ \sum_{\substack{i_1<\ldots<i_k \\ j_1<\ldots<j_l}} a_{i_1\ldots i_k} db_{j_1\ldots j_l} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l} \\
&= \left( \sum_{i_1<\ldots<i_k} da_{i_1\ldots i_k} \wedge dx_{i_1\ldots} \wedge dx_{i_k} \right) \wedge \left( \sum b_{j_1\ldots j_l} dx_{j_1} \wedge \ldots \wedge dx_{j_l} \right) \\
&+ (-1)^k \left( \sum_{i_1<\ldots<i_k} a_{i_1\ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k} \right) \wedge \left( \sum_{j_i<\ldots<j_l} db_{j_1\ldots j_l} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l} \right) \\
&= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.
\end{aligned}
$$

Notice that the sign $(-1)^k$ appeared because we had to make $k$ transposition to move $db_{j_1\ldots j_l}$ to its place. ■

**Proposition 3.1.2.** *For any differential $k$-form $w$ we have*

$$ddw = 0.$$

*Proof.* Let $w = \sum_{i_1<\ldots<i_k} a_{i_1\ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. Then we have

$$dw = \sum_{i_1<\ldots<i_k} da_{i_1\ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

Applying Proposition 3.1.1 we get

$$
\begin{aligned}
ddw = \sum_{i_1<\ldots<i_k} & dda_{i_1\ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} - da_{i_1\ldots i_k} \wedge ddx_{i_1} \wedge \ldots \wedge dx_{i_k} + \ldots \\
& + (-1)^k da_{i_1\ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge ddx_{i_k}.
\end{aligned}
$$

But $ddf = 0$ for any function $f$ as was shown above. Hence all terms in this sum are equal to 0, i.e. $ddw = 0$. ■

**Definition.** A $k$-form $\omega$ is called *closed* if $d\omega = 0$. It is called *exact* if there exists a $(k-1)$-form $\theta$ such that $d\theta = \omega$. The form $\theta$ is called the *primitive* of the form $\omega$. The previous theorem can be reformulated as follows:

**Corollary 3.1.3.** *Every exact form is closed.*

The converse is not true in general. For instance, take a differential 1-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

on the punctured plane $U = \mathbb{R}^2 \setminus 0$ (i.e the plane $\mathbb{R}^2$ with the deleted origin). It is easy to calculate that $d\omega = 0$, i.e $\omega$ is closed. On the other hand it is not exact. Indeed, let us write down this form in polar coordinates $(r, \varphi)$. We have

$$x = r\cos\varphi,\ y = r\sin\varphi.$$

Hence,

$$\omega = \frac{1}{r^2}\left(r\cos\varphi(\sin\varphi dr + r\cos\varphi d\varphi) - r\sin\varphi(cos\varphi dr - r\sin\varphi d\varphi)\right) = d\varphi\ .$$

If there were a function $H$ on $U$ such that $dH = \omega$, then we would have to have $H = \varphi$+const, but this is impossible because the polar coordinate $\varphi$ is not a continuous univalent function on $U$. Hence $\omega$ is not exact.

However, a closed form is exact if it is defined on the whole vector space $V$.

**Proposition 3.1.4.** *Operators $f^*$ and $d$ commute, i.e. for any differential $k$-form $w \in \Omega^k(W)$, and a smooth map $f : U \to W$ we have*

$$df^*w = f^*dw$$

.

*Proof.* Suppose first that $k = 0$, i.e. $w$ is a function $\varphi : W \to \mathbb{R}$. Then $f^*\varphi = \varphi \circ f$. Then $d(\varphi \circ f) = f^* d\varphi$. Indeed, for any point $x \in U$ and a vector $X \in V_x$ we have

$$d(\varphi \circ f)(X) = d\varphi(d_x f(X))$$

(chain rule)

But $d\varphi(d_x f(X)) = f^*(d\varphi(X))$.

Consider now the case of arbitrary $k$-form $w$,

$$w = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

Then

$$f^* w = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} \circ f df_{i_1} \wedge \ldots \wedge df_{i_k}$$

where $f_1, \ldots, f_n$ are coordinate functions of the map $f$. Using the previous theorem and taking into account that $d(df_i) = 0$, we get

$$d(f^* w) = \sum_{i_1 < \ldots < i_k} d(a_{i_1 \ldots i_k} \circ f) \wedge df_{i_1} \wedge \ldots \wedge df_{i_k}.$$

On the other hand

$$dw = \sum_{i_1 < \ldots < i_k} da_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

and therefore

$$f^* dw = \sum_{i_1 < \ldots < i_k} f^*(da_{i_1 \ldots i_k}) \wedge df_{i_1} \wedge \ldots \wedge df_{i_k}.$$

But according to what is proven above, we have

$$f^* da_{i_1 \ldots i_k} = d(a_{i_1 \ldots i_k} \circ f)$$

Thus,

$$f^*dw = \sum_{i_1<\ldots<i_k} d(a_{i_1\ldots i_k} \circ f) \wedge df_{i_1} \wedge \ldots \wedge df_{i_k} = df^*w$$

■

The above theorem shows, in particular, that the definition of the exterior differential is independent of the choice of the coordinate. Moreover, one can even use non-linear (curvilinear) coordinate systems, like polar coordinates on the plane.

**Remark 3.1.5.** We will show later (see Lemma 5.1.2) that one can give another equivalent definition of the operator $d$ *without using any coordinates at all.*

## 3.2 Curvilinear coordinate systems

A (non-linear) *coordinate system* on a domain $U$ in an $n$-dimensional space $V$ is a smooth map $f = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ such that

1. For each point $x \in U$ the differentials $d_xf_1, \ldots, d_xf_n \in (V_x)^*$ are linearly independent.
2. $f$ is injective, i.e. $f(x) \neq f(y)$ for $x \neq y$.

Thus a *coordinate map* $f$ associates $n$ coordinates $y_1 = f_1(x), \ldots, y_n = f_n(x)$ with each point $x \in U$. The inverse map $f^{-1} : U' \to U$ is called *the parameterization*. Here $U' = f(U) \subset \mathbb{R}^n$ is the image of $U$ under the map $f$. If one already has another set of coordinates $x_1 \ldots x_n$ on $U$, then the coordinate map $f$ expresses new coordinates $y_1 \ldots y_n$ through the old one, while the parametrization map expresses the old coordinate through the new one. Thus the statement

$$g^*dw = dg^*w$$

applied to the parametrization map $g$ just tells us that the formula for the exterior differential is the same in the new coordinates and in the old one.

Consider a space $\mathbb{R}^n$ with coordinates $(u_1, \dots, u_n)$. The $j$-th coordinate line is given by equations $u_i = c_i, i = 1, \dots, n; i \neq j$. Given a domain $U' \subset \mathbb{R}^n$ consider a parameterization map $g : U' \to U \subset V$. The images $g\{u_i = c_i, i \neq j\}) \subset U$ of coordinates lines $\{u_i = c_i, i \neq j\} \subset U'$ are called coordinate lines in $U$ with respect to the curvilinear coordinate system $(u_1, \dots, u_n)$. For instance, coordinate lines for polar coordinates in $\mathbb{R}^2$ are concentric circles and rays, while coordinate lines for spherical coordinates in $\mathbb{R}^3$ are rays from the origin, and latitudes and meridians on concentric spheres.

## 3.3 More about vector fields

Similarly to the case of linear coordinates, given any curvilinear coordinate system $(u_1, \dots, u_n)$ in $U$, one denotes by

$$\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}$$

the vector fields which correspond to the partial derivatives with respect to the coordinates $u_1, \dots, u_n$. In other words, the vector field $\frac{\partial}{\partial u_i}$ is tangent to the $u_i$-coordinate lines and represents the the velocity vector of the curves $u_1 = \mathrm{const}_1, \dots, u_{i-1} = \mathrm{const}_{i-1}, u_{i+1} = \mathrm{const}_{i+1}, \dots, u_n = \mathrm{const}_n$ , parameterized by the coordinate $u_i$.

For instance, for spherical coordinates $(r, \theta, \varphi)$ in $\mathbb{R}^3$ the vector fields

$$\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \text{ and } \frac{\partial}{\partial \varphi}$$

are mutually orthogonal. We also have $||\frac{\partial}{\partial r}|| = 1$. However the length of vector fields $\frac{\partial}{\partial \varphi}$ and $\frac{\partial}{\partial \theta}$ vary. When $r$ and $\theta$ are fixed and $\varphi$ varies, then the corresponding point $(r, \theta, \varphi)$ is moving along a meridian of radius $r$ with a constant angular speed 1. Hence,

$$||\frac{\partial}{\partial \varphi}|| = r\,.$$

When $r$ and $\varphi$ are fixed and $\theta$ varies, then the point $(r, \varphi, \theta)$ is moving along a latitude of radius $r \sin \varphi$ with a constant angular speed 1. Hence,

$$||\frac{\partial}{\partial \theta}|| = r \sin \varphi\,.$$

The chain rule allows us to express the vector fields $\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_n}$ through the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. Indeed, for any function $f : U \to \mathbb{R}$ we have

$$\frac{\partial f}{\partial u_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial u_i},$$

and, therefore,

$$\frac{\partial}{\partial u_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial u_i} \frac{\partial}{\partial x_j},$$

For instance, suppose we are given spherical coordinates $(r, \varphi, \theta)$ in $\mathbb{R}^3$. The spherical coordinates are related to the cartesian coordinates $(x, y, z)$ by the formulas

$$\begin{aligned} x &= r \sin\varphi \cos\theta, \\ y &= r \sin\varphi \sin\theta, \\ z &= r \cos\varphi . \end{aligned}$$

Hence we derive the following expression of the vector fields $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$ through the vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ :

$$\begin{aligned} \frac{\partial}{\partial r} &= \sin\varphi \cos\theta \frac{\partial}{\partial x} + \sin\varphi \sin\theta \frac{\partial}{\partial y} + \cos\varphi \frac{\partial}{\partial z}, \\ \frac{\partial}{\partial \theta} &= -r \sin\varphi \sin\theta \frac{\partial}{\partial x} + r \sin\varphi \cos\theta \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial \varphi} &= r \cos\varphi \cos\theta \frac{\partial}{\partial x} + r \cos\varphi \sin\theta \frac{\partial}{\partial y} - r \sin\varphi \frac{\partial}{\partial z}. \end{aligned}$$

## 3.4 Case $n = 3$. Summary of isomorphisms

Let $U$ be a domain in the 3-dimensional space $V$. We will consider 5 spaces associated with $U$.

$\Omega^0(U) = C^\infty(U)$—the space of 0-forms, i.e. the space of smooth functions;

$\Omega^k(U)$ for $k = 1, 2, 3$—the spaces of differential $k$-forms on $U$;

$\mathrm{Vect}(U)$—the space of vector fields on $U$.

Let us fix a volume form $w \in \Omega^3(U)$ that is any nowhere vanishing differential 3-form. In coordinates $w$ can be written as

$$w = f(x) dx_1 \wedge dx_2 \wedge dx_3$$

where the function $f : U \to \mathbb{R}$ is never equal to 0. The choice of the form $w$ allows us to define the following isomorphisms.

1. $\Lambda_w : C^\infty(U) \to \Omega^3(U), \quad \Lambda_w(h) = hw$ for any function $h \in C^\infty(U)$.
2. $\lrcorner_w : \mathrm{Vect}(U) \to \Omega^2(U) \quad \lrcorner_w(v) = v \lrcorner w$.

Sometimes we will omit the subscript $w$ an write just $\Lambda$ and $\lrcorner$.

Our third isomorphism depends on a choice of a scalar product $<,>$ in $V$. Let us fix a scalar product. This enables us to define an isomorphism

$$\mathcal{D} = \mathcal{D}_{<,>} : \mathrm{Vect}(U) \to \Omega^1(U)$$

which associates with a vector field $v$ on $U$ a differential 1-form $\mathcal{D}(v) = \langle v, . \rangle$. Let us write down the coordinate expressions for all these isomorphisms. Fix a cartesian coordinate system $(x_1, x_2, x_3)$ in $V$ so that the scalar product $\langle x, y \rangle$ in these coordinates equals $x_1y_1+x_2y_2+x_3y_3$. Suppose also that $w = dx_1 \wedge dx_2 \wedge dx_3$. Then $\Lambda(h) = hdx_1 \wedge dx_2 \wedge dx_3$.

$$\lrcorner(v) = v_1 dx_2 \wedge dx_3 + v_2 dx_3 \wedge dx_1 + v_3 dx_1 \wedge dx_2$$

where $v_1, v_2, v_3$ are coordinate functions of the vector field $v$.

$$\mathcal{D}(v) = v_1 dx_1 + v_2 dx_2 + v_3 dx_3.$$

If $V$ is an *oriented* Euclidean space then one also has isomorphisms

$$\star : \Omega^k(V) \to \Omega^{3-k}(V), \quad k = 0, 1, 2, 3.$$

If $w$ is the volume form on $V$ for which the unit cube has volume 1 and which define the given orientation of $V$ (equivalently, if $w = x_1 \wedge x_2 \wedge x_3$ for any Cartesian positive coordinate system on $V$), then

$$\lrcorner_w(v) = \star\mathcal{D}(v), \quad \text{and} \quad \Lambda_w = \star : \Omega^0(V) \to \Omega^3(V)\,.$$

## 3.5 Gradient, curl and divergence of a vector field

The above isomorphism, combined with the operation of exterior differentiation, allows us to define the following operations on the vector fields. First recall that for a function $f \in C^\infty(U)$,

$$\text{grad} f = \mathcal{D}^{-1}(df).$$

Now let $v \in$ Vect $(U)$ be a vector field. Then its divergence div $v$ is the function defined by the formula

$$\text{div } v = \Lambda^{-1}(d(\lrcorner v))$$

In other words, we take the 2-form $v \lrcorner w$ ($w$ is the volume form) and compute its exterior differential $d(v \lrcorner w)$. The result is a 3-form, and, therefore is proportional to the volume form $w$, i.e. $d(v \lrcorner w) = hw$. This proportionality coefficient (which is a function; it varies from point to point) is simply the divergence: div $v = h$.

Given a vector field $v$, its *curl* is as another vector field curl $v$ defined by the formula

$$\text{curl } v := \lrcorner^{-1} d(\mathcal{D}v) = \mathcal{D}^{-1} * d(\mathcal{D}v).$$

If one fixes a cartesian coordinate system in $V$ such that $w = dx_1 \wedge dx_2 \wedge dx_3$ and $\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3$ then we get the following formulas

$$\begin{aligned}
\text{grad } f &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \\
\text{div } v &= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \\
\text{curl } v &= \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)
\end{aligned}$$

where $v = (v_1, v_2, v_3)$.

We will discuss the geometric meaning of these operations later in Section 5.5.

# 4 Integration of differential forms and functions

## 4.1 One-dimensional Riemann integral for functions and differential 1-forms

A *partition* $\mathcal{P}$ of an interval $[a, b]$ is a finite sequence $a = t_0 < t_1 < \cdots < t_N = b$. We will denote by $T_j, j = 0, \ldots, N$ the vector $t_{j+1} - t_j \in \mathbb{R}_{t_j}$ and by $\Delta_j$ the interval $[t_j, t_{j+1}]$. The length $t_{j+1} - t_j = ||T_j||$ of the interval $\Delta_j$ will be denoted by $\delta_j$. The number $\max\limits_{j=1,\ldots,N} \delta_j$ is called the *fineness* or the *size* of the partition $\mathcal{P}$ and will be denoted by $\delta(\mathcal{P})$.

Let us first recall the definition of (Riemann) integral of a function of one variable. Given a function $f : [a, b] \to \mathbb{R}$ we will form a *lower* and *upper* integral sums corresponding to the partition $\mathcal{P}$:

$$\begin{aligned}
L(f; \mathcal{P}) &= \sum_0^{N-1} (\inf_{[t_j, t_{j+1}]} f)(t_{j+1} - t_j)\,, \\
U(f; \mathcal{P}) &= \sum_0^{N-1} (\sup_{[t_j, t_{j+1}]} f)(t_{j+1} - t_j)\,, \qquad (4.1)
\end{aligned}$$

The function is called *Riemann integrable* if

$$\sup_{\mathcal{P}} L(f;\mathcal{P}) = \inf_{\mathcal{P}} U(f;\mathcal{P}),$$

and in this case this number is called the (Riemann) *integral of the function $f$ over the interval* $[a,b]$. The integrability of $f$ can be equivalently reformulated as follows. Let us choose a set $C = \{c_1,\dots,c_{N-1}\}$, $c_j \in \Delta_j$, and consider an integral sum

$$I(f;\mathcal{P},C) = \sum_{0}^{N-1} f(c_j)(t_{j+1}-t_j),\ c_j \in \Delta_j. \tag{4.2}$$

Then the function $f$ is integrable if there exists a limit $\lim\limits_{\delta(\mathcal{P})\to 0} I(f,\mathcal{P},C)$. In this case this limit is equal to the integral of $f$ over the interval $[a,b]$. Let us emphasize that if we already know that the function is integrable, then to compute the integral one can choose *any* sequence of integral sum, provided that their fineness goes to 0. In particular, sometimes it is convenient to choose $c_j = t_j$, and in this case we will write $I(f;\mathcal{P})$ instead of $I(f;\mathcal{P},C)$.

The integral has different notations. It can be denoted sometimes by $\int\limits_{[a,b]} f$, but the most common notation for this integral is $\int\limits_a^b f(x)dx$. This notation hints that we are integrating here the differential form $f(x)dx$ rather than a function $f$. Indeed, given a differential form $\alpha = f(x)dx$ we have $f(c_j)(t_{j+1}-t_j) = \alpha_{c_j}(T_j)$,[3] and hence

$$I(\alpha;\mathcal{P},C) = I(f;\mathcal{P},C) = \sum_{0}^{N-1} \alpha_{c_j}(T_j),\ c_j \in \Delta_j. \tag{4.3}$$

We say that a differential 1-form $\alpha$ is *integrable* if there exists a limit $\lim\limits_{\delta(\mathcal{P})\to 0} I(\alpha,\mathcal{P},C)$, which is called in this case the *integral of the differential* 1*-form $\alpha$ over the oriented interval* $[a,b]$ and will be denoted by $\int\limits_{\overrightarrow{[a,b]}} \alpha$, or simply $\int\limits_a^b \alpha$. By definition, we say that $\int\limits_{\overleftarrow{[a,b]}} \alpha = -\int\limits_{\overrightarrow{[a,b]}} \alpha$. This agrees with the definition $\int\limits_{\overleftarrow{[a,b]}} \alpha = \lim\limits_{\delta(\mathcal{P})\to 0} \sum\limits_{1}^{N-1} \alpha_{c_j}(-T_j)$, and with the standard calculus rule $\int\limits_b^a f(x)dx = -\int\limits_a^b f(x)dx$.

[3]Here we parallel transported the vector $T_j$ from the point $t_j$ to the point $c_j \in [t_j, t_{j+1}]$.

Let us recall that a map $\phi : [a,b] \to [c,d]$ is called a *diffeomorphism* if it is smooth and has a smooth inverse map $\phi^{-1} : [c,d] \to [a,b]$. This is equivalent to one of the following:

- $\phi(a) = c; \phi(b) = d$ and $\phi' > 0$ everywhere on $[a,b]$. In this case we say that $\phi$ preserves orientation.
- $\phi(a) = d; \phi(b) = c$ and $\phi' < 0$ everywhere on $[a,b]$. In this case we say that $\phi$ reverses orientation.

**Theorem 4.1.1.** *Let $\phi : [a,b] \to [c,d]$ be a diffeomorphism. Then if a 1-form $\alpha = f(x)dx$ is integrable over $[c,d]$ then its pull-back $f^*\alpha$ is integrable over $[a,b]$, and we have*

$$\int\limits_{\overrightarrow{[a,b]}} \phi^*\alpha = \int\limits_{\overrightarrow{[c,d]}} \alpha, \tag{4.4}$$

*if $\phi$ preserves the orientation and*

$$\int\limits_{\overrightarrow{[a,b]}} \phi^*\alpha = \int\limits_{\overleftarrow{[c,d]}} \alpha = -\int\limits_{\overrightarrow{[c,d]}} \alpha,$$

*if $\phi$ reverses the orientation.*

**Remark 4.1.2.** We will show later a stronger result:

$$\int\limits_{\overrightarrow{[a,b]}} \phi^*\alpha = \int\limits_{\overrightarrow{[c,d]}} \alpha$$

for *any* $\phi : [a,b] \to [c,d]$ with $\phi(a) = c, \phi(b) = d$, which is not necessarily a diffeomorphism.

**Proof.** We consider only the orientation preserving case, and leave the orientation reversing one to the reader. Choose any partition $\mathcal{P} = \{a = t_0 < \cdots < t_{N-1} < t_N = b\}$ of the interval $[a,b]$ and choose any set $C = \{c_0, \ldots, c_{N-1}\}$ such that $c_j \in \Delta_j$. Then the points $\widetilde{t}_j = \phi(t_j) \in [c,d]$, $j = 0, \ldots N$ form a partition of $[c,d]$. Denote this partition by $\widetilde{\mathcal{P}}$, and denote $\widetilde{\Delta}_j := [\widetilde{t}_j, \widetilde{t}_{j+1}] \subset [c,d]$, $\widetilde{c}_j = \phi(c_j)$, $\widetilde{C} = \phi(C) = \{\widetilde{c}_0, \ldots, \widetilde{c}_{N-1}\}$. Then we have

$$I(\phi^*\alpha,\mathcal{P},C)=\sum_0^{N-1}\phi^*\alpha_{c_j}(T_j)=\sum_0^{N-1}\alpha_{\widetilde{c}_j}(d\phi(T_j))=$$
$$\sum_0^{N-1}\alpha_{\widetilde{c}_j}(\phi'(c_j)\delta_j). \tag{4.5}$$

Recall that according to the mean value theorem there exists a point $d_j\in\Delta_j$, such that

$$\widetilde{T}_j=\widetilde{t}_{j+1}-\widetilde{t}_j=\phi(t_{j+1})-\phi(t_j)=\phi'(d_j)(t_{j+1}-t_j).$$

Note also that the function $\phi'$ is *uniformly continuous*, i.e. for any $\epsilon>0$ there exists $\delta>0$ such that for any $t,t'\in[a,b]$ such that $|t-t'|<\delta$ we have $|\phi'(t)-\phi'(t')|<\epsilon$. Besides, the function $\phi'$ is bounded above and below by some positive constants: $m<\phi'<M$. Hence $m\delta_j<\widetilde{\delta}_j<M\delta_j$ for all $j=1,\dots,N-1$. Hence, if $\delta(\mathcal{P})<\delta$ then we have

$$\left|I(\phi^*\alpha,\mathcal{P},C)-I(\alpha;\widetilde{\mathcal{P}},\widetilde{C})\right|=\left|\sum_0^{N-1}\alpha_{\widetilde{c}_j}\big((\phi'(c_j)-\phi'(d_j))\delta_j\big)\right|=$$
$$\le\frac{\epsilon}{m}\left|\sum_1^{N-1}f(\widetilde{c}_j)\widetilde{\delta}_j\right|=\frac{\epsilon}{m}\left|I(\alpha;\widetilde{\mathcal{P}},\widetilde{C})\right|. \tag{4.6}$$

When $\delta(\mathcal{P})\to 0$ we have $\widetilde{\delta}(\mathcal{P})=0$, and hence by assumption $I(\alpha;\widetilde{\mathcal{P}},\widetilde{C})\to\int\limits_c^d\alpha$, but this implies that $I(\phi^*\alpha,\mathcal{P},C)-I(\alpha;\widetilde{\mathcal{P}},\widetilde{C})\to 0$, and thus $\phi^*\alpha$ is integrable over $[a,b]$ and

$$\int\limits_a^b\phi^*\alpha=\lim_{\delta(\mathcal{P})\to 0}I(\phi^*\alpha,\mathcal{P},C)=\lim_{\delta(\widetilde{\mathcal{P}})\to 0}I(\alpha,\widetilde{\mathcal{P}},\widetilde{C})=\int\limits_c^d\alpha.$$

■

If we write $\alpha=f(x)dx$, then $\phi^*\alpha=f(\phi(x))\phi'(x)dx$ and the formula (4.1.1) takes a familiar form of the change of variables formula from the 1-variable calculus:

$$\int\limits_c^d f(x)dx=\int\limits_a^b f(\phi(x))\phi'(x)dx.$$

## 4.2 Integration of differential 1-forms along curves

### Curves as paths

A *path*, or *parametrically given curve* in a domain $U$ in a vector space $V$ is a map $\gamma : [a,b] \to U$. We will assume in what follows that all considered paths are differentiable. Given a differential 1-form $\alpha$ in $U$ we define the *integral of* $\alpha$ *over* $\gamma$ by the formula

$$\int_\gamma \alpha = \int_{[a,b]} \gamma^* \alpha.$$

**Example 4.2.1.** Consider the form $\alpha = dz - ydx + xdy$ on $\mathbb{R}^3$. Let $\gamma : [0, 2\pi] \to \mathbb{R}^3$ be a *helix* given by parametric equations $x = R\cos t, y = R\sin t, z = Ct$. Then

$$\int_\gamma \alpha = \int_0^{2\pi} (Cdt + R^2(sin^2 tdt + \cos^2 tdt)) = \int_0^{2\pi} (C + R^2)dt = 2\pi(C + R^2).$$

Note that $\int_\gamma \alpha = 0$ when $C = -R^2$. One can observe that in this case the curve $\gamma$ is tangent to the plane field $\xi$ given by the Pfaffian equation $\alpha = 0$.

**Proposition 4.2.2.** *Let a path $\widetilde{\gamma}$ be obtained from $\gamma : [a,b] \to U$ by a reparameterization, i.e. $\widetilde{\gamma} = \gamma \circ \phi$, where $\phi : [c,d] \to [a,b]$ is an orientation preserving diffeomorphism. Then $\int_{\widetilde{\gamma}} \alpha = \int_\gamma \alpha$.*

Indeed, applying Theorem 4.1.1 we get

$$\int_{\widetilde{\gamma}} \alpha = \int_c^d \widetilde{\gamma}^* \alpha = \int_c^d \phi^*(\gamma^* \alpha) \int_a^b \gamma^* \alpha = \int_\gamma \alpha.$$

A vector $\gamma'(t) \in V_{\gamma(t)}$ is called the *velocity* vector of the path $\gamma$.

### Curves as 1-dimensional submanifolds

A subset $\Gamma \subset U$ is called a *1-dimensional submanifold* of $U$ if for any point $x \in \Gamma$ there is a neighborhood $U_x \subset U$ and a diffeomorphism $\Phi_x : U_x \to \Omega_x \subset \mathbb{R}^n$, such that $\Phi_x(x) = 0 \in \mathbb{R}^n$

and $\Phi_x(\Gamma \cap U_x)$ either coincides with $\{x_2 = \dots x_n = 0\} \cap \Omega_x$, or with $\{x_2 = \dots x_n = 0,\ x_1 \geq 0\} \cap \Omega_x$. In the latter case the point $x$ is called a *boundary* point of $\Gamma$. In the former case it is called an *interior* point of $\Gamma$.

A 1-dimensional submanifold is called *closed* if it is compact and has no boundary. An example of a closed 1-dimensional manifold is the circle $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$.

WARNING. The word *closed* is used here in a different sense than when one speaks about closed *subsets*. For instance, a circle in $\mathbb{R}^2$ is both, a closed subset and a closed 1-dimensional submanifold, while a closed interval is a closed subset but not a closed submanifold: it has 2 boundary points. An open interval in $\mathbb{R}$ (or any $\mathbb{R}^n$) is a submanifold without boundary but it is not closed because it is not compact. A line in a vector space is a 1-dimensional submanifold which is a closed subset of the ambient vector space. However, it is not compact, and hence not a closed submanifold.

**Proposition 4.2.3.** *1. Suppose that a path $\gamma : [a, b] \to U$ is an embedding. This means that $\gamma'(t) \neq 0$ for all $t \in [a, b]$ and $\gamma(t) \neq \gamma(t')$ if $t \neq t'$.*[4] *Then $\Gamma = \gamma([a, b])$ is 1-dimensional compact submanifold with boundary (and possibly with corners).*

2. *Suppose $\Gamma \subset U$ is given by equations $F_1 = 0, \dots, F_{n-1} = 0$ where $F_1, \dots, F_{n-1} : U \to \mathbb{R}$ are smooth functions such that for each point $x \in \Gamma$ the differential $d_x F_1, \dots, d_x F_{n-1}$ are linearly independent. Then $\Gamma$ is a 1-dimensional submanifold of $U$.*

3. *Any compact connected 1-dimensional submanifold $\Gamma \subset U$ can be parameterized either by an embedding $\gamma : [a, b] \to \Gamma \hookrightarrow U$ if it has non-empty boundary, or by an embedding $\gamma : S^1 \to \Gamma \hookrightarrow U$ if it is closed.*

**Proof.** *Will be added.* ■

In the case where $\Gamma$ is closed we will usually parameterize it by a path $\gamma : [a, b] \to \Gamma \subset U$ with $\gamma(a) = \gamma(b)$. For instance, we parameterize the circle $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ by a path $[0, 2\pi] \mapsto (\cos t, \sin t)$. Such $\gamma$, of course, cannot be an embedding, but we will require

[4] If only the former property is satisfied that $\gamma$ is called an *immersion*.

that $\gamma'(t) \neq 0$ and that for $t \neq t'$ we have $\gamma(t) \neq \gamma(t')$ *unless* one of these points is $a$ and the other one is $b$. We will refer to 1-dimensional submanifolds simply as *curves*, respectively closed, with boundary etc.

Given a curve $\Gamma$ its *tangent line* at a point $x \in \Gamma$ is a subspace of $V_x$ generated by the velocity vector $\gamma'(t)$ for any local parameterization $\gamma : [a, b] \to \Gamma$ with $\gamma(t) = x$. If $\Gamma$ is given implicitly, as in 4.2.3.2, then the tangent line is defined in $V_x$ by the system of linear equations $d_x F_1 = 0, \dots, d_x F_{n-1} = 0$.

*Orientation* of a curve $\Gamma$ is the continuously depending on points orientation of all its tangent lines. If the curve is given as a path $\gamma : [a, b] \to \Gamma \subset U$ such that $\gamma'(t) \neq 0$ for all $t \in [a, b]$ than it is canonically oriented. Indeed, the orientation of its tangent line $l_x$ at a point $x = \gamma(t) \in \Gamma$ is defined by the velocity vector $\gamma'(t) \in l_x$.

It turns out that one can define an integral of a differential form $\alpha$ over an oriented compact curve directly without referring to its parameterization. For simplicity we will restrict our discussion to the case when the form $\alpha$ is continuous.

Let $\Gamma$ be a compact connected oriented curve. A *partition* of $\Gamma$ is a sequence of points $\mathcal{P} = \{z_0, z_1, \dots, z_N\}$ ordered according to the orientation of the curve and such that the boundary points of the curve (if they exist) are included into this sequence. If $\Gamma$ is closed we assume that $z_N = z_0$. The *fineness* $\delta(\mathcal{P})$ of $\mathcal{P}$ is by definition is $\max\limits_{j=0,\dots,N-1} \operatorname{dist}(z_j, z_{j+1})$ (we assume here that $V$ a Euclidean space).

**Definition 4.2.4.** *Let $\alpha$ be a differential 1-form and $\Gamma$ a compact connected oriented curve. Let $\mathcal{P} = \{z_0, \dots, z_N\}$ be its partition. Then we define*

$$\int\limits_{\Gamma} \alpha = \lim_{\delta(\mathcal{P}) \to 0} I(\alpha, \mathcal{P}),$$

*where* $I(\alpha, \mathcal{P}) = \sum\limits_{0}^{N-1} \alpha_{z_j}(Z_j), \quad Z_j = z_{j+1} - z_j \in V_{z_j}$.

When $\Gamma$ is a closed submanifold then one sometimes uses the notation $\oint\limits_{\Gamma} \alpha$ instead of $\int\limits_{\Gamma} \alpha$.

**Proposition 4.2.5.** *If one chooses a parameterization $\gamma : [a, b] \to \Gamma$ which respects the given orientation of $\Gamma$ then*

$$\int\limits_{\gamma} \alpha = \int\limits_{a}^{b} \gamma^* \alpha = \int\limits_{\Gamma} \alpha.$$

**Proof.** Indeed, let $\widetilde{\mathcal{P}} = \{t_0, \ldots, t_N\}$ be a partition of $[a, b]$ such that $\gamma(t_j) = z_j,\ j = 0, \ldots, N$.

$$I(\gamma^*\alpha, \widetilde{\mathcal{P}}) = \sum_{1}^{N-1} \gamma^*\alpha_{t_j}(T_j) = \sum_{1}^{N-1} \alpha_{z_j}(U_j),$$

where $U_j = d_{t_j}\gamma(T_j)) \in V_{z_j}$ is a tangent vector to $\Gamma$ at the point $z_j$. Let us evaluate the difference $U_j - Z_j$. Choosing some Cartesian coordinates in $V$ we denote by $\gamma_1, \ldots, \gamma_n$ the coordinate functions of the path $\gamma$. Then using the mean value theorem for each of the coordinate functions we get $\gamma_i(t_{j+1}) - \gamma_i(t_j) = \gamma_i'(c_j^i)\delta_j$ for some $c_j^i \in \Delta_j,\ i = 1, \ldots, n;\ j = 0, \ldots, N-1$. Thus

$$Z_j = \gamma(t_{j+1}) - \gamma(t_j) = (\gamma_1'(c_j^1), \ldots, \gamma_i'(c_j^n))\delta_j.$$

On the other hand, $U_j = d_{t_j}\gamma(T_j)) \in V_{z_j} = \gamma'(t_j)\delta_j$. Hence,

$$||Z_j - U_j|| = \delta_j \sqrt{\sum_{1}^{n} (\gamma_i'(c_j^i) - \gamma_i'(t_j))^2}.$$

Note that if $\delta(\mathcal{P}) \to 0$ then we also have $\delta(\widetilde{\mathcal{P}}) \to 0$, and hence using smoothness of the path $\gamma$ we conclude that for any $\epsilon > 0$ there exists $\delta > 0$ such that $||Z_j - U_j|| < \epsilon\delta_j$ for all $j = 1, \ldots, N$. Thus

$$\sum_{1}^{N-1} \alpha_{z_j}(\widetilde{T}_j) - \sum_{1}^{N-1} \alpha_{z_j}(Z_j) \underset{\delta(\mathcal{P})}{\to} 0,$$

and therefore

$$\int\limits_{a}^{b} \gamma^*\alpha = \lim_{\delta(\widetilde{\mathcal{P}}) \to 0} I(\gamma^*\alpha, \widetilde{\mathcal{P}}) = \lim_{\delta(\mathcal{P}) \to 0} I(\alpha, \mathcal{P}) = \int\limits_{\Gamma} \alpha.$$

■

## 4.3 Integrals of closed and exact differential 1-forms

**Theorem 4.3.1.** *Let $\alpha = df$ be an exact 1-form in a domain $U \subset V$. Then for any path $\gamma : [a,b] \to U$ which connects points $A = \gamma(a)$ and $B = \gamma(b)$ we have*

$$\int_\gamma \alpha = f(B) - f(A).$$

*In particular, if $\gamma$ is a loop then $\oint_\gamma \alpha = 0$.*

Similarly for an oriented curve $\Gamma \subset U$ with boundary $\partial\Gamma = B - A$ we have

$$\int_\Gamma \alpha = f(B) - f(A).$$

**Proof.** We have $\int_\gamma df = \int_a^b \gamma^* df = \int_a^b d(f \circ \gamma) = f(\gamma(b)) - f(\gamma(a)) = f(B) - f(A)$. ■

It turns out that closed forms are *locally exact.* A domain $U \subset V$ is called *star-shaped* with respect to a point $a \in V$ if with any point $x \in U$ it contains the whole interval $I_{a,x}$ connecting $a$ and $x$, i.e. $I_{a,x} = \{a + t(x-a);\ t \in [0,1]\}$. In particular, any convex domain is star-shaped.

**Proposition 4.3.2.** *Let $\alpha$ be a closed 1-form in a star-shaped domain $U \subset V$. Then it is exact.*

**Proof.** Define a function $F : U \to \mathbb{R}$ by the formula

$$F(x) = \int_{\overrightarrow{I_{a,x}}} \alpha, \quad x \in U,$$

where the intervals $I_{a,x}$ are oriented from 0 to $x$.

We claim that $dF = \alpha$. Let us identify $V$ with the $\mathbb{R}^n$ choosing $a$ as the origin $a = 0$. Then $\alpha$ can be written as $\alpha = \sum_1^n P_k(x) dx_k$, and $I_{0,x}$ can be parameterized by

$$t \mapsto tx,\ t \in [0,1].$$

Hence,

$$F(x) = \int\limits_{\overrightarrow{I_{0,x}}} \alpha = \int\limits_0^1 \sum_1^n P_k(tx) x_k dt. \tag{4.7}$$

Differentiating the integral over $x_j$ as parameters, we get

$$\frac{\partial F}{\partial x_j} = \int\limits_0^1 \sum_{k=1}^n t x_k \frac{\partial P_k}{\partial x_j}(tx) dt + \int\limits_0^1 P_j(tx) dt.$$

But $d\alpha = 0$ implies that $\frac{\partial P_k}{\partial x_j} = \frac{\partial P_j}{\partial x_k}$, and using this we can further write

$$\frac{\partial F}{\partial x_j} = \int\limits_0^1 \sum_{k=1}^n t x_k \frac{\partial P_j}{\partial x_k}(tx) dt + \int\limits_0^1 P_j(tx) dt = \int\limits_0^1 \int\limits_0^1 t \frac{dP_j(tx)}{dt} dt + \int\limits_0^1 P_j(tx) dt$$

$$= (tP_j(tx))|_0^1 - \int\limits_0^1 P_j(tx) dt + \int\limits_0^1 P_j(tx) dt = P_j(tx)$$

Thus

$$dF = \sum_{j=1}^n \frac{\partial F}{\partial x_j} dx_j = \sum_{j=1}^n P_j(x) dx = \alpha$$

## 4.4 Integration of functions over domains in high-dimensional spaces

In this section we will discuss integration of bounded functions over bounded sets in a vector space $V$. We will fix a basis $e_1, \ldots, e_n$ and the corresponding coordinate system $x_1, \ldots, x_n$ in the space and thus will identify $V$ with $\mathbb{R}^n$. Let $\eta$ denote the volume form $x_1 \wedge \ldots x_n$. As it will be clear below, the definition of an integral will not depend on the choice of a coordinate system but only on the background volume form, or rather its absolute value because the orientation of $V$ will be irrelevant.

We will need a special class of parallelepipeds in $V$, namely those which are generated by vectors proportional to basic vectors, or in other words, parallelepipeds with edges parallel to the coordinate axes. We will also allow these parallelepipeds to be parallel transported anywhere in the space. Let us denote

$$P(a_1, b_1; a_2, b_2; \dots; a_n, b_n) := \{a_i \le x_i \le b_i;\ i = 1, \dots, n\} \subset \mathbb{R}^n.$$

We will refer to $P(a_1, b_1; a_2, b_2; \dots; a_n, b_n)$ as a *special parallelepiped*, or *rectangle.*

Let us fix one rectangle $P := P(a_1, b_1; a_2, b_2; \dots; a_n, b_n)$. Following the same scheme as we used in the 1-dimensional case, we define a *partition* $\mathcal{P}$ of $P$ as a product of partitions $a_1 = t_0^1 < \dots < t_{N_1}^1 = b_1, \quad \dots, a_n = t_0^n < \dots < t_{N_n}^n = b_n$, of intervals $[a_1, b_1], \dots, [a_n, b_n]$. For simplicity of notation we will always assume that each of the coordinate intervals is partitioned into the same number of intervals, i.e. $N_1 = \dots = N_n = N$. This defines a partition of $P$ into $N^n$ smaller rectangles $P_{\mathbf{j}} = \{t_{j_1}^1 \le x_1 \le t_{j_1+1}^1, \dots, t_{j_n}^n \le x_n \le t_{j_n+1}^n\}$, where $\mathbf{j} = (j_1, \dots, j_n)$ and each index $j_k$ takes values between 0 and $N-1$. Let us define

$$\mathrm{Vol}(P_{\mathbf{j}}) := \prod_{k=1}^{n} (t_{j_k+1}^k - t_{j_k}^k). \tag{4.8}$$

This agrees with the definition of the volume of a parallelepiped which we introduced earlier (see formula (1.4) in Section 1.14). We will also denote $\delta_{\mathbf{j}} := \max\limits_{k=1,\dots,n} (t_{j_k+1}^k - t_{j_k}^k)$ and $\delta(\mathcal{P}) := \max\limits_{\mathbf{j}}(\delta_{\mathbf{j}})$. Let us fix a point $c_{\mathbf{j}} \in P_{\mathbf{j}}$ and denote by $C$ the set of all such $c_{\mathbf{j}}$. Given a function $f : P \to \mathbb{R}$ we form an integral sum

$$I(f; \mathcal{P}, C) = \sum_{\mathbf{j}} f(c_{\mathbf{j}}) \mathrm{Vol}(P_{\mathbf{j}}) \tag{4.9}$$

where the sum is taken over all elements of the partition. If there exists a limit $\lim\limits_{\sigma(\mathcal{P}) \to 0} I(f; \mathcal{P}, C)$ then the function $f : P \to \mathbb{R}$ is called *integrable* (in the sense of Riemann) over $P$, and this limit is called the *integral of $f$ over $P$*. There exist several different notations for this integral: $\int\limits_P f$, $\int\limits_P f dV$, $\int\limits_P f d\mathrm{Vol}$, etc. In the particular case of $n = 2$ one often uses notation $\int\limits_P f dA$, or $\iint\limits_P f dA$. Sometime, the functions we integrate may depend on a parameter, and in these cases it is important to indicate with respect to which variable we integrate. Hence, one also uses the notation like $\int\limits_P f(x, y) dx^n$, where the index $n$ refers to the dimension of the space over which we integrate. One also use the notation $\underbrace{\int \dots \int}_{P} f(x_1, \dots x_n) dx_1 \dots dx_n$, which are reminiscent both of the integral $\int_P f(x_1, \dots x_n) dx_1 \wedge \dots \wedge dx_n$ which will be defined later in Section 4.6 and the notation $\int\limits_{a_n}^{b_n} \dots \int\limits_{a_1}^{b_1} f(x_1, \dots x_n) dx_1 \dots dx_n$ for n interated integral which will be discussed in Section 4.5.

Alternatively and equivalently the integrability can be defined via upper and lower integral sum, similar to the 1-dimensional case. Namely, we define

$$U(f; \mathcal{P}) = \sum_{\mathbf{j}} M_{\mathbf{j}}(f) \mathrm{Vol}(P_{\mathbf{j}}), \quad L(f; \mathcal{P}) = \sum_{\mathbf{j}} m_{\mathbf{j}}(f) \mathrm{Vol}(P_{\mathbf{j}}),$$

where $M_{\mathbf{j}}(f) = \sup\limits_{P_{\mathbf{j}}} f, m_{\mathbf{j}}(f) = \inf\limits_{P_{\mathbf{j}}} f$, and say that the function $f$ is integrable over $P$ if $\inf\limits_{\mathcal{P}} U(f; \mathcal{P}) = \sup\limits_{\mathcal{P}} L(f, \mathcal{P})$.

Note that $\inf\limits_{\mathcal{P}} U(f; \mathcal{P})$ and $\sup\limits_{\mathcal{P}} L(f, \mathcal{P})$ are sometimes called *upper* and *lower* integrals, respectively, and denoted by $\overline{\int\limits_P} f$ and $\underline{\int\limits_P} f$. Thus a function $f : P \to \mathbb{R}$ is integrable iff $\overline{\int\limits_P} f = \underline{\int\limits_P} f$.

Let us list some properties of Riemann integrable functions and integrals.

**Proposition 4.4.1.** *Let $f, g : P \to \mathbb{R}$ be integrable functions. Then*

1. *$af + bg$, where $a, b \in \mathbb{R}$ is integrable and $\int\limits_P af + bg = a \int\limits_P f + b \int\limits_P g$;*
2. *If $f \le g$ then $\int\limits_P f \le \int\limits_P g$;*
3. *$h = \max(f, g)$ is integrable; in particular the functions $f_+ := \max(f, 0)$ and $f_- := \max(-f, 0)$ and $|f| = f_+ + f_-$ are integrable;*
4. *$fg$ is integrable.*

**Proof.** Parts 1 and 2 are straightforward and we leave them to the reader as an exercise. Let us check 3 and 4.

3. Take any partition $\mathcal{P}$ of $P$. Note that

$$M_{\mathbf{j}}(h) - m_{\mathbf{j}}(h) \le \max(M_{\mathbf{j}}(f) - m_{\mathbf{j}}(f), M_{\mathbf{j}}(g) - m_{\mathbf{j}}(g)) . \tag{4.10}$$

Indeed, we have $M_{\mathbf{j}}(h) = \max(M_{\mathbf{j}}(f), M_{\mathbf{j}}(g))$ and $m_{\mathbf{j}}(h) = \max(m_{\mathbf{j}}(f), m_{\mathbf{j}}(g))$. Suppose for the determinacy that $\max(M_{bj}(f), M_{\mathbf{j}}(g)) = M_{\mathbf{j}}(f)$. We also have $m_{\mathbf{j}}(h) \geq m_{\mathbf{j}}(f)$. Thus

$$M_{\mathbf{j}}(h) - m_{\mathbf{j}}(h) \leq M_{\mathbf{j}}(f) - m_{\mathbf{j}}(f) \leq \max(M_{\mathbf{j}}(f) - m_{\mathbf{j}}(f), M_{\mathbf{j}}(g) - m_{\mathbf{j}}(g)).$$

Then using (4.10) we have

$$\begin{aligned} U(h;\mathcal{P}) - L(h;\mathcal{P}) = &\sum_{\mathbf{j}} (M_{\mathbf{j}}(h) - m_{\mathbf{j}}(h)) \mathrm{Vol}(P_{\mathbf{j}}) \leq \\ &\max\left(\sum_{\mathbf{j}} (M_{\mathbf{j}}(f) - m_{\mathbf{j}}(f), M_{\mathbf{j}}(g) - m_{\mathbf{j}}(g)\right) \mathrm{Vol}(P_{\mathbf{j}}) = \\ &\max\left(U(f;\mathcal{P}) - L(f;\mathcal{P}), U(f;\mathcal{P}) - L(f;\mathcal{P})\right). \end{aligned}$$

By assumption the right-hand side can be made arbitrarily small for an appropriate choice of the partition $\mathcal{P}$, and hence $h$ is integrable.

4. We have $f = f_+ - f_-$, $g = g_+ - g_-$ and $fg = f_+g_+ + f_-g_- - f_+g_- - f_-g_+$. Hence, using 1 and 3 we can assume that the functions $f, g$ are non-negative. Let us recall that the functions $f, g$ are by assumption bounded, i.e. there exists a constant $C > 0$ such that $f, g \leq C$. We also have $M_{\mathbf{j}}(fg) \leq M_{\mathbf{j}}(f)M_{\mathbf{j}}(g)$ and $m_{\mathbf{j}}(fg) \geq m_{\mathbf{j}}(f)m_{\mathbf{j}}(g)$. Hence

$$\begin{aligned} &U(fg;\mathcal{P}) - L(fg;\mathcal{P}) = \sum_{\mathbf{j}} \left(M_{\mathbf{j}}(fg) - m_{\mathbf{j}}(fg)\right) \mathrm{Vol}(P_{\mathbf{j}}) \leq \\ &\sum_{\mathbf{j}} (M_{\mathbf{j}}(f)M_{\mathbf{j}}(g) - m_{\mathbf{j}}(f)m_{\mathbf{j}}(g)) \mathrm{Vol}(P_{\mathbf{j}}) = \\ &\sum_{\mathbf{j}} \left(M_{\mathbf{j}}(f)M_{\mathbf{j}}(g) - m_{\mathbf{j}}(f)M_{\mathbf{j}}(g) + m_{\mathbf{j}}(f)M_{\mathbf{j}}(g) - m_{\mathbf{j}}(f)m_{\mathbf{j}}(g)\right) \mathrm{Vol}(P_{\mathbf{j}}) \leq \\ &\sum_{\mathbf{j}} \left((M_{\mathbf{j}}(f) - m_{\mathbf{j}}(f))M_{\mathbf{j}}(g) + m_{\mathbf{j}}(f)(M_{\mathbf{j}}(g) - m_{\mathbf{j}}(g))\right) \mathrm{Vol}(P_{\mathbf{j}}) \leq \\ &C(U(f;\mathcal{P}) - L(f;\mathcal{P}) + U(g;\mathcal{P}) - L(g;\mathcal{P})). \end{aligned}$$

By assumption the right-hand side can be made arbitrarily small for an appropriate choice of the partition $\mathcal{P}$, and hence $fg$ is integrable. ■

Consider now a bounded subset $K \subset \mathbb{R}^n$ and choose a rectangle $P \supset K$. Given any function $f : K \to \mathbb{R}$ one can always extend it to $P$ as equal to 0. A function $f : K \to \mathbb{R}$ is called *integrable over* $K$ if this trivial extension $\overline{f}$ is integrable over $P$, and we define $\int\limits_K f dV := \int \overline{f} dV$. When this will not be confusing we will usually keep the notation $f$ for the above extension.

We further define the volume

$$\mathrm{Vol}(K) = \int\limits_K 1 dV = \int\limits_P \chi_K dV,$$

provided that this integral exists. In this case we call the set $K$ *measurable in the sense of Riemann*, or just measurable.[5] Here $\chi_K$ is the *characteristic* or *indicator* function of $K$, i.e. the function which is equal to 1 on $K$ and 0 elsewhere. In the 2-dimensional case the volume is called the area, and in the 1-dimensional case the length.

**Remark 4.4.2.** For any bounded set $A$ there is defined a *lower* and *upper* volumes,

$$\underline{\mathrm{Vol}}(A) = \underline{\int} \chi_A dV \leq \overline{\mathrm{Vol}}(A) = \overline{\int} \chi_A dV.$$

The set is measurable iff $\underline{\mathrm{Vol}}(A) = \overline{\mathrm{Vol}}(A)$. If $\overline{\mathrm{Vol}}(A) = 0$ then $\underline{\mathrm{Vol}}(A) = 0$, and hence $A$ is measurable and $\mathrm{Vol}(A) = 0$.

**Exercise 4.4.3.** *Prove that for the rectangles this definition of the volume coincides with the one given by the formula* (4.8).

The next proposition lists some properties of the volume.

**Proposition 4.4.4.** 1. *Volume is monotone, i.e. if $A, B \subset P$ are measurable and $A \subset B$ then* $\mathrm{Vol}(A) \leq \mathrm{Vol}(B)$.

2. *If sets $A, B \subset P$ are measurable then $A \cap B$, $A \setminus B$ and $A \cup B$ are measurable as well and we have*

$$\mathrm{Vol}(A \cup B) = \mathrm{Vol}(A) + \mathrm{Vol}(B) - \mathrm{Vol}(A \cap B).$$

[5] There exists a more general and more common notion of measurability in the sense of Lebesgue. Any Riemann measurable set is also measurable in the sense of Lebesgue, but not the other way around.

3. *If $A$ can be covered by a measurable set of arbitrarily small total volume then* $\mathrm{Vol}(A) = 0$. *Conversely, if* $\mathrm{Vol}(A) = 0$ *then for any* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for any partition* $\mathcal{P}$ *with* $\delta(\mathcal{P}) < \delta$ *the elements of the partition which intersect $A$ have arbitrarily small total volume.*

4. *$A$ is measurable iff* $\mathrm{Vol}\,(\partial A) = 0$.

**Proof.** The first statement is obvious. To prove the second one, we observe that $\chi_{A\cup B} = \max(\chi_A, \chi_B)$, $\chi_{A\cap B} = \chi_A\chi_B$, $\max(\chi_A, \chi_B) = \chi_A + \chi_B - \chi_A\chi_B$, $\xi_{A\setminus B} = \xi_A - xi_{A\cap B}$ and then apply Proposition 4.4.1. To prove 4.4.4.3 we first observe that if a set $B$ is measurable and $\mathrm{Vol}B < \epsilon$ then then for a sufficiently fine partition $\mathcal{P}$ we have $U(\chi_B;\mathcal{P}) < \mathrm{Vol}B+\epsilon < 2\epsilon$. Since $A \subset B$ then $\xi_A \leq \chi_B$, and therefore $U(\chi_A, \mathcal{P}) \leq U(\chi_B, \mathcal{P}) < 2\epsilon$. Thus, $\inf_{\mathcal{P}} U(\chi_A, \mathcal{P}) = 0$ and therefore $A$ is measurable and $\mathrm{Vol}(A) = 0$. Conversely, if $\mathrm{Vol}(A) = 0$ then for any $\epsilon > 0$ for a sufficiently fine partition $\mathcal{P}$ we have $U(\chi_A;\mathcal{P}) < \epsilon$. But $U(\chi_A;\mathcal{P})$ is equal to the sum of volumes of elements of the partition which have non-empty intersection with $A$.

Finally, let us prove 4.4.4.4. Consider any partition $\mathcal{P}$ of $P$ and form lower and upper integral sums for $\chi_A$. Denote $M_{\mathbf{j}} := M_{\mathbf{j}}(\chi_A)$ and $m_{\mathbf{j}} = m_{\mathbf{j}}(\chi_A)$. Then all numbers $M_{\mathbf{j}}, m_{\mathbf{j}}$ are equal to either 0 or 1. We have $M_{\mathbf{j}} = m_{\mathbf{j}} = 1$ if $P_{\mathbf{j}} \subset A$; $M_{\mathbf{j}} = m_{\mathbf{j}} = 0$ if $P_{\mathbf{j}} \cap A = \varnothing$ and $M_{\mathbf{j}} = 1, m_{\mathbf{j}} = 0$ if $P_{\mathbf{j}}$ has non-empty intersection with both $A$ and $P \setminus A$. In particular,

$$B(\mathcal{P}) := \bigcup_{\mathbf{j}; M_{\mathbf{j}} - m_{\mathbf{j}} = 1} P_{\mathbf{j}} \supset \partial A.$$

Hence, we have

$$U(\chi_A;\mathcal{P}) - L(\chi_A;\mathcal{P}) = \sum_{\mathbf{j}} (M_{\mathbf{j}} - m_{\mathbf{j}})\mathrm{Vol}(P_{\mathbf{j}}) = \mathrm{Vol}B(\mathcal{P}).$$

Suppose that $A$ is measurable. Then there exists a partition such that $U(\chi_A;\mathcal{P}) - L(\chi_A;\mathcal{P}) < \epsilon$, and hence $\partial A$ is can be covered by the set $B(\mathcal{P})$ of volume $< \epsilon$. Thus applying part 3 we conclude that $\mathrm{Vol}(\partial A) = 0$. Conversely, we had seen below that if $\mathrm{Vol}(\partial A) = 0$ then there exists a partition such that the total volume of the elements intersecting $\partial A$ is $< \epsilon$. Hence, for this partition we have $L(\chi_A;\mathcal{P}) \leq U(\chi_A;\mathcal{P}) < \epsilon$, which implies the integrability of $\chi_A$, and hence measurability of $A$. ■

**Corollary 4.4.5.** *If a bounded set $A \subset V$ is measurable then its interior* $\mathrm{Int}\, A$ *and its closure* $\overline{A}$ *are also measurable and we have in this case*

$$\mathrm{Vol} A = \mathrm{Vol}\,\mathrm{Int}\, A = \mathrm{Vol}\overline{A}.$$

**Proof.** 1. We have $\partial\overline{A} \subset \partial A$ and $\partial(\mathrm{Int}\, A) \subset \partial A$. Therefore, $\mathrm{Vol}\,\partial\overline{A} = \mathrm{Vol}\,\partial\,\mathrm{Int}\, A = 0$, and therefore the sets $\overline{A}$ and $\mathrm{Int}\, A$ are measurable. Also $\mathrm{Int}\, A \cup \partial A = \overline{A}$ and $\mathrm{Int}\, A \cap \partial A = \varnothing$. Hence, the additivity of the volume implies that $\mathrm{Vol}\overline{A} = \mathrm{Vol}\,\partial\,\mathrm{Int}\, A + \mathrm{Vol}\,\partial A = \mathrm{Vol}\,\partial\,\mathrm{Int}\, A$. On the other hand, $\mathrm{Int}\, A \subset A \subset \overline{A}$. and hence the monotonicity of the volume implies that $\mathrm{Vol\ Int}\, A \leq \mathrm{Vol}\, A \leq \mathrm{Vol}\,\overline{A}$. Hence, $\mathrm{Vol}\, A = \mathrm{Vol\ Int}\, A = \mathrm{Vol}\,\overline{A}$. ■

**Exercise 4.4.6.** If $\mathrm{Int}\, A$ or $\overline{A}$ are measurable then this does not imply that $A$ is measurable. For instance, if $A$ is the set of rational points in interval $I = [0,1] \subset \mathbb{R}$ then $\mathrm{Int}\, A = \varnothing$ and $\overline{A} = I$. However, show that $A$ is not Riemann measurable.
2. A set $A$ is called *nowhere dense* if $\mathrm{Int}\, A = \varnothing$. Prove that if $A$ is nowhere dense than either $\mathrm{Vol} A = 0$, or $A$ is not measurable in the sense of Riemann. Find an example of a non-measurable nowhere dense set.

We will further need the following lemma. Let us recall that given a compact set $C \subset V$, we say that a map $f : C \to W$ is smooth if it extends to a smooth map defined on an open neighborhood $U \supset C$. Here $V, W$ are vector spaces.

**Lemma 4.4.7.** *Let $A \subset V$ be a compact set of volume* $0$ *and $f : V \to W$ a $C^1$-smooth map, where* $\dim W \geq \dim V$. *Then* $\mathrm{Vol} f(A) = 0$.

**Proof.** The $C^1$-smoothness of $f$ and compactness of $A$ imply that there exists a constant $K$ such that $||d_x f(h)|| \leq K||h||$ for any $x \in A$ and $h \in V_x$. In particular, the image $d_x f(P)$ of every rectangle $P$ of size $\delta$ in $V_x$, $x \in A$, is contained in a cube of size $K\delta$ in $W_{f(x)}$.

$C^1$-smoothness of $f$ also implies that for any $\epsilon > 0$ there exists such $\delta > 0$ that if $x \in A$ and $||h|| \leq \delta$ then $||f(x+h) - f(x) - d_x f(h)|| \leq \epsilon ||h||$. This implies that if we view $P$ as a subset of $V$, rather than $V_x$, then the image $f(P)$ is contained in a cube in $W$ of size $2K\delta$ if $\delta$ is small enough, and if $P \cap A \neq \varnothing$. Let us denote dimensions of $V$ and $W$ by $n$ and $m$, respectively. By assumption $A$ can be covered by $N$ cubes of size $\delta$ such that the total volume of these cubes is equal to $N\delta^n \leq \epsilon$. Hence, $f(A)$ can be covered by $N$ cubes of size $2K\delta$ of total volume

$$N(2K\delta)^m = N\delta^n (2K)^m \delta^{m-n} = \epsilon (2K)^m \delta^{m-n} \underset{\epsilon \to 0}{\to} 0,$$

because $m \geq n$. ■

**Corollary 4.4.8.** *Let $A \subset V$ be a compact domain and $f : A \to W$ a $C^1$-smooth map. Suppose that $n = \dim V < m = \dim W$. Then* $\mathrm{Vol}(f(A)) = 0$.

Indeed, $f$ can be extended to a smooth map defined on a neighborhood of $A \times 0$ in $V \times \mathbb{R}$ (e.g. as independent of the new coordinate $t \in \mathbb{R}$). But $\mathrm{Vol}_{n+1}(A \times 0) = 0$ and $m \geq n+1$. Hence, the required statement follows from Lemma 4.4.7.

**Remark 4.4.9.** The statement of Corollary 4.4.8 is wrong for continuous maps. For instance, there exists a continuous map $h : [0,1] \to \mathbb{R}^2$ such that $h([0,1])$ is the square $\{0 \leq x_1, x_1 \leq 1\}$. (This is a famous Peano curve passing through every point of the square.)

Corollary 4.4.8 is a simplest special case of *Sard's theorem* which asserts that the set of critical values of a sufficiently smooth map has volume 0. More precisely,

**Proposition 4.4.10.** (A. SARD, 1942) *Given a $C^k$-smooth map $f : A \to W$ (where $A$ is a compact subset of $V$, $\dim V = n, \dim W = m$) let us denote by*

$$\Sigma(f) : \{x \in C;\ \mathrm{rank}\, d_x f < m\}.$$

*Then if $k \geq \max(n - m + 1, 1)$ then* $\mathrm{Vol}_m(f(\Sigma(f)) = 0$.

If $m > n$ then $\Sigma(f) = A$, and hence the statement is equivalent to Corollary 4.4.8.

**Proof.** We prove the proposition only for the case $m = n$. The proof in this case is similar to the proof of Corollary 4.4.7. $C^1$-smoothness of $f$ and compactness of $\Sigma(f)$ imply that there exists a constant $K$ such that

$$||d_x f(h)|| \leq K||h|| \tag{4.11}$$

for any $x \in A$ and $h \in V_x$. $C^1$-smoothness of $f$ also implies that for any $\epsilon > 0$ there exists such $\delta > 0$ that if $x \in A$ and $||h|| \leq \delta$ then

$$||f(x+h) - f(x) - d_x f(h)|| \leq \epsilon||h||. \tag{4.12}$$

Take a partition $\mathcal{P}$ of a rectangle $P \supset A$ by $N^n$ smaller rectangles of equal size. Let $B$ be the union of the rectangles intersecting $\Sigma(f)$. For any such rectangle $P_{\mathbf{j}} \subset B$ choose a point $c_{\mathbf{j}} \in \Sigma(f)$. Viewing $P_{\mathbf{j}}$ as a subset of $V_{c_{\mathbf{j}}}$ we can take its image $\widetilde{P}_{\mathbf{j}} = d_{c_{\mathbf{j}}}(P_{bj}) \subset W_{f(c_{\mathbf{j}})}$. Then the parallelepiped $\widetilde{P}_{\mathbf{j}}$ is contained in a subspace $L \subset W_{f(c_{\mathbf{j}})}$ of dimension $r = \text{rank}(d_{c_{\mathbf{j}}} f) < m$. In view of (4.4) it also contained in a ball of radius $\frac{K\sqrt{r}}{N}$ centered at the point $f(c_{\mathbf{j}})$. On the other hand, the inequality (4.12) implies that if $N$ is large enough then for any point $u \in f(P_{\mathbf{j}})$ there is a point in $\widetilde{u} \in \widetilde{P}_{\mathbf{j}}$ such $||u - \widetilde{u}|| \leq \frac{\epsilon}{N}$. This means that $f(P_{\mathbf{j}})$ is contained in a parallelepiped centered at $c_{\mathbf{j}}$ and generated by $r$ orthogonal vectors of length $\frac{C_1}{N}$ parallel to the subspace $L$ and $n - r > 0$ vectors of length $\frac{C_2\epsilon}{N}$ which are orthogonal to $L$, where $C_1, C_2$ are some positive constants. The volume of this parallelepiped is equal to

$$\frac{C_1^r C_2^{n-r} \epsilon^{n-r}}{N^n} = \frac{C_3 \epsilon^{n-r}}{N^n},$$

and hence $\text{Vol}(f(P_{\mathbf{j}}) < \frac{C_3\epsilon^{n-r}}{N^n}$. The set $B$ contains no more that $N^n$ cubes $P_{\mathbf{j}}$, and hence

$$\text{Vol}(f(\Sigma(f)) \leq \text{Vol} f(C) \leq N^n \frac{C_3 \epsilon^{n-r}}{N^n} = C_3 \epsilon^{n-r} \underset{\epsilon \to 0}{\to} 0.$$

■

We say that some property holds *almost everywhere* (we will abbreviate a.e.) if it holds in the complement of a set of volume 0. For instance, we say that a bounded function $f : P \to \mathbb{R}$

is almost everywhere continuous (or a.e. continuous) if it is continuous in the complement of a set $A \subset P$ of volume 0. For instance, a *characteristic function of any measurable set is a.e. continuous.* Indeed, it is constant away from the set $\partial A$ which according to Proposition 4.4.4.4 has volume 0.

**Proposition 4.4.11.** *Suppose that the bounded functions $f, g : P \to \mathbb{R}$ coincide a.e. Then if $f$ is integrable, then so is $g$ and we have $\int\limits_P f = \int\limits_P g$.*

**Proof.** Denote $A = \{x \in P : f(x) \neq g(x)\}$. By our assumption, $\mathrm{Vol} A = 0$. Hence, for any $\epsilon$ there exists a $\delta > 0$ such that for every partition $\mathcal{P}$ with $\delta(\mathcal{P}) \leq \delta$ the union $B_\delta$ of all rectangles of the partition which have non-empty intersection with $A$ has volume $<$ *eps*. The functions $f, g$ are bounded, i.e. there exists $C > 0$ such $-C \leq |f(x)|, |g(x)| \leq C$ for all $x \in P$. Due to integrability of $f$ we can choose $\delta$ small enough so that $|U(f, \mathcal{P}) - L(f, \mathcal{P})| \leq \epsilon$ when $\delta(\mathcal{P}) \leq \delta$. Then we have

$$|U(g, \mathcal{P}) - U(f, \mathcal{P})| = \left| \sum_{J:\, P_J \subset B_\delta} \sup_{P_J} g - \sup_{P_J} f) \right| \leq 2C \mathrm{Vol} B_\delta \leq 2C\epsilon.$$

Similarly, $|L(g, \mathcal{P}) - L(f, \mathcal{P})| \leq 2C\epsilon$, and hence

$$\begin{aligned} |U(g, \mathcal{P}) - L(g, \mathcal{P})| &\leq |U(g, \mathcal{P}) - U(f, \mathcal{P})| + |U(f, \mathcal{P}) - L(f, \mathcal{P})| + |L(f, \mathcal{P}) - L(g, \mathcal{P})| \\ &\leq \epsilon + 4C\epsilon \underset{\delta \to 0}{\longrightarrow} 0, \end{aligned}$$

and hence $g$ is integrable and

$$\int\limits_P g = \lim_{\delta(\mathcal{P}) \to 0} U(g, \mathcal{P}) = \lim_{\delta(\mathcal{P}) \to 0} U(f, \mathcal{P}) = \int\limits_P f.$$

■

**Proposition 4.4.12.** *1. Suppose that a function $f : P \to \mathbb{R}$ is a.e. continuous. Then $f$ is integrable.*

*2. Let $A \subset V$ be compact and measurable, $f : U \to W$ a $C^1$-smooth map defined on a neighborhood $U \supset A$. Suppose that $\dim W = \dim V$. Then $f(A)$ is measurable.*

**Proof.** 1. Let us begin with a

WARNING. One could think that in view of Proposition 4.4.11 it is sufficient to consider only the case when the function $f$ is continuous. However, this is not the case, because for a given a.e. continuos function one cannot, in general, find a continuos function $g$ which coincides with $f$ a.e.

Let us proceed with the proof. Given a partition $\mathcal{P}$ we denote by $J_A$ the set of multi-indices $\mathbf{j}$ such that $\operatorname{Int} P_{\mathbf{j}} \cap A \neq \varnothing$, and by $\overline{J}_A$ the complementary set of multi-indices, i.e. for each $\mathbf{j} \in \overline{J}_A$ we have $P_{\mathbf{j}} \cap A = \varnothing$. Let us denote $C := \bigcup\limits_{\mathbf{j}\in J_A} P_{\mathbf{j}}$. According to Proposition 4.4.4.3 for any $\epsilon > 0$ there exists a partition $\mathcal{P}$ such that $\mathrm{Vol}(C) = \sum\limits_{\mathbf{j}\in J_A} \mathrm{Vol}(P_{\mathbf{j}}) < \epsilon$. By assumption the function $f$ is continuous over a compact set $B = \bigcup\limits_{\mathbf{j}\in \overline{J}_A} P_{\mathbf{j}}$, and hence it is uniformly continuous over it. Thus there exists $\delta > 0$ such that $|f(x) - f(x')| < \epsilon$ provided that $x, x' \in B$ and $||x - x'|| < \delta$. Thus we can further subdivide our partition, so that for the new finer partition $\mathcal{P}'$ we have $\delta(\mathcal{P}') < \delta$. By assumption the function $f$ is bounded, i.e. there exists a constant $K > 0$ such that $M_{\mathbf{j}}(f) - m_{\mathbf{j}}(f) < K$ for all indices $\mathbf{j}$. Then we have

$$
\begin{aligned}
U(f;\mathcal{P}') - L(f,\mathcal{P}') = & \sum_{\mathbf{j}} (M_{\mathbf{j}}(f) - m_{\mathbf{j}}(f))\mathrm{Vol}(P_{\mathbf{j}}) = \\
& \sum_{\mathbf{j};P_{\mathbf{j}}\subset B} (M_{\mathbf{j}}(f) - m_{\mathbf{j}}(f))\mathrm{Vol}(P_{\mathbf{j}}) + \sum_{\mathbf{j};P_{\mathbf{j}}\subset C} (M_{\mathbf{j}}(f) - m_{\mathbf{j}}(f))\mathrm{Vol}(P_{\mathbf{j}}) < \\
& \epsilon \mathrm{Vol} B + K \mathrm{Vol} C < \epsilon(\mathrm{Vol} P + K).
\end{aligned}
$$

Hence $\inf\limits_{\mathcal{P}} U(f;\mathcal{P}) = \sup\limits_{\mathcal{P}} L(f;\mathcal{P})$, i.e. the function $f$ is integrable.

2. If $x$ is an interior point of $A$ and $detDf(x) \neq 0$ then the inverse function theorem implies that $f(x) \in \operatorname{Int} f(A)$. Denote $C = \{x \in A; \det Df(x) = 0\}$. Hence, $\partial f(A) \subset f(\partial A) \cup f(C)$. But $\mathrm{Vol}(\partial A) = 0$ because $A$ is measurable and $\mathrm{Vol}\, f(C) = 0$ by Sard's theorem 4.4.10. Therefore, $\mathrm{Vol}\, \partial f(A) = 0$ and thus $f(A)$ is measurable. ■

The following lemma provides a way of computing the volume via packing by balls rather then cubes. An *admissible set of balls in* $A$ is any finite set of disjoint balls $B_1, \dots, B_K \subset A$

**Lemma 4.4.13.** *Let $A$ be a measurable set. Then* Vol$A$ *is the supremum of the total volume of admissible sets of balls in $A$. Here the supremum is taken over all admissible sets of balls in $A$.*

**Proof.** Let us denote this supremum by $\beta$. The monotonicity of volume implies that $\beta \leq \mathrm{Vol}A$. Suppose that $\beta < VolA$. Let us denote by $\mu_n$ the volume of an $n$-dimensional ball of radius 1 (we will compute this number later on). This ball is contained in a cube of volume $2^n$. It follows then that the ratio of the volume of any ball to the volume of the cube to which it is inscribed is equal to $\frac{\mu_n}{2^n}$. Choose an $\epsilon < \frac{\mu_n}{2^n}(\mathrm{Vol}A - \beta)$. Then there exists a finite set of disjoint balls $B_1, \dots, B_K \subset A$ such that $\mathrm{Vol}(\bigcup\limits_1^K B_j) > \beta - \epsilon$. The volume of the complement $C = A \setminus \bigcup\limits_1^K B_j$ satisfies

$$\mathrm{Vol}C = \mathrm{Vol}A - \mathrm{Vol}\left(\bigcup_1^K B_j\right) > \mathrm{Vol}A - \beta.$$

Hence there exists a partition $\mathcal{P}$ of $P$ by cubes such that the total volume of cubes $Q_1, \dots, Q_L$ contained in $C$ is $> \mathrm{Vol}A - \beta$. Let us inscribe in each of the cubes $Q_j$ a ball $\widetilde{B}_j$. Then $B_1, \dots, B_K, \widetilde{B}_1, \dots, \widetilde{B}_L$ is an admissible set of balls in $A$. Indeed, all these balls are disjoint and contained in $A$. The total volume of this admissible set is equal to

$$\sum_1^K \mathrm{Vol}B_j + \sum_1^L \mathrm{Vol}\widetilde{B}_i \geq \beta - \epsilon + \frac{\mu_n}{2^n}(\mathrm{Vol}A - \beta) > \beta,$$

in view of our choice of $\epsilon$, but this contradicts to our assumption $\beta < \mathrm{Vol}A$. Hence, we have $\beta = \mathrm{Vol}A$. ■

**Lemma 4.4.14.** *Let $A \subset V$ be any measurable set in a Euclidean space $V$. Then for any linear orthogonal transformation $F : V \to V$ the set $F(A)$ is also measurable and we have* $\mathrm{Vol}(F(A)) = \mathrm{Vol}(A)$.

**Proof.** First note that if $\mathrm{Vol}A = 0$ then the claim follows from Lemma 4.4.7. Indeed, an orthogonal transformation is, of course a smooth map.

Let now $A$ be an arbitrary measurable set. Note that $\partial F(A) = F(\partial A)$. Measurability of $A$ implies $\mathrm{Vol}(\partial A) = 0$. Hence, as we just have explained, $\mathrm{Vol}(\partial F(A)) = \mathrm{Vol}(F(\partial A)) = 0$, and hence $F(A)$ is measurable. According to Lemma 4.4.13 the volume of a measurable set can be computed as a supremum of the total volume of disjoint inscribed balls. But the orthogonal transformation $F$ moves disjoint balls to disjoint balls of the same size, and hence $\mathrm{Vol}A = \mathrm{Vol}F(A)$. ■

Next proposition shows that the volume of a parallelepiped can be computed by formula (1.4) from Section 1.14.

**Proposition 4.4.15.** *Let $v_1, \dots, v_n \in V$ be linearly independent vectors. Then*

$$\mathrm{Vol}\, P(v_1, \dots, v_n) = |x_1 \wedge \dots \wedge x_n(v_1 \dots, v_n)|. \tag{4.13}$$

**Proof.** The formula (4.13) holds for rectangles, i.e. when vectors $v_j = c_j e_j$ for some non-zero numbers $c_j$, $j = 1, \dots n$. Using Lemma 4.4.14 we conclude that it also holds for any orthogonal basis. Indeed, any such basis can be moved by an orthogonal transformation to a basis of the above form $c_j e_j$, $j = 1, \dots n$. Lemma 4.4.14 ensures that the volume does not change under the orthogonal transformation, while Proposition 1.10.2 implies the same about $|x_1 \wedge \dots \wedge x_n(v_1 \dots, v_n)|$.

The Gram-Schmidt orthogonalization process shows that one can pass from any basis to an orthogonal basis by a sequence of following elementary operations: reordering of basic vectors, and *shears*, i.e. an addition to the last vector a linear combination of the other ones:

$$v_1, \dots, v_{n-1}, v_n \mapsto v_1, \dots, v_{n-1}, v_n + \sum_1^{n-1} \lambda_j v_j.$$

Note that the reordering of vectors $v_1, \dots, v_n$ changes neither $\mathrm{Vol}P(v_1, \dots, v_n)$, nor the absolute value $|x_1 \wedge \dots \wedge x_n(v_1 \dots, v_n)|$. On the other hand, a shear does not change

$$x_1 \wedge \dots \wedge x_n(v_1 \dots, v_n).$$

It remains to be shown that a shear does not change the volume of a parallelepiped. We will consider here only the case $n = 2$ and will leave to the reader the extension of the argument to the general case.

Let $v_1, v_2$ be two orthogonal vectors in $\mathbb{R}^2$. Because we already prove the invariance of volume under orthogonal transformations, we can assume that $v_1 = (a, 0)$, $v_2 = (0, b)$ for $a, b > 0$. Let $v_2' = v_2 + \lambda v_1 = (a', b)$, where $a' = a + \lambda b$. Let us partition the rectangle $P = P(v_1, v_2)$ into $N^2$ smaller rectangles $P_{i,j}$, $i, j = 0, \dots, N-1$, of equal size. We number the rectangles in such a way that the first index corresponds to the first coordinate, so that the rectangles $P_{00}, \dots, P_{N-1,0}$ form the lower layer, $P_{01}, \dots, P_{N-1,1}$ the second layer, etc. Let us now shift the rectangles in $k$-th layer horizontally by the vector $(\frac{k\lambda b}{N}, 0)$. Then the total volume of the rectangles, denoted $\widetilde{P}_{ij}$ remains the same, while when $N \to \infty$ the volume of part of the parallelogram $P(v_1, v_2')$ that *is not covered* by rectangles $\widetilde{P}_{i,j}$, $i, j = 0, \dots, N-1$ converges to 0.

■

## 4.5 Fubini's Theorem

Let us consider $\mathbb{R}^n$ as a direct product of $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ for some $k = 1, \dots, n-1$. We will denote coordinates in $\mathbb{R}^k$ by $x = (x_1, \dots, x_k)$ and coordinates in $\mathbb{R}^{n-k}$ by $y = (y_1, \dots, y_{n-k})$, so the coordinates in $\mathbb{R}^n$ are denoted by $(x_1, \dots, x_k, y_1, \dots, y_{n-k})$. Given rectangles $P_1 \subset \mathbb{R}^k$ and $P_2 \subset \mathbb{R}^{n-k}$ their product $P = P_1 \times P_2$ is a rectangle in $\mathbb{R}^n$.

The following theorem provides us with a basic tool for computing multiple integrals.

**Theorem 4.5.1.** *Suppose that a function $f : P \to \mathbb{R}$ is integrable over $P$. Given a point $x \in P_1$ let us define a function $f_x : P_2 \to \mathbb{R}$ by the formula $f_x(y) = f(x, y)$, $y \in P_2$. Then*

$$\int\limits_P f dV_n = \int\limits_{P_1} \left( \underline{\int\limits_{P_2}} f_x dV_{n-k} \right) dV_k = \int\limits_{P_1} \left( \overline{\int\limits_{P_2}} f_x dV_{n-k} \right) dV_k.$$

*In particular, if the function $f_x$ is integrable for all (or almost all) $x \in P_1$ then one has*

$$\int\limits_P f dV_n = \int\limits_{P_1} \left( \int\limits_{P_2} f_x dV_{n-k} \right) dV_k.$$

Here by writing $dV_k, dV_{n-k}$ and $dV_n$ we emphasize the integration with respect to the $k$-, $(n-k)$- and $n$-dimensional volumes, respectively.

**Proof.** Choose any partition $\mathcal{P}_1$ of $P_1$ and $\mathcal{P}_2$ of $P_2$. We will denote elements of the partition $\mathcal{P}_1$ by $P_1^{\mathbf{j}}$ and elements of the partition $\mathcal{P}_2$ by $P_2^{\mathbf{i}}$. Then products of $P^{\mathbf{j},\mathbf{i}} = P_1^{\mathbf{j}} \times P_2^{\mathbf{i}}$ form a partition $\mathcal{P}$ of $P = P_1 \times P_2$. Let us denote

$$\overline{I}(x) := \overline{\int\limits_{P_2}} f_x,\ \underline{I}(x) := \underline{\int\limits_{P_2}} f_x,\ x \in P_1.$$

Let us show that

$$L(f,\mathcal{P}) \le L(\underline{I},\mathcal{P}_1) \le U(\overline{I},\mathcal{P}_1) \le U(f,\mathcal{P}). \tag{4.14}$$

Indeed, we have

$$L(f,\mathcal{P}) = \sum_{\mathbf{j}}\sum_{\mathbf{i}} m_{\mathbf{j},\mathbf{i}}(f)\mathrm{Vol}_n P^{\mathbf{j},\mathbf{i}}.$$

Here the first sum is taken over all multi-indices $\mathbf{j}$ of the partition $\mathcal{P}_1$, and the second sum is taken over all multi-indices $\mathbf{i}$ of the partition $\mathcal{P}_2$. On the other hand,

$$L(\underline{I},\mathcal{P}_1) = \sum_{\mathbf{j}} \inf_{x\in P_1^{\mathbf{j}}} \left( \underline{\int\limits_{P_2}} f_x dV_{n-k} \right) \mathrm{Vol}_k P_1^{\mathbf{j}}.$$

Note that for every $x \in P_1^{\mathbf{j}}$ we have

$$\underline{\int\limits_{P_2}} f_x dV_{n-k} \ge L(f_x;\mathcal{P}_2) = \sum_{\mathbf{i}} m_{\mathbf{i}}(f_x)\mathrm{Vol}_{n-k}(P_2^{\mathbf{i}}) \ge \sum_{\mathbf{i}} m_{\mathbf{i},\mathbf{j}}(f)\mathrm{Vol}_{n-k}(P_2^{\mathbf{i}}),$$

and hence

$$\inf_{x\in P_1^{\mathbf{j}}} \underline{\int\limits_{P_2}} f_x dV_{n-k} \ge \sum_{\mathbf{i}} m_{\mathbf{i},\mathbf{j}}(f)\mathrm{Vol}_{n-k}(P_2^{\mathbf{i}}).$$

Therefore,

$$L(\underline{I},\mathcal{P}_1) \ge \sum_{\mathbf{j}}\sum_{\mathbf{i}} m_{\mathbf{i},\mathbf{j}}(f)\mathrm{Vol}_{n-k}(P_2^{\mathbf{i}})\mathrm{Vol}_k(P_1^{\mathbf{j}}) = \sum_{\mathbf{j}}\sum_{\mathbf{i}} m_{\mathbf{j},\mathbf{i}}(f)\mathrm{Vol}_n(P^{\mathbf{j},\mathbf{i}}) = L(f,\mathcal{P}).$$

Similarly, one can check that $U(\overline{I}, \mathcal{P}_1) \leq U(f, \mathcal{P})$. Together with an obvious inequality $L(\underline{I}, \mathcal{P}_1) \leq U(\overline{I}, \mathcal{P}_1)$ this completes the proof of (4.14). Thus we have

$$\max(U(\overline{I}, \mathcal{P}_1) - L(\overline{I}, \mathcal{P}_1), U(\underline{I}, \mathcal{P}_1) - L(\underline{I}, \mathcal{P}_1)) \leq U(\overline{I}, \mathcal{P}_1) - L(\underline{I}, \mathcal{P}_1) \leq U(f, \mathcal{P}) - L(f, \mathcal{P}).$$

By assumption for appropriate choices of partitions, the right-hand side can be made $< \epsilon$ for any a priori given $\epsilon > 0$. This implies the integrability of the function $\underline{I}(x)$ and $\overline{I}(x)$ over $P_1$. But then we can write

$$\int\limits_{P_1} \underline{I}(x) dV_{n-k} = \lim_{\delta(\mathcal{P}_1) \to 0} L(\underline{I}; \mathcal{P}_1)$$

and

$$\int\limits_{P_1} \overline{I}(x) dV_{n-k} = \lim_{\delta(\mathcal{P}_1) \to 0} U(\overline{I}; \mathcal{P}_1).$$

We also have

$$\lim_{\delta(\mathcal{P}) \to 0} L(f; \mathcal{P}) = \lim_{\delta(\mathcal{P}) \to 0} U(f; \mathcal{P}) = \int\limits_P f dV_n.$$

Hence, the inequality (4.14) implies that

$$\int\limits_P f dV_n = \int\limits_{P_1} \left( \underline{\int\limits_{P_2}} f_x dV_{n-k} \right) dV_k = \int\limits_{P_1} \left( \overline{\int\limits_{P_2}} f_x dV_{n-k} \right) dV_k.$$

■

**Corollary 4.5.2.** *Suppose $f : P \to \mathbb{R}$ is a continuous function. Then*

$$\int\limits_P f = \int\limits_{P_1} \int\limits_{P_2} f_x = \int\limits_{P_2} \int\limits_{P_1} f_y.$$

Thus if we switch back to the notation $x_1, \dots, x_n$ for coordinates in $\mathbb{R}^n$, and if $P = \{a_1 \leq x_1 \leq b_1, \dots, a_n \leq x_n \leq b_n\}$ then we can write

$$\int\limits_P f = \int\limits_{a_n}^{b_n} \left( \dots \left( \int\limits_{a_1}^{b_1} f(x_1, \dots, x_n) dx_1 \right) \dots \right) dx_n \, . \tag{4.15}$$

The integral in the right-hand side of (4.15) is called an *iterated integral*. Note that the order of integration is irrelevant there. In particular, for continuous functions one can change the order of integration in the iterated integrals.

■

## 4.6 Integration of $n$-forms over domains in $n$-dimensional space

Differential forms are much better suited to be integrated than functions. For integrating a function, one needs a measure. To integrate a differential form, one needs nothing except an orientation of the domain of integration.

Let us start with the integration of a $n$-form over a domain in a $n$-dimensional space. Let $\omega$ be a $n$-form on a domain $U \subset V$, $\dim V = n$.

Let us fix now an orientation of the space $V$. Pick any coordinate system $(x_1 \dots x_n)$ that agrees with the chosen orientation.

We proceed similar to the way we defined an integral of a function. Let us fix a rectangle $P = P(a_1, b_1; a_2, b_2; \dots; a_n, b_n) = \{a_i \le x_i \le b_i;\ i = 1, \dots, n\}$. Choose its partition $\mathcal{P}$ by $N^n$ smaller rectangles $P_{\mathbf{j}} = \{t^1_{j_n} \le x_1 \le t^1_{j_n+1}, \dots, t^n_{j_1} \le x_1 \le t^n_{j_1+1}\}$, where $\mathbf{j} = (j_1, \dots, j_n)$ and each index $j_k$ takes values between 0 and $N-1$. Let us fix a point $c_{\mathbf{j}} \in P_{\mathbf{j}}$ and denote by $C$ the set of all such $c_{\mathbf{j}}$. We also denote by $t_{\mathbf{j}}$ the point with coordinates $t^1_{j_1}, \dots, t^n_{j_n}$ and by $T_{\mathbf{j},m} \in V_{c_{\mathbf{j}}}$, $m = 1, \dots, n$ the vector $t_{\mathbf{j}+1_m} - t_{\mathbf{j}}$, parallel-transported to the point $c_{\mathbf{j}}$. Here we use the notation $\mathbf{j} + 1_m$ for the multi-index $j_1, \dots, j_{m-1}, j_m + 1, j_{m+1}, \dots, j_n$. Thus the vector $T_{\mathbf{j},m}$ is parallel to the $m$-th basic vector and has the length $|t_{j_m+1} - t_{j_m}|$.

Given a differential $n$-form $\alpha$ on $P$ we form an integral sum

$$I(\alpha; \mathcal{P}, C) = \sum_{\mathbf{j}} \alpha(T_1^{\mathbf{j}}, T_2^{\mathbf{j}}, \dots, T_{\mathbf{j},n}), \tag{4.16}$$

where the sum is taken over all elements of the partition. We call an $n$-form $\alpha$ *integrable* if there exists a limit $\lim\limits_{\delta(\mathcal{P}) \to 0} I(\alpha; \mathcal{P}, C)$ which we denote by $\int\limits_P \alpha$ and call the *integral of $\alpha$ over*

$P$. Note that if $\alpha = f(x)dx_1 \wedge \cdots \wedge dx_n$ then the integral sum $I(\alpha, \mathcal{P}, C)$ from (4.16) coincides with the integral sum $I(f; \mathcal{P}, C)$ from (4.9) for the function $f$. Thus the integrability of $\alpha$ is the same as integrability of $f$ and we have

$$\int\limits_P f(x)dx_1 \wedge \cdots \wedge dx_n = \int\limits_P f dV. \tag{4.17}$$

Note, however, that the equality (4.17) holds *only if the coordinate system* $(x_1, \ldots, x_n)$ *defines the given orientation of the space* $V$. The integral $\int\limits_P f(x)dx_1 \wedge \cdots \wedge dx_n$ changes its sign with a change of the orientation while the integral $\int\limits_P f dV$ is not sensitive to the orientation of the space $V$.

**Theorem 4.6.1.** *Let $f : P \to P$ be an orientation preserving diffeomorphism. Then for any integrable $n$-form $\alpha$ the form $f^*\alpha$ is also integrable and we have*

$$\int\limits_P \alpha = \int\limits_P f^*\alpha. \tag{4.18}$$

For an orientation reversing diffeomorphism one has $\int\limits_P \alpha = -\int\limits_P f^*\alpha$.

**Remark 4.6.2.** *We will prove later (see Corollary 6.3.6 that if $f : P \to P$ is any smooth map (and not necessarily a diffeomorphism) such that $f(x) = x$ for all $x \in \partial P$, then the equation* (4.18) *holds, i.e.* $\int\limits_P \alpha = \int\limits_P f^*\alpha.$.

Let $\alpha = g(x)dx_1 \wedge \cdots \wedge dx_n$. Then $f^*\alpha = g \circ f \det Df dx_1 \wedge \cdots \wedge dx_n$, and hence the formula (4.18) can be rewritten as

$$\int\limits_P g(x_1, \ldots, x_n)dx_1 \wedge \cdots \wedge dx_n = \int\limits_P g \circ f \det Df \, dx_1 \wedge \cdots \wedge dx_n.$$

Here

$$\det Df = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial f_n}{\partial x_1} & \ldots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

is the determinant of the Jacobian matrix of $f = (f_1, \dots, f_n)$.

Hence, in view of formula (4.17) we get the following *change of variables* formula for multiple integrals of functions.

**Corollary 4.6.3.** [CHANGE OF VARIABLES IN A MULTIPLE INTEGRAL] *Let $g : P \to \mathbb{R}$ be an integrable function and $f : P \to P$ a diffeomorphism. Then the function $g \circ f$ is also integrable and*

$$\int\limits_P g dV = \int\limits_P g \circ f |\det Df| dV \,. \tag{4.19}$$

**Proof of Theorem 4.6.1.** We can choose a coordinate system in $V$ such that $P$ is the unit cube $P = \{0 \le x_i \le 1, j = 1, \dots, n\}$. Let us write $\alpha = g\, dx_1 \wedge \dots dx_n$. To emphasize the main ideas of the proof we will present here the proof only for the most important special case when $g$ is a *bounded a.e. continuous function.* Under this assumption, the form $f^*\alpha$ is also a.e. continuous, and hence integrable.

Let us first analyze the case when $g$ is continuous. Let us choose a partition $\mathcal{P}$ of $P$ by $N^n$ equal cubes. Thus $\delta(\mathcal{P}) = \frac{1}{N}$ and partition points and take a set $C = \{c_{\mathbf{j}}\}$, where $c_{\mathbf{j}} = (t_{j_1,1}, \dots, t_{j_n,n}) = \left(\frac{j_1}{N}, \dots, \frac{j_n}{N}\right)$ is one of the corners of the rectangle $P_{\mathbf{j}}$. Consider the corresponding integral sum

$$I(f^*\alpha; \mathcal{P}, C) = \sum_{\mathbf{j}} f^*\alpha_{c_{\mathbf{j}}}(T_1^{\mathbf{j}}, \dots, T_{\mathbf{j},n}) = \sum_{\mathbf{j}} \alpha_{\widetilde{c}_{\mathbf{j}}}(\widetilde{T}_1^{\mathbf{j}}, \dots, \widetilde{T}_{\mathbf{j},n}),$$

where we denoted

$$\widetilde{c}_j = f(c_{\mathbf{j}}),\ \widetilde{T}_{\mathbf{j},k} = d_{c_{\mathbf{j}}} f(T_{\mathbf{j},k}),\ k = 1, \dots, n.$$

Choose $\epsilon > 0$. Then for a sufficiently large $N$ we have

$$|I(f^*\alpha; \mathcal{P}, C) - \int\limits_P f^*\alpha| < \epsilon. \tag{4.20}$$

Let us denote $\widetilde{P}_{\mathbf{j}} := f(P_{\mathbf{j}})$. Though the domain $\widetilde{P}_{\mathbf{j}}$ is not a parallelepiped it is close, when the rectangle $P_{\mathbf{j}}$ is small, to the parallelepiped $\widehat{P}_{\mathbf{j}} = d_{c_{\mathbf{j}}}(P_{\mathbf{j}})$ viewed as a subset of $V$, rather than of $V_{\widetilde{c}_{\mathbf{j}}}$. In other words,

$$\widehat{P}_{\mathbf{j}} = \widetilde{c}_{\mathbf{j}} + P(\widetilde{T}_1^{\mathbf{j}}, \ldots, \widetilde{T}_{\mathbf{j},n}) = \{\widetilde{c}_{\mathbf{j}} + d_{c_{\mathbf{j}}}(h),\ h = P(T_1^{\mathbf{j}}, T_2^{\mathbf{j}}, \ldots, T_{\mathbf{j},n})\}.$$

More precisely, we have

**Lemma 4.6.4.** *If $N$ is sufficiently large then*

$$\mathrm{Vol}(\widehat{P}_{\mathbf{j}} \triangle \widetilde{P}_{\mathbf{j}}) < \frac{C\epsilon}{N^n},$$

*where $\widehat{P}_{\mathbf{j}} \triangle \widetilde{P}_{\mathbf{j}}$ is the symmetric difference of two sets:*

$$\widehat{P}_{\mathbf{j}} \triangle \widetilde{P}_{\mathbf{j}} = \left(\widehat{P}_{\mathbf{j}} \setminus \widetilde{P}_{\mathbf{j}}\right) \cup \left(\widetilde{P}_{\mathbf{j}} \setminus \widehat{P}_{\mathbf{j}}\right).$$

**Proof of Lemma 4.6.4.** In view of smoothness of the map $f$ we have

$$||f(x+h) - f(x) - d_x(f)(h)|| < \epsilon||h|| \tag{4.21}$$

for any point $x \in P$ and any $h \in V_x$ such that $x + h \in P$ and $||h|| < \delta$, provided that $\delta$ is chosen large enough. Hence, if we choose[6] $N > \frac{\sqrt{n}}{\delta}$ then for any any $x = c_{\mathbf{j}} + h \in P_{\mathbf{j}}$ we have

$$|f(x) - f(c_{\mathbf{j}}) - d_{c_{\mathbf{j}}} f(h)| < \epsilon||h||.$$

This implies that any point $y \in \widehat{P}_{\mathbf{j}} \triangle \widetilde{P}_{\mathbf{j}}$ lies at a distance at most $\frac{\epsilon}{N}$ from the boundary of $\widehat{P}_{\mathbf{j}}$. We also observe that smoothness of $f$ guarantees existence of a constant $K > 0$ such that

$$||d_x f(h)|| \leq K||h|| \tag{4.22}$$

for any $x \in P, h \in V_x$. Consider one of the faces $F$ of the parallelelpiped $\widehat{P}\mathbf{j}$. Let $F_\epsilon$ be the set of points which lie at a distance at most $\frac{\epsilon}{N}$ from $F$. We claim that $\mathrm{Vol}(F_\epsilon) \leq \frac{C_1\epsilon}{N^n}$ for some positive constant $C_1$. Indeed, according to (4.22) $F$ is contained in an $(n-1)$-dimensional cube with the side $\frac{K}{N}$ in the same hyperplane as $F$. Hence, the orthogonal projection of $F_\epsilon$ this hyperplane is contained in a cube of side $\frac{K+2\epsilon}{N}$, while the $F$ itself is contained in a

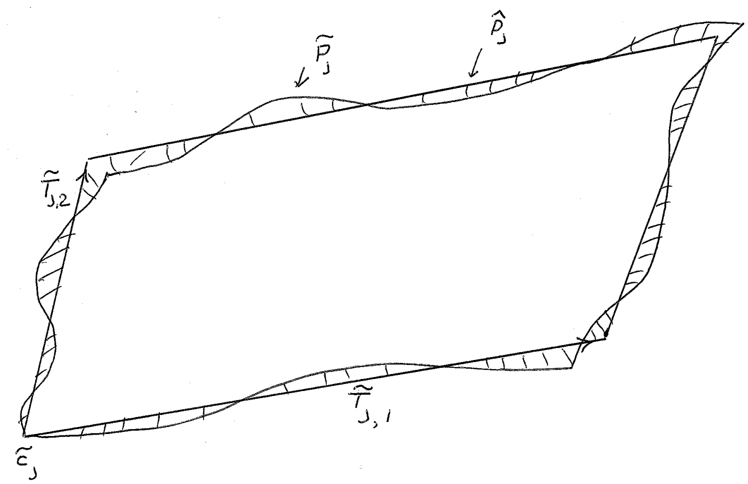


Figure 1: The shaded area is $\widetilde{P}_{\mathbf{j}} \triangle \widehat{P}_{\mathbf{j}}$

rectangular (i.e. a parallelepiped with orthogonal edges) whose one side has length $\frac{2\epsilon}{N}$ and all the others of length $\frac{K+2\epsilon}{N}$.

Thus,

$$\mathrm{Vol} F_\epsilon < \frac{2\epsilon(K+2\epsilon)^{n-1}}{N^n} \leq \frac{C_1 \epsilon}{N^n}$$

for some positive constant $C_1 > 0$. The parallelepipeds $\widehat{P}_{\mathbf{j}}$ has $2n$ faces, and hence

$$\mathrm{Vol}\left(\widehat{P}_{\mathbf{j}} \triangle \widetilde{P}_{\mathbf{j}}\right) < \frac{2nC_1\epsilon}{N^n} = \frac{C\epsilon}{N^n}.$$

■

Let us now evaluate the difference $I(f^*\alpha; \mathcal{P}, C) - \int_P \alpha$. We have

$$\int_{\widetilde{P}_{\mathbf{j}}} \alpha = \int_{\widehat{P}_{\mathbf{j}} \cap \widetilde{P}_{\mathbf{j}}} \alpha + \int_{\widetilde{P}_{\mathbf{j}} \setminus \widehat{P}_{\mathbf{j}}} \alpha$$

[6] This condition ensures that the distance between any two points of $P_{\mathbf{j}}$ is $< \delta$.

and

$$\int_{\widehat{P}_{\mathbf{j}}} \alpha = \int_{\widehat{P}_{\mathbf{j}} \cap \widetilde{P}_{\mathbf{j}}} \alpha + \int_{\widehat{P}_{\mathbf{j}} \setminus \widetilde{P}_{\mathbf{j}}} \alpha \,.$$

Thus

$$\left| \int_{\widetilde{P}_{\mathbf{j}}} \alpha - \int_{\widehat{P}_{\mathbf{j}}} \alpha \right| \le \int\limits_{\widehat{P}_{\mathbf{j}} \triangle \widetilde{P}_{\mathbf{j}}} |g| dV \le \max_{P} |g| \mathrm{Vol}\left(\widehat{P}_{\mathbf{j}} \triangle \widetilde{P}_{\mathbf{j}}\right) \le \frac{\max_{P} |g| C\epsilon}{N^n} = \frac{C_2 \epsilon}{N^n}. \tag{4.23}$$

Using additivity of the integral we have

$$\int\limits_{P} \alpha = \sum_{\mathbf{j}} \int_{\widetilde{P}_{\mathbf{j}}} \alpha \tag{4.24}$$

Hence, (4.23) implies that

$$\left| \int\limits_{P} \alpha - \sum_{\mathbf{j}} \int_{\widehat{P}_{\mathbf{j}}} \alpha \right| \le \sum_{\mathbf{j}} \left| \int_{\widetilde{P}_{\mathbf{j}}} \alpha - \int_{\widehat{P}_{\mathbf{j}}} \alpha \right| \le N^n \frac{C_2 \epsilon}{N^n} = C_2 \epsilon. \tag{4.25}$$

In view of (4.22) we have $||\widetilde{T}_{\mathbf{j},k}|| \le K||T_{\mathbf{j},k}|| = \frac{K}{N}$ for all $\mathbf{j}$ and $k = 1, \ldots .n$. Together with inequality (4.21) and continuity of $g$ this imply that for a sufficiently large $N$,

$$\max_{\mathbf{j}} \left( \max_{\widetilde{P}_{\mathbf{j}}} g - \min_{\widetilde{P}_{\mathbf{j}}} g \right) < \epsilon.$$

Let us also recall that in view of Proposition 4.4.15 we have

$$\alpha_{c_{\mathbf{j}}}(\widetilde{T}_1^{\mathbf{j}}, \ldots, \widetilde{T}_{\mathbf{j},n}) = g(c_{\mathbf{j}}) dx_1 \wedge \cdots \wedge dx_n(\widetilde{T}_1^{\mathbf{j}}, \ldots, \widetilde{T}_{\mathbf{j},n}) = g(c_{\mathbf{j}}) \mathrm{Vol}(\widehat{P}_{\mathbf{j}}) = \int\limits_{P_{\mathbf{j}}} g(c_{\mathbf{j}}) dV, \tag{4.26}$$

and hence using (4.22) we get

$$\left| \left( \int\limits_{\widehat{P}_{\mathbf{j}}} \alpha \right) - \alpha(\widetilde{T}_1^{\mathbf{j}}, \ldots, \widetilde{T}_{\mathbf{j},n}) \right| \le \int\limits_{\widehat{P}_{\mathbf{j}}} |g(x) - g(c_{\mathbf{j}})| dV < \epsilon \mathrm{Vol} \widehat{P}_{\mathbf{j}} \le \ \epsilon \frac{K^n}{N^n}. \tag{4.27}$$

Thus we get

$$\left| I(f^*\alpha; \mathcal{P}, C) - \int\limits_{P} \alpha \right| \le \left| \sum_{\mathbf{j}} \alpha_{\widetilde{c}_{\mathbf{j}}}(\widetilde{T}_1^{\mathbf{j}}, \ldots, \widetilde{T}_{\mathbf{j},n}) - \sum_{\mathbf{j}} \int\limits_{\widehat{P}_{\mathbf{j}}} \alpha \right| + \left| \int\limits_{P} \alpha - \sum_{\mathbf{j}} \int_{\widehat{P}_{\mathbf{j}}} \alpha \right| \le$$

$$K^n \epsilon + C_2 \epsilon = C_3 \epsilon. \tag{4.28}$$

But $\epsilon$ can be chosen arbitrarily small, and hence combining (4.20) and (4.28) we conclude that

$$\int\limits_P f^*\alpha = \int\limits_P \alpha.$$

Let us finally sketch the proof for the case when the function $h$ is *almost everywhere* continuous. Let $A$ be the discontinuity locus of the function $h$. By assumption, $\mathrm{Vol}(A) = 0$. Choose an open set $B \supset A$ such that $\mathrm{Vol}B < \epsilon$. The complement $P \setminus B$ is compact, the function $h|_{P\setminus B} : P \setminus B \to \mathbb{R}$ is continuous, and hence it is uniformly continuous.

Given a partition $\mathcal{P}$ of $P$ we denote by $J_{\mathrm{bad}}$ the set of multi-indices $\mathbf{j}$ for which $P_{\mathbf{j}} \cap B \neq \varnothing$ and by $J_{\mathrm{good}}$ the complementary set of indices. Respectively, the integral sum $I(f^*\alpha; \mathcal{P}, C)$ can be split as

$$I(f^*\alpha; \mathcal{P}, C) = I_{\mathrm{bad}}(f^*\alpha; \mathcal{P}, C) + I_{\mathrm{good}}(f^*\alpha; \mathcal{P}, C),$$

where

$$I_{\mathrm{good}}(f^*\alpha; \mathcal{P}, C) = \sum_{\mathbf{j}\in J_{\mathrm{good}}} f^*\alpha_{c_{\mathbf{j}}}(T_1^{\mathbf{j}}, \dots, T_{\mathbf{j},n}),$$
$$I_{\mathrm{bad}}(f^*\alpha; \mathcal{P}, C) = \sum_{\mathbf{j}\in J_{\mathrm{bad}}} f^*\alpha_{c_{\mathbf{j}}}(T_1^{\mathbf{j}}, \dots, T_{\mathbf{j},n}).$$

We will further denote $P_{\mathrm{good}} := \bigcup\limits_{\mathbf{j}\in J_{\mathrm{good}}} P_{\mathbf{j}}$, $P_{\mathrm{bad}} := \bigcup\limits_{\mathbf{j}\in J_{\mathrm{bad}}} P_{\mathbf{j}}$, $\widetilde{P}_{\mathrm{good}} = f(P_{\mathrm{good}})$. $\widetilde{P}_{\mathrm{bad}} = f(P_{\mathrm{bad}})$. Hence, we can write

$$\int\limits_P \alpha = \int\limits_{P_{\mathrm{good}}} \alpha + \int\limits_{P_{\mathrm{bad}}} \alpha.$$

Arguing as in the case of continuous $\alpha$ we can show that by choosing $N$ sufficiently large one can ensure that

$$\left| I_{\mathrm{good}}(f^*\alpha; \mathcal{P}, C) - \int\limits_{\widetilde{P}_{\mathrm{good}}} \alpha \right| < C_1\epsilon. \tag{4.29}$$

On the other hand we can also arrange that

$$|I_{\mathrm{bad}}(f^*\alpha; \mathcal{P}, C)| < C_2\epsilon. \tag{4.30}$$

$$\left| \int_{\widetilde{P}_{\text{bad}}} \alpha \right| < C_3 \epsilon. \tag{4.31}$$

Combining inequalities (4.29), (4.30) and (4.31) we get the bound

$$\left| I(f^*\alpha; \mathcal{P}, C) - \int_{\mathcal{P}} \alpha \right| < C_4 \epsilon.$$

and the claim follows as in the case of continuous $\alpha$. ■

**Discussion.**

1. One can modify the formulation of Theorem 4.6.1 by considering a diffeomorphism $f : P_1 \to P_2$ between two different rectangles, provided that both are oriented and $f$ preserves their orientation. Then for any integrable differential $n$-form $\alpha$ on $P_2$ we have

   $$\int_{P_1} f^*\alpha = \int_{P_2} \alpha.$$

   We leave it to the reader to formally deduce this claim from Theorem 4.6.1.

2. The condition that $f : P \to P$ is a diffeomorphism can be relaxed by requiring that $f$ is $1-1$ map and that $\det Df \geq 0$, i.e. that the differential of $f$ preserves orientation everywhere where it is non-degenerate. The above proof goes in this case without any modifications.

3. One can also prove the following version of Theorem 4.6.1 for partially defined maps.

**Proposition 4.6.5.** *Let $A \subset V$ be a compact measurable set, $U \supset A$ its open neighborhood and $f : U \to V$ any injective map with non-negative Jacobian $\det Df$. Let $\alpha$ be an integrable differential $n$-form given on a neighborhood of the image $f(A) \subset V$. Then the form $f^*\alpha$ is integrable over $A$ and*

$$\int_{A} f^*\alpha = \int_{f(A)} \alpha.$$

**Sketch of a proof.** Take a large rectangle $P$ which contains both sets, $A$ and $f(A)$, and extend the form $\alpha$ to $P \subset V$ as equal to 0 outside $f(A)$. Take a sufficiently fine partition $\mathcal{P}$ of $P$ such that the rectangles which intersect $A$ are contained in $U$, and such that the total volume of rectangles intersecting $\partial A$ is $< \epsilon$. Then the proof of Theorem 4.6.1 goes through in this case with the only modification that we consider only those rectangles which intersect $A$. ■

**Corollary 4.6.6.** *Let $\alpha$ be an integrable $n$-form over a rectangle $P$. Let $A_1, \dots, A_K$ be compact measurable sets in $V$, and $U_1 \supset A, \dots, U_K \supset A_K$ be their neighborhoods. Let $f_j : U_j \to P$, $j = 1, \dots, K$ be smooth maps such that*

- *$f_j(A_j) \subset P$;*
- *$f_j$ has non-negative Jacobian determinant $\det Df_j$;*
- *$f_j$ is injective;*
- *$\bigcup\limits_1^k f_j(A_j) = P$;*
- *$\mathrm{Vol}\,(f_j(A_j) \cap f_i(A_i)) = 0$ for all $i \neq j$.*

*Then*

$$\int\limits_P \alpha = \sum_1^n \int\limits_{a_j} f^*\alpha.$$

3. In the case when the form $\alpha$ is smooth over $P$ we will prove later, as a corollary of Stokes' theorem that *for any smooth map $P \to P$ such that $f(x) = x$ for $x \in \partial P$ we have* $\int\limits_P f^*\alpha = \int\limits_P \alpha$.

## 4.7 Manifolds and submanifolds

### 4.7.1 Manifolds

Manifolds of dimension $n$ are spaces which are locally look like open subsets of $\mathbb{R}^n$ but globally could be much more complicated. We give a precise definition below.

Let $U, U' \subset \mathbb{R}^n$ be open sets. A map $f : U \to U'$ is called a *homeomorpism* if it is continuous one-to-one map which has a continuous inverse $f^{-1} : U' \to U$.

A map $f : U \to U'$ is called a $C^k$*-diffeomorpism*, $k = 1, \ldots, \infty$, if it is $C^k$-smooth, one-to-one map which has a $C^k$-smooth inverse $f^{-1} : U' \to U$. Usually we will omit the reference to the class of smoothness, and just call $f$ a *diffeomorphism*, unless it will be important to emphasize the class of smoothness.

A set $M$ is called an $n$-dimensional $C^k$-smooth (resp. topological) *manifold* if there exist subsets $U_\lambda \subset X$, $\lambda \in \Lambda$, where $\Lambda$ is a finite or countable set of indices, and for every $\lambda \in \Lambda$ a map $\Psi_\lambda : U_\lambda \to \mathbb{R}^n$ such that

M1. $M = \bigcup\limits_{\lambda \in \Lambda} U_\lambda$.

M2. The image $G_\lambda = \Psi_\lambda(U_\lambda)$ is an open set in $\mathbb{R}^n$.

M3. The map $\Psi_\lambda$ viewed as a map $U_\lambda \to G_\lambda$ is one-to-one.

M4. For any two sets $U_\lambda, U_\mu, \lambda, \mu \in \Lambda$ the images $\Psi_\lambda(U_\lambda \cup U_\mu), \Psi_\mu(U_\lambda \cup U_\mu) \subset \mathbb{R}^n$ are open and the map

$$h_{\lambda\mu} := \Psi_\mu^{-1} \circ \Psi_\lambda : \Psi_\lambda(U_\lambda \cup U_\mu) \to \Psi_\mu(U_\lambda \cup U_\mu) \subset \mathbb{R}^n$$

is a $C^k$-diffeomorphism (resp. homeomorphism).

Sets $U_\lambda$ are called *coordinate neighborhoods* and maps $\Phi_\lambda : U_\lambda \to \mathbb{R}$ are called *coordinate maps.* The pairs $(U_\lambda, \Phi_\lambda)$ are also called *local coordinate charts.* The inverse maps $\Psi_\lambda = \Phi_\lambda^{-1} : \Phi_\lambda(U_\lambda) \to U_\lambda$ are called *(local) parameterization maps.* An *atlas* is a collection $\mathfrak{A} = \{U_\lambda, \Phi_\lambda\}_{\lambda \in \Lambda}$ of all coordinate charts.

One says that two atlases $\mathfrak{A} = \{U_\lambda, \Phi_\lambda\}_{\lambda\in\Lambda}$ and $\mathfrak{A}' = \{U'_\gamma, \Phi'_\gamma\}_{\gamma\in\Gamma}$ on the same manifold $X$ are *equivalent*, or that they *define the same smooth structure on* $X$ if their union $\mathfrak{A}\cup\mathfrak{A}' = \{(U_\lambda, \Phi_\lambda), (U'_\gamma, \Phi'_\gamma)\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is again an atlas on $X$. In other words, two atlases define the same smooth structure if transition maps from local coordinates in one of the atlases to the local coordinates in the other one are given by smooth functions.

A subset $G \subset M$ is called *open* if for every $\lambda \in \Lambda$ the image $\Phi_\lambda(G \cap U_\lambda) \subset \mathbb{R}^n$ is open. In particular, coordinate charts $U_\lambda$ themselves are open, and we can equivalently say that a set $G$ is open if its intersection with every coordinate chart is open. By a *neighborhood* of a point $a \in M$ we will mean any open subset $U \subset M$ such that $a \in U$.

Given two smooth manifolds $M$ and $\widetilde{M}$ of dimension $m$ and $n$ then for $k = 0, \dots, l$ a map $f : M \to \widetilde{M}$ is called *smooth* if for every point $a \in M$ there exist local coordinate charts $(U_\lambda, \Phi_\lambda)$ in $M$ and $(\widetilde{U}_\lambda, \widetilde{\Phi}_\lambda)$ in $\widetilde{M}_\lambda$, such that $a \in U_\lambda, f(U_\lambda \subset \widetilde{U}_\lambda$ and the composition map

$$\Phi_\lambda(U_\lambda) \xrightarrow{\Psi_\lambda} U_\lambda \xrightarrow{f} \widetilde{U}_\lambda \xrightarrow{\widetilde{\Phi}_\lambda} \mathbb{R}^n$$

is smooth. In other words, a map is smooth, if it is smooth when expressed in local coordinates.

A map $f : M \to N$ is called a *diffeomorphism* if it is smooth, one-to-one, and the inverse map is also smooth.

Note that in view of the chain rule the $C^k$-smoothness is independent of the choice of local coordinate charts $(U_\lambda, \Phi_\lambda)$ and $(\widetilde{U}_\lambda, \widetilde{\Phi}_\lambda)$. Note that for $C^k$-smooth manifolds one can talk only about $C^l$-smooth maps for $l \le k$. For topological manifolds one can talk only about continuous maps. One-to-one continuous maps with continuous inverses are called *homeomorphisms*.

Note that any point $a$ of an $n$-dimensional manifold $M$ has a neighborhood $B$ diffeomorphic to an open ball $B_1(0) \subset \mathbb{R}^n$.

### 4.7.2 Gluing construction

The construction which is described in this section is called *gluing* or *quotient* construction. It provides a rich source of examples of manifolds. We discuss here only very special cases of this construction.

a) Let $M$ be a manifold and $U, U'$ its two open disjoint subsets. Let us moreover assume that each point $x \in M$ has a neighborhood which does not intersect at least one of the sets $U$ and $U'$.[7] Consider a diffeomorphism $f : U \to U'$.

Let us denote by $M/\{f(x) \sim x\}$ the set obtained from $M$ by identifying each point $x \in U$ with its image $f(x) \in U'$. In other words, a point of $M/\{f(x) \sim x\}$ is either a point from $x \in M \setminus (U \cup U')$, or a pair of points $(x, f(x))$, where $x \in U$. Note that there exists a canonical projection $\pi : M \to M/\{f(x) \sim x\}$. Namely $\pi(x) = x$ if $x \notin U \cup U'$, $\pi(x) = (x, f(x))$ if $x \in U$ and $\pi(x) = (f^{-1}(x), x)$ if $x \in U'$. By our assumption each point $x \in M$ has a coordinate neighborhood $G_x \ni x$ such that $f(G_x \cap U) \cap G_x = \varnothing$. In particular, the projection $\pi|_{G_x} : G_x \to \widetilde{G}_x = \pi(G_x)$ is one-to-one. We will declare by definition that $\widetilde{G}_x$ is a coordinate neighborhood of $\pi(x) \in M/\{f(x) \sim x\}$ and define a coordinate map $\widetilde{\Phi} : \widetilde{G}_x \to \mathbb{R}^n$ by the formula $\widetilde{\Phi} = \Phi \circ \pi^{-1}$. It is not difficult to check that this construction define a structure of an $n$-dimensional manifold on the set $M/\{f(x) \sim x\}$. We will call the resulted manifold *the quotient manifold* of $M$, or say that $M/\{f(x) \sim x\}$ is obtained from *$M$ by gluing $U$ with $U'$ with the diffeomorphism* $\Phi$.

Though the described above gluing construction always produce a manifold, the result could be quite pathological, if no additional care is taken. Here is an example of such pathology.

**Example 4.7.1.** Let $M = I \cup I'$ be the union of two disjoint open intervals $I = (0, 2)$ and $I' = (3, 5)$. Then $M$ is a 1-dimensional manifold. Denote $U := (0, 1) \subset I, U' := (3, 4) \subset I'$. Consider a diffeomorphism $f : U \to U'$ given by the formula $f(t) = t + 3,\ t \in U$. Let

[7] Here is an example when this condition *is not satisfied*: $M = (0, 2), U = (0, 1), U' = (1, 2)$. In this case any neighborhood of the point 1 intersect both sets, $U$ and $U'$.

$\widetilde{M} = M/\{f(x) \sim x\}$ be the corresponding quotient manifold. In other words, $\widetilde{M}$ is the result of gluing the intervals $I$ and $I'$ along their open sub-intervals $U$ and $U'$. Note that the points $1 \in I$ and $4 \in I'$ are not identified, but $1-\epsilon, 4-\epsilon$ are identified for an arbitrary small $\epsilon > 0$. This means that any neighborhood of 1 and any neighborhood of 4 have non-empty intersection.

In order to avoid such pathological examples one usually (but not always) requires that manifolds satisfy an additional axiom, called *Hausdorff property*:

M5. Any two distinct points $x, y \in M$ have non-intersecting neighborhoods $U \ni x, G \ni Y$.

In what follows we always assume that the manifolds *satisfy the Hausdorff property* M5.

b) Let $M$ be a manifold, $f: M \to M$ be a diffeomorphism. Suppose that $f$ satisfies the following property: *There exists a positive integer $p$ such that for any point $x \in M$ we have $f^p(x) = \underbrace{f \circ f \circ \cdots \circ f}_{p}(x) = x$, but the points $x, f(x) \dots, f^{p-1}(x)$ are all disjoint.* The set $\{x, f(x) \dots, f^{p-1}(x)\} \subset M$ is called the *trajectory* of the point $x$ under the action of $f$. It is clear that trajectory of two different points either coincide or disjoint. Then one can consider the *quotient space* $X/f$, whose points are trajectories of points of $M$ under the action of $f$. Similarly to how it was done in a) one can define on $M/f$ a structure of an $n$-dimensional manifold.

c) Here is a version of the construction in a) for the case when trajectory of points are infinite. Let $f: M \to M$ be a diffeomorphism which satisfies the following property: *for each point $x \in M$ there exists a neighborhood $U \setminus X \ni x$ such that all sets*

$$\dots, f^{-2}(U_x), f^{-1}(U_x), U_x, f(U_x), f^2(U_x), \dots$$

*are mutually disjoint.* In this case the *trajectory* $\{\dots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \dots\}$ of each point is infinite. As in the case b), the trajectories of two different points either coincide or disjoint. The set $M/f$ of all trajectories can again be endowed with a structure of a manifold of the same dimension as $M$.

### 4.7.3 Examples of manifolds

1. **$n$-dimensional sphere $S^n$.** Consider the unit sphere $S^n = \{||x|| = \sqrt{\sum\limits_1^{n+1} x_j^2} = 1\} \subset \mathbb{R}^{n+1}$. Let introduce on $S^n$ the structure of an $n$-dimensional manifold. Let $N = (0, \dots, 1)$ and $S = (0, \dots, -1)$ be the North and South poles of $S^n$, respectively.

Denote $U_- = S^n \setminus S, U_+ == S^n \setminus N$ and consider the maps $p_\pm : U_\pm \to \mathbb{R}^n$ given by the formula

$$p_\pm(x_1, \dots, x_{n+1}) = \frac{\sqrt{\sum\limits_1^n x_j^2}}{1 \mp x_{n+1}}(x_1, \dots, x_n). \tag{4.32}$$

The maps $p_+ : U_+ \to \mathbb{R}_n$ and $p_- : U_- \to \mathbb{R}^n$ are called *stereographic projections* from the North and South poles, respectively. It is easy to see that stereographic projections are one-to one maps. Note that $U_+ \cap U_- = S^n \setminus \{S, N\}$ and both images, $p_+(U_+ \cap U_-)$ and $p_-(U_+ \cap U_-)$ coincide with $\mathbb{R}^n \setminus 0$. The map $p_- \circ p_+^{-1} : \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$ is given by the formula

$$p_- \circ p_+^{-1}(x) = \frac{x}{||x||^2}, \tag{4.33}$$

and therefore it is a diffeomorphism $\mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$.

Thus, the atlas which consists of two coordinate charts $(U_+, p_+)$ and $(U_-, p_-)$ defines on $S^n$ a structure of an $n$-dimensional manifold. One can equivalently defines the manifold $S^n$ as follows. Take two disjoint copies of $\mathbb{R}^n$, let call them $\mathbb{R}^n_1$ and $\mathbb{R}^n_2$. Denote $M = \mathbb{R}^n_1 \cup \mathbb{R}^n_2$, $U = \mathbb{R}^n_1 \setminus 0$ and $U' = \mathbb{R}^n_2 \setminus 0$. Let $f : U \to U'$ be a difeomorphism defined by the formula $f(x) = \frac{x}{||x||^2}$, as in (4.33). Then $S^n$ can be equivalently described as the quotient manifold $M/f$.

Note that the 1-dimensional sphere is the circle $S^1$. It can be alternatively defined as follows. Consider the map $T : \mathbb{R} \to \mathbb{R}$ given by the formula $T(x) = x + 1$, $x \in \mathbb{R}$. It satisfies the condition from 4.7.2 and hence, one can define the manifold $\mathbb{R}/T$. This manifold is diffeomorphic to $S^1$.

2. **Real projective space.** The real projective space $\mathbb{R}P^n$ is the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin. One introduces on $\mathbb{R}P^n$ a structure of an $n$-dimensional manifold

as follows. For each $j = 1, \dots, n+1$ let us denote by $U_j$ the set of lines which are not parallel to the affine subspace $\Pi_j = \{x_j = 1\}$. Clearly, $\bigcup\limits_1^{n+1} U_j = \mathbb{R}P^n$. There is a natural one-to one map $\pi_j : U_j \to \Pi_j$ which associates with each line $\mu \in U_j$ the unique intersection point of $\mu$ with $\Pi_j$. Furthermore, each $\Pi_j$ can be identified with $\mathbb{R}^n$, and hence pairs $(U_j, \pi_j)$, $j = 1, \dots, n+1$ can be chosen as an atlas of coordinate charts. We leave it to the reader to check that this atlas indeed define on $\mathbb{R}P^n$ a structure of a smooth manifold, i.e. that the transition maps between different coordinate charts are smooth.

**Exercise 4.7.2.** Let us view $S^n$ as the unit sphere in $\mathbb{R}^{n+1}$. Consider a map $p : S^n \to \mathbb{R}P^n$ which associates to a point of $S^n$ the line passing through this point and the origin. Prove that this two-to-one map is smooth, and moreover a *local diffeomorphism*, i.e. that the restriction of $p$ to a sufficiently small neighborhood of each point is a diffeomorphism. Use it to show that $\mathbb{R}P^n$ is diffeomorphic to the quotient space $S^n/f$, where $f : S^n \to S^n$ is the antipodal map $f(x) = -x$.

3. **Products of manifolds and $n$-dimensional tori.** Given two manifolds, $M$ and $N$ of dimension $m$ and $n$, respectively, one can naturally endow the *direct product*

$$M \times N = \{(x, y);\ x \in M, y \in N\}$$

with a structure of a manifold of dimension $m+n$. Let $\{(U_\lambda, \Phi_\lambda)\}_{\lambda \in \Lambda}$ and $\{(V_\gamma, \Psi_\gamma)\}_{\gamma \in \Gamma}$ be atlases for $M$ and $N$, so that $\Phi_\lambda : U_\lambda \to U'_\lambda \subset \mathbb{R}^m$, $\Psi_\mu : V_\mu \to V'_\mu \subset \mathbb{R}^n$ are diffeomorphisms on open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$. Then the smooth structure on $M \times N$ can be defined by an atlas

$$\{(U_\lambda \times V_\gamma, \Phi_\lambda \times \Psi_\gamma)\}_{\lambda \in \Lambda, \gamma \in \Gamma},$$

where we denote by $\Phi_\lambda \times \Psi_\gamma : U_\lambda \times V_\gamma \to U'_\lambda \times V'_\gamma \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ are diffeomorphisms defined by the formula $(x, y) \mapsto (\Phi_\lambda(x)\Psi_\mu(y))$ for $x \in U_\lambda$ and $y \in V_\mu$.

One can similarly define the direct product of any finite number of smooth manifolds. In particular the *$n$-dimensional torus* $T^n$ is defined as the product of $n$ circles: $T^n =$

$\underbrace{S^1 \times \cdots \times S^1}_{n}$. Let us recall, that the circle $S^1$ is diffeomorphic to $\mathbb{R}/T$, i.e. a point of $S^1$ is a real number up to adding any integer. Hence, the points of the torus $T^n$ can viewed as the points of $\mathbb{R}^n$ up to adding any vector with all integer coordinates.

### 4.7.4 Submanifolds of an $n$-dimensional vector space

Let $V$ be an $n$-dimensional vector space. A subset $A \subset V$ is called a *$k$-dimensional submanifold* of $V$, or simply a *$k$-submanifold* of $V$, $0 \le k \le n$, if for any points $a \in A$ there exists a local coordinate chart $(U_a, u = (u_1, \dots, u_n) \to \mathbb{R}^n)$ such that $u(a) = 0$ (i.e. the point $a$ is the origin in this coordinate system) and

$$A \cap U_a = \{u = (u_1, \dots, u_n) \in U_a;\ u_{k+1} = \cdots = u_n = 0\}. \tag{4.34}$$

We will always assume the local coordinates at least as smooth as necessary for our purposes (but at least $C^1$-smooth), but more precisely, one can talk of $C^m$- submanifolds if the implied coordinate systems are at least $C^m$-smooth.

**Example 4.7.3.** *Suppose a subset $A \subset U \subset V$ is given by equations $F_1 = \cdots = F_{n-k} = 0$ for some $C^m$-smooth functions $F_1, \dots, F_{n-k}$ on $U$. Suppose that for any point $a \in A$ the differentials $d_aF_1, \dots, d_aF_{n-k} \in V_a^*$ are linearly independent. Then $A \subset U$ is a $C^m$-smooth submanifold.*

Indeed, for each $a \in A$ one can choose a linear functions $l_1, \dots, l_k \in V_a^*$ such that together with $d_aF_1, \dots, d_aF_{n-k} \in V_a^*$ they form a basis of $V^*$. Consider functions $L_1, \dots, L_k : V \to \mathbb{R}$, defined by $L_j(x) = l_j(x - a)$ so that $d_a(L_j) = l_j$, $j = 1, \dots, k$. Then the Jacobian $\det D_aF$ of the map $F : (L_1, \dots, L_k, F_1, \dots, F_{n-k}) : U \to \mathbb{R}^n$ does not vanish at $a$, and hence the inverse function theorem implies that this map is invertible in a smaller neighborhood $U_a \subset U$ of the point $a \in A$. Hence, the functions $u_1 = L_1, \dots, u_k = L_k, u_{k+1} = F_1, \dots, u_n = F_{n-k}$ can be chosen as a local coordinate system in $U_a$, and thus $A \cap U_a = \{u_{k+1} = \cdots = u_n = 0\}$.

Note that the map $u' = (u_1, \dots, u_k)$ maps $U_a^A = U_a \cap A$ onto an open neighborhood $\widetilde{U} = u(U_a) \cap \mathbb{R}^k$ of the origin in $\mathbb{R}^k \subset \mathbb{R}^n$, and therefore $u' = (u_1, \dots u_k)$ defines a local coordinates, so that the pair $(U_a^A, u')$ is a *coordinate chart*. The restriction $\widetilde{\phi} = \phi|_{\widetilde{U}}$ of the parameterization map $\phi = u^{-1}$ maps $\widetilde{U}$ onto $U_a^A$. Thus $\widetilde{\phi}$ a *parameterization* map for the neighborhood $U_a^A$. The atlas $\{(U_a^A, u')\}a \in A$ defines on $a$ a structure of a $k$-dimensional manifold. The complementary dimension $n-k$ is called the *codimension* of the submanifold $A$. We will denote dimension and codimension of $A$ by $\dim A$ and $\operatorname{codim} A$, respectively.

As we already mentioned above in Section 4.2 1-dimensional submanifolds are usually called curves. We will also call 2-dimensional submanifolds *surfaces* and codimension 1 submanifolds *hypersurfaces*. Sometimes $k$-dimensional submanifolds are called $k$-surfaces. Submanifolds of codimension 0 are open domains in $V$.

An important class form *graphical* $k$-submanifolds. Let us recall that given a map $f : B \to \mathbb{R}^{n-k}$, where $B$ is a subset $B \subset \mathbb{R}^k$, then *graph* is the set

$$\Gamma_f = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n;\ x \in B, y = f(x)\}.$$

A $(C^m)$-submanifold $A \subset V$ is called *graphical* with respect to a splitting $\Phi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to V$, if there exist an open set $U \subset \mathbb{R}^k$ and a $(C^m)$-smooth map $f : U \to \mathbb{R}^{n-k}$ such that

$$A = \Phi(\Gamma_f).$$

In other words, $A$ is graphical if there exists a coordinate system in $V$ such that

$$A = \{x = (x_1, \dots, x_n); (x_1, \dots x_k) \in U, x_j = f_j(x_1, \dots, x_k),\ j = k+1, \dots, n\}.$$

for some open set $U \subset \mathbb{R}^k$ and smooth functions, $f_{k+1}, \dots, f_n : U \to \mathbb{R}$.

For a graphical submanifold there is a global coordinate system given by the projection of the submanifold to $\mathbb{R}^k$.

It turns out that that *any submanifolds locally is graphical.*

**Proposition 4.7.4.** *Let $A \subset V$ be a submanifold. Then for any point $a \in A$ there is a neighborhood $U_a \ni a$ such that $U_a \cap A$ is graphical with respect to a splitting of $V$. (The splitting may depend on the point $a \in A$).*

We leave it to the reader to prove this proposition using the implicit function theorem.

One can generalize the discussion in this section and define *submanifolds of any manifold $M$, and not just the vector space $V$*. In fact, the definition (4.34) can be used without any changes to define submanifolds of an arbitrary smooth manifold.

A map $f : M \to Q$ is called an *embedding* of a manifold $M$ into another manifold $Q$ if it is a diffeomorphism of $M$ onto a submanifold $A \subset Q$. In other words, $f$ is an embedding if the image $A = f(M)$ is a submanifold of $Q$ and the map $f$ viewed as a map $M \to A$ is a diffeomorphism. One can prove that any $n$-dimensional manifold can be embedded into $\mathbb{R}^N$ with a sufficiently large $N$ (in fact $N = 2n + 1$ is always sufficient).

Hence, one can think of *manifold* as *submanifold of some* $\mathbb{R}^n$ given up to a diffeomorphism, i.e. ignoring how this submanifold is embedded in the ambient space.

In the exposition below we mostly restrict our discussion to submanifolds of $\mathbb{R}^n$ rather than general abstract manifolds.

### 4.7.5 Submanifolds with boundary

A slightly different notion is of a *submanifold with boundary.* A subset $A \subset V$ is called a *$k$-dimensional submanifold with boundary*, or simply a *$k$-submanifold of $V$ with boundary*, $0 \leq k < n$, if for any points $a \in A$ there is a neighborhood $U_a \ni a$ in $V$ and local (curvilinear) coordinates $(u_1, \dots, u_n)$ in $U_a$ with the origin at $a$ if *one of two conditions* is satisfied: condition (4.34), or the following condition

$$A \cap U_a = \{u = (u_1, \dots, u_n) \in U_a;\ u_1 \geq 0, u_{k+1} = \dots = u_n = 0\}. \tag{4.35}$$

In the latter case the point $a$ is called a *boundary* point of $A$, and the set of all boundary points is called the *boundary* of $A$ and is denoted by $\partial A$.

**Exercise 4.7.5.** *Prove that if $A$ is $k$-submanifold with boundary then $\partial A$ is a $(k-1)$-dimensional submanifold (without boundary).*

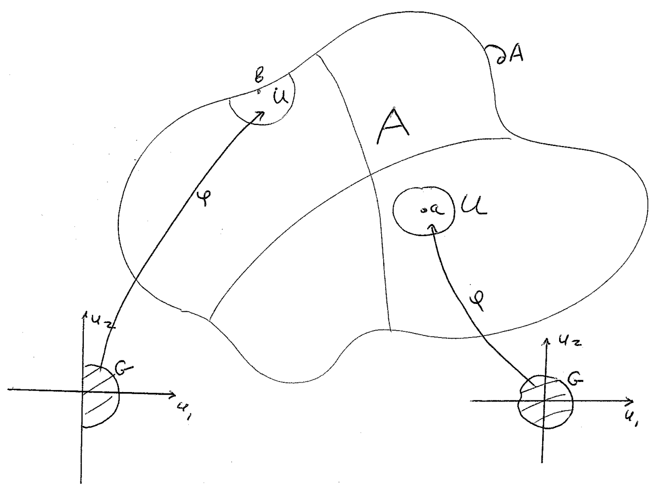


Figure 2: The parameterization $\phi$ introducing local coordinates near an interior point $a$ and a boundary point $b$.

**Remark 4.7.6.**

1. A slightly awkward nuance in the above definition is that a submanifold with boundary is not a submanifold! It would be, probably, less confusing to write this as a 1 word *submanifold-with-boundary*, but of course nobody does that.

2. As it was already pointed out when we discussed 1-dimensional submanifolds with boundary, the boundary of a $k$-submanifold with boundary is not the same as its set-theoretic boundary, though traditionally the same notation $\partial A$ is used. Usually this should be clear from the context, what the notation $\partial A$ stands for in each concrete case. We will explicitly point this difference out when it could be confusing.

A compact manifold (without boundary) is called *closed*. The boundary of any compact manifold with boundary is closed, i.e. $\partial(\partial A) = \varnothing$.

**Example 4.7.7.** An open ball $B^n_r = B^n_R(0) = \{\sum\limits_1^n x_j^2 < 1\} \subset \mathbb{R}^n$ is a codimension 0 submanifold, A closed ball $D^n_r = D^n_R(0) = \{\sum\limits_1^n x_j^2 \le 1\} \subset \mathbb{R}^n$ ia codimension 0 submanifold with boundary. Its boundary $\partial D^n_R$ is an $(n-1)$-dimensional sphere $S^{n-1}_R = \{\sum\limits_1^n x_j^2 = 1\} \subset \mathbb{R}^n$. It is a closed hypersurface. For $k = 0, 1 \dots n-1$ let us denote by $L^k$ the subspace $L_k = \{x_{k+1} = \dots = x_n = 0\} \subset \mathbb{R}^n$. Then the intersections

$$B^k_R = B^n_R \cap L^k, D^k_R = D^n_R \cap L^k, \text{ and } S^{k-1}_R = S^{n-1}_R \cap L^k \subset \mathbb{R}^n$$

are, respectively a $k$-dimensional submanifold, a $k$-dimensional submanifold with boundary and a closed $(k-1)$-dimensional submanifold of $\mathbb{R}^n$. Among all above examples there is only one (which one?) for which the manifold boundary is the same as the set-theoretic boundary.

A neighborhood of a boundary point $a \in \partial A$ can be always locally parameterized by the semi-open upper-half ball

$$B_+(0) = \{x = (x_1, \dots, x_k) \in \mathbb{R}^k;\ x_1 \geq 0, \sum_1^n x_j^2 < 1\}.$$

We will finish this section by defining submanifolds with *piece-wise smooth boundary*. A subset $A \subset V$ is called a $k$-dimensional submanifold of $V$ with *piecewise smooth boundary* or with *boundary with corners*, $0 \leq k < n$, if for any points $a \in A$ there is a neighborhood $U_a \ni a$ in $V$ and local (curvi-linear) coordinates $(u_1, \dots, u_n)$ in $U_a$ with the origin at $a$ if one of three condittions satisfied: conditions (4.34), (4.35) or the following condition

$$A \cap U_a = \{u = (u_1, \dots, u_n) \in U_a;\ l_1(u) \geq 0, \dots, l_m(u) \geq 0, u_{k+1} = \dots = u_n = 0\}, \quad (4.36)$$

where $m > 1$ and $l_1, \dots, l_m \in (\mathbb{R}^k)^*$ are linear functions In the latter case the point $a$ is called a *corner* point of $\partial A$.

Note that the system of linear inequalities $l_1(u) \geq 0, \dots, l_m(u) \geq 0$ defines a *convex cone* in $\mathbb{R}^k$. Hence, near a corner point of its boundary the manifold is diffeomorphic to a convex cone. Thus convex polyhedra and their diffeomorphic images are important examples of submanifolds with boundary with corners.

## 4.8 Tangent spaces and differential

Suppose we are given two local parameterizations $\phi : G \to A$ and $\widetilde{\phi} : \widetilde{G} \to A$. Suppose that $0 \in G \cap \widetilde{G}$ and $\phi(0) = \widetilde{\phi}(0) = a \in A$. Then there exists a neighborhood $U \ni a$ in $A$ such that $U \subset \phi(G) \cap \widetilde{\phi}(\widetilde{G})$.

Denote $G_1 := \phi^{-1}(U), \widetilde{G}_1 := \widetilde{\phi}^{-1}(\widetilde{U})$. Then one has two coordinate charts on $U$: $u = (u_1, \dots, u_k) = (\phi|_{G_1})^{-1} : U \to G_1$, and $\widetilde{u} = (\widetilde{u}_1, \dots, \widetilde{u}_k) = \left(\widetilde{\phi}|_{\widetilde{G}_1}\right)^{-1} : U \to \widetilde{G}_1$.

Denote $h := u \circ \widetilde{\phi}|_{\widetilde{G}_1} = \widetilde{G}_1 \to G_1$. We have

$$\widetilde{\phi} = \phi \circ u \circ \widetilde{\phi} = \phi \circ h,$$

and hence the differentials $d\phi_0$ and $d\widetilde{\phi}_0$ of parameterizations $\phi$ and $\widetilde{\phi}$ at the origin map $\mathbb{R}^k_0$ isomorphically onto the same $k$-dimensional linear subspace $T \subset V_a$. Indeed, $d_0\widetilde{\phi} = d_0\phi \circ d_0 h$. Thus the space $T = d_0\phi(\mathbb{R}^k_0) \subset V_a$ is independent of the choice of parameterization. It is called the *tangent space* to the submanifold $A$ at the point $a \in A$ and will be denoted by $T_aA$. If $A$ is a submanifold with boundary and $a \in \partial A$ then there are defined both the $k$-dimensional tangent space $T_aA$ and its $(k-1)$-dimensional subspace $T_a(\partial A) \subset T_aA$ tangent to the boundary.

**Example 4.8.1.**

1. Suppose a submanifold $A \subset V$ is globally parameterized by an embedding $\phi : G \to A \hookrightarrow V$, $G \subset \mathbb{R}^k$. Suppose the coordinates in $\mathbb{R}^k$ are denoted by $(u_1, \dots, u_k)$. Then the tangent space $T_aA$ at a point $a = \phi(b), b \in G$ is equal to the span
$$\mathrm{Span}\left(\frac{\partial \phi}{\partial u_1}(a), \dots, \frac{\partial \phi}{\partial u_k}(a)\right).$$

2. In particular, suppose a submanifold $A$ is graphical and given by equations
$$x_{k+1} = g_1(x_1, \dots, x_k), \dots, x_n = g_{n-k}(x_1, \dots, x_k), \ \ (x_1, \dots, x_k) \in G \subset \mathbb{R}^k.$$
Take points $b = (b_1, \dots b_k) \in G$ and $a = (b_1, \dots, b_k, g_1(b), \dots, g_{n-k}(b)) \in A$. Then

$T_aA = \mathrm{Span}\,(T_1, \dots T_k)\,,$ where

$$T_1 = \left( \underbrace{1, 0, \dots, 0}_{k}, \frac{\partial g_1}{\partial x_1}(b), \dots, \frac{\partial g_{n-k}}{\partial x_1}(b) \right),$$

$$T_1 = \left( \underbrace{0, 1, \dots, 0}_{k}, \frac{\partial g_1}{\partial x_2}(b), \dots, \frac{\partial g_{n-k}}{\partial x_2}(b) \right),$$

$$\dots$$

$$T_1 = \left( \underbrace{0, 0, \dots, 1}_{k}, \frac{\partial g_1}{\partial x_k}(b), \dots, \frac{\partial g_{n-k}}{\partial x_k}(b) \right).$$

3. Suppose a hypersurface $\Sigma \subset \mathbb{R}^n$ is given by an equation $\Sigma = \{F = 0\}$ for a smooth function $F$ defined on an neighborhood of $\Sigma$ and such that $d_aF \neq 0$ for any $a \in \Sigma$. In other words, the function $F$ has no critical points on $\Sigma$. Take a point $a \in \Sigma$. Then $T_a\Sigma \subset \mathbb{R}^n_a$ is given by a linear equation

$$\sum_1^n \frac{\partial F}{\partial x_j}(a) h_j = 0, \quad h = (h_1, \dots, h_n) \in \mathbb{R}^n_a.$$

Note that sometimes one is interested to define $T_a\Sigma$ as an affine subspace of $\mathbb{R}^n = \mathbb{R}^n_0$ and not as a linear subspace of $\mathbb{R}^n_a$. We get the required equation by shifting the origin:

$$T_a\Sigma = \{x = (x_1, \dots, x_n) \subset \mathbb{R}^n;\ \sum_1^n \frac{\partial F}{\partial x_j}(a)(x_j - a_j) = 0\}.$$

If for some parameterization $\phi : G \to A$ with $\phi(0) = a$ the composition $f \circ \phi$ is differentiable at 0, and the linear map

$$d_0(f \circ \phi) \circ (d_0\phi)^{-1} : T_aA \to W_{f(a)}$$

is called the *differential* of $f$ at the point $a$ and denoted, as usual, by $d_af$. Similarly one can define $C^m$-smooth maps $A \to W$.

**Exercise 4.8.2.** Show that a map $f : A \to W$ is differentiable at a point $a \in A$ iff for some neighborhood $U$ of $a$ in $V$ there exists a map $F : U \to W$ that is differentiable at $a$ and such that $F|_{U\cap A} = f|_{U\cap A}$, and we have $dF|_{T_aA} = d_a f$. As it follows from the above discussion the map $dF|_{T_aA}$ is independent of this extension. Similarly, any $C^m$-smooth map of $A$ locally extends to a $C^m$-smooth map of a neighborhood of $A$ in $V$.

Suppose that the image $f(A)$ of a smooth map $f : A \to W$ is contained in a submanifold $B \subset W$. In this case the image $d_af(T_aA)$ is contained in $T_{f(a)}B$. Hence, given a smooth map $f : A \to B$ between two submanifolds $A \subset V$ and $B \subset W$ its differential at a point $a$ can be viewed as a linear map $d_af : T_aA \to T_{f(a)}B$.

Let us recall, that given two submanifolds $A \subset V$ and $B \subset W$ (with or without boundary), a smooth map $f : A \to B$ is called a diffeomorphism if there exists a smooth inverse map $: B \to A$, i.e. $f \circ g : B \to B$ and $g \circ f : A \to A$ are both identity map. The submanifolds $A$ and $B$ are called in this case diffeomorphic.

**Exercise 4.8.3.** 1. Let $A, B$ be two diffeomorphic submanifolds. Prove that

   (a) if $A$ is path-connected then so is $B$;

   (b) if $A$ is compact then so is $B$;

   (c) if $\partial A = \varnothing$ then $\partial B = \varnothing$;

   (d) $\dim A = \dim B$; [8]

2. Give an example of two diffeomorphic submanifolds, such that one is bounded and the other is not.

3. Prove that any closed connected 1-dimensional submanifold is diffeomorphic to the unit circle $S^1 = \{x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$.

## 4.9 Vector bundles and their homomorphisms

Let us put the above discussion in a bit more global and general setup.

[8] In fact, we will prove later that even *homeomorphic* manifolds should have the same dimension.

A collection of all tangent spaces $\{T_aA\}_{a\in A}$ to a submanifold $A$ is called its *tangent bundle* and denoted by $TA$ or $T(A)$. This is an example of a more general notion of a *vector bundle of rank* $r$ over a set $A \subset V$. One understands by this a family of $r$-dimensional vector subspaces $L_a \subset V_a$, parameterized by points of $A$ and *continuously* (or $C^m$-smoothly) depending on $a$. More precisely one requires that each point $a \in A$ has a neighborhood $U \subset A$ such that there exist linear independent vector fields $v_1(a), \dots, v_r(a) \in L_a$ which continuously (smoothly, etc.) depend on $a$.

Besides the tangent bundle $T(A)$ over a $k$-submanifold $A$ an important example of a vector bundle over a submanifold $A$ is its *normal bundle* $NA = N(A)$, which is a vector bundle of rank $n-k$ formed by orthogonal complements $N_aA = T_a^\perp A \subset V_a$ of the tangent spaces $T_aA$ of $A$. We assume here that $V$ is Euclidean space.

A vector bundle $L$ of rank $k$ over $A$ is called *trivial* if one can find $k$ continuous linearly independent vector fields $v_1(a), \dots, v_k(a) \in L_a$ defined for *all* $a \in A$. The set $A$ is called the *base* of the bundle $L$.

An important example of a trivial bundle is the bundle $TV = \{V_a\}_{a\in V}$.

**Exercise 4.9.1.** Prove that the tangent bundle to the unit circle $S^1 \subset \mathbb{R}^2$ is trivial. Prove that the tangent bundle to $S^2 \subset \mathbb{R}^3$ is not trivial, but the tangent bundle to the unit sphere $S^3 \subset \mathbb{R}^4$ is trivial. (The case of $S^1$ is easy, of $S^3$ is a bit more difficult, and of $S^2$ even more difficult. It turns out that the tangent bundle $TS^{n-1}$ to the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is trivial if and only if $n = 2, 4$ and 8. The *only if* part is a very deep topological fact which was proved by F. Adams in 1960.

Suppose we are given two vector bundles, $L$ over $A$ and $\widetilde{L}$ over $\widetilde{A}$ and a continuous (resp. smooth) map $\phi : A \to \widetilde{A}$. By a continuous (resp. smooth) *homomorphism* $\Phi : L \to \widetilde{L}$ which *covers the map* $\phi : A \to \widetilde{A}$ we understand a continuous (resp. smooth) family of linear maps $\Phi_a : L_a \to \widetilde{L}_{\phi(a)}$. For instance, a $C^m$-smooth map $f : A \to B$ defines a $C^{m-1}$-smooth homomorphism $df : TA \to TB$ which covers $f : A \to B$. Here $df = \{d_af\}_{a\in A}$ is the family of linear maps $d_af : T_aA \to T_{f(a)}B$, $a \in A$.

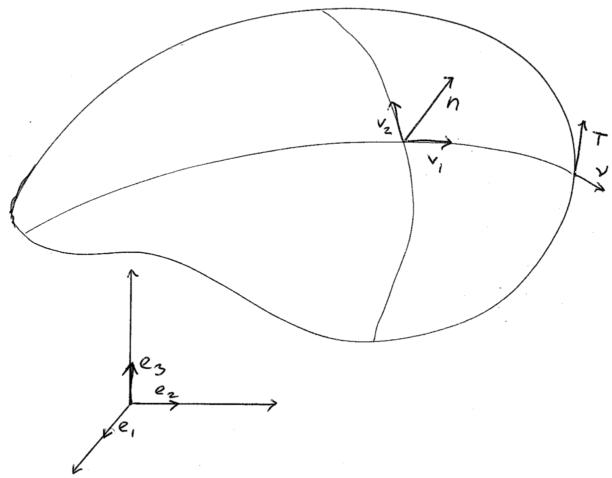


Figure 3: The orientation of the surface is induced by its co-orientation by the normal vector **n**. The orientation of the boundary us induced by the orientation of the surface.

## 4.10 Orientation

By an *orientation* of a vector bundle $L = \{L_a\}_{a \in A}$ over $A$ we understand continuously depending on $a$ orientation of all vector spaces $L_a$. An orientation of a submanifold $k$ is the same as an orientation of its tangent bundle $T(A)$. A *co-orientation* of a $k$- submanifold $A$ is an orientation of its normal bundle $N(A) = T^{\perp}A$ in $V$. Note that not all bundles are *orientable*, i.e. some bundles admit no orientation. But if $L$ is orientable and the base $A$ is connected, then $L$ admits exactly two orientations. Here is a simplest example of a non-orientable rank 1 bundle of the circle $S^1 \subset \mathbb{R}^2$. Let us identify a point $a \in S^1$ with a complex number $a = e^{i\phi}$, and consider a line $l_a \in \mathbb{R}^2_a$ directed by the vector $e^{\frac{i\phi}{2}}$. Hence, when the point completes a turn around $S^1$ the line $l_a$ rotates by the angle $\pi$. We leave it to the reader to make a precise argument why this bundle is not orientable . In fact, rank 1 bundles are orientable if and only if they are trivial.

If the ambient space $V$ is oriented then co-orientation and orientation of a submanifold $A$ determine each other according to the following rule. For each point $a$, let us choose any basis $v_1, \dots, v_k$ of $T_a(A)$ and any basis $w_1, \dots, w_{n-k}$ of $N_a(A)$. Then $w_1, \dots, w_{n-k}, v_1, \dots, v_k$ is a basis of $V_a = V$. Suppose one of the bundles, say $N(A)$, is oriented. Let us assume that the basis $w_1, \dots, w_{n-k}$ defines this orientation. Then we orient $T_aA$ by the basis $v_1, \dots, v_k$ if the basis $w_1, \dots, w_{n-k}, v_1, \dots, v_k$ defines the given orientation of $V$, and we pick the opposite orientation of $T_aA$ otherwise.

**Example 4.10.1. (Induced orientation of the boundary of a submanifold.)** Suppose $A$ is an oriented manifold with boundary. Let us co-orient the boundary $\partial A$ by orienting the rank 1 normal bundle to $T(\partial A)$ in $T(A)$ by the unit ourtward normal to $T(\partial A)$ in $T(A)$ vector field. Then the above rule determine an orientation of $T(\partial A)$, and hence of $\partial A$.

## 4.11 Partition of unity and cut-off functions

Let us recall that the *support of a function* $\theta$ is the closure of the set of points where it is not equal to 0. We denote the support by $\mathrm{Supp}(\theta)$. We say that $\theta$ is *supported* in an open set $U$ if $\mathrm{Supp}(\theta) \subset U$.

**Lemma 4.11.1.** *There exists a $C^\infty$ function $\rho : \mathbb{R} \to [0, \infty)$ with the following properties:*

- $\rho(x) \equiv 0,\ |x| \geq 1$;
- $\rho(x) = \rho(-x)$;
- $\rho(x) > 0$ *for* $|x| < 1$.

**Proof.** There are a lot of functions with this property. For instance, one can be constructed as follows. Take the function

$$h(x) = \begin{cases} e^{-\frac{1}{x^2}} & , \quad x > 0 \\ 0 & , \quad x \leq 0. \end{cases} \tag{4.37}$$

The function $e^{-\frac{1}{x^2}}$ has the property that all its derivatives at 0 are equal to 0, and hence the function $h$ is $C^\infty$-smooth. Then the function $\rho(x) := h(1+x)h(1-x)$ has the required properties. ■

**Lemma 4.11.2.** Existence of cut-off functions *Let $C \subset V$ be compact set and $U \supset C$ its open neighborhood. Then there exists a $C^\infty$-smooth function $\sigma_{C,U} : V \to [0,\infty)$ with its support in $U$ which is equal to 1 on $C$*

**Proof.** Let us fix a Euclidean structure in $V$ and a Cartesian coordinate system. Thus we can identify $V$ with $\mathbb{R}^n$ with the standard dot-product. Given a point $a \in V$ and $\delta > 0$ let us denote by $\psi_{a,\delta}$ the *bump* function on $V$ defined by

$$\psi_{a,\delta}(x) := \rho\left(\frac{||x-a||^2}{\delta^2}\right), \tag{4.38}$$

where $\rho : \mathbb{R}\vec{}[0,\infty)$ is the function constructed in Lemma 4.11.1. Note that $\psi_{a,\delta}(x)$ is a $C^\infty$-function with $\mathrm{Supp}\,(\psi_{a,\delta}) = D_\delta := \overline{B_\delta(a)}$ and such that $\psi_{a,\delta}(x) > 0$ for $x \in B_\delta(a)$.

Let us denote by $U_\epsilon(C)$ the $\epsilon$-neighborhood of $C$, i.e.

$$U_\epsilon(C) = \{x \in V; \exists y \in C, ||y-x|| < \epsilon.\}$$

There exists $\epsilon > 0$ such that $U_\epsilon(C) \subset U$. Using compactness of $C$ we can find finitely many points $z_1,\dots,z_N \in C$ such that the balls $B_\epsilon(z_1),\dots,B_\epsilon(z_N) \subset U$ cover $\overline{U_{\frac{\epsilon}{2}}(C)}$, i.e. $\overline{U_{\frac{\epsilon}{2}}}(C) \subset \bigcup\limits_1^N B_\epsilon(z_j)$. Consider a function

$$\sigma_1 := \sum_1^N \psi_{z_i,\frac{\epsilon}{2}} : V \to \mathbb{R}.$$

The function $\psi_1$ is positive on $\overline{U}_{\frac{\epsilon}{2}}(C)$ and has $\mathrm{Supp}(\psi_1) \subset U$.

The complement $E = V \setminus U_{\frac{\epsilon}{2}}(C)$ is a closed but unbounded set. Take a large $R > 0$ such that $B_R(0) \supset \overline{U}$. Then $E_R = D_R(0) \setminus U_{\frac{\epsilon}{2}}(C)$ is compact. Choose finitely many points $x_1,\dots,x_M \in E_R$ such that $\bigcup\limits_1^M B_{\frac{\epsilon}{4}}(x_i) \supset E_R$. Notice that $\bigcup\limits_1^M B_{\frac{\epsilon}{4}}(x_i) \cap C = \varnothing$. Denote

$$\sigma_2 := \sum_1^M \psi_{x_i,\frac{\epsilon}{4}}.$$

Then the function $\sigma_2$ is positive on $V_R$ and vanishes on $C$. Note that the function $\sigma_1 + \sigma_2$ is positive on $B_R(0)$ and it coincides with $\sigma_1$ on $C$. Finally, define the function $\sigma_{C,U}$ by the formula

$$\sigma_{C,U} := \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

on $B_(R)(0)$ and extend it to the whole space $V$ as equal to 0 outside the ball $B_R(0)$. Then $\sigma_{C,U} = 1$ on $C$ and $\mathrm{Supp}(\sigma_{C,U}) \subset U$, as required. ■.

Let $C \subset V$ be a compact set. Consider its finite covering by open sets $U_1, \dots, U_N$, i.e.

$$\bigcup_1^N U_j \supset C.$$

We say that a finite sequence $\theta_1, \dots, \theta_K$ of $C^\infty$-functions defined on some open neighborhood $U$ of $C$ in $V$ forms a *partition of unity* over $C$ subordinated to the covering $\{U_j\}_{j=1,\dots,N}$ if

- $\sum\limits_1^K \theta_j(x) = 1$ for all $x \in C$;
- Each function $\theta_j$, $j = 1, \dots, K$ is supported in one of the sets $U_i$, $i = 1, \dots, K$.

**Lemma 4.11.3.** *For any compact set $C$ and its open covering $\{U_j\}_{j=1,\dots,N}$ there exists a partition of unity over $C$ subordinated to this covering.*

**Proof.** In view of compactness of there exists $\epsilon > 0$ and finitely many balls $B_\epsilon(z_j)$ centered at points $z_j \in C$, $j = 1, \dots, K$, such that $\bigcup\limits_1^K B_\epsilon(z_j) \supset C$ and each of these balls is contained in one of the open sets $U_j$, $j = 1, \dots, N$. Consider the functions $\psi_{z_j,\epsilon}$ defined in (4.38). We have $\sum\limits_1^K \psi_{z_j,\epsilon} > 0$ on some neighborhood $U \supset C$. Let $\sigma_{C,U}$ be the cut-off function constructed in Lemma 4.11.2. For $j = 1, \dots, K$ we define

$$\theta_j(x) = \begin{cases} \frac{\psi_{z_j,\epsilon}(x)\sigma_{C,U}(x)}{\sum\limits_1^K \psi_{z_j,\epsilon}(x)}, & \text{if } x \in U, \\ 0, & \text{otherwise} \end{cases}.$$

Each of the functions is supported in one of the open sets $U_j$, $j = 1, \dots, N$, and we have for every $x \in C$

$$\sum_1^K \theta_j(x) = \frac{\sum\limits_1^K \psi_{z_j,\epsilon}(x)}{\sum\limits_1^K \psi_{z_j,\epsilon}(x)} = 1.$$

■

## 4.12 Approximation of continuous functions by smooth ones

**Theorem 4.12.1.** *Let $C \subset V$ be a compact domain with smooth boundary. Then any continuous function $f : C \to \mathbb{R}$ can be $C^0$-approximated by $C^\infty$- smooth functions, i.e. for any $\epsilon > 0$ there exists a $C^\infty$-smooth function $g : C \to \mathbb{R}$ such that $|f(x) - g(x)| < \epsilon$ for any $x \in C$. Moreover, if the function $f$ is already $C^\infty$-smooth in a neighborhood of a closed subset $B \subset \operatorname{Int} C$, then one can arrange that the function $g$ coincides with $f$ over $A$.*

**Lemma 4.12.2.** *There is a continuous extension of $f$ to $V$.*

**Sketch of the proof.** Let $\mathbf{n}$ be the outward normal vector field to the boundary $\partial C$. If the boundary is $C^\infty$-smooth then so is the vector field $\mathbf{n}$. Consider a map $\nu : \partial C \times [0,1] \to V$ given by the formula $\nu(x,t) = x + t\,\mathbf{n}, x \in \partial C, t \in [0,1]$. The differential of $\nu$ at the points of $\partial C \times 0$ has rank $n$. (*Exercise: prove this.*) Hence by the inverse function theorem for a sufficiently small $\epsilon > 0$ the map $\nu$ is a diffeomorphism of $\partial C \times [0, \epsilon)$ onto $U \setminus \operatorname{Int} C$ for some open neighborhood $U \supset C$. Consider a function $F : \partial C \times [0, \epsilon) \to \mathbb{R}$, defined by the formula

$$F(x,t) = \left(1 - \frac{2t}{\epsilon}\right) f(x)$$

if $t \in [0, \frac{\epsilon}{2}]$ and $f(x,t) = 0$ if $t \in (\frac{\epsilon}{2}, \epsilon)$. Now we can extend $f$ to $U$ by the formula $f(y) = F\nu^{-1}(y)$ if $y \in U \setminus C$, and setting it to 0 outside $U$. ■

Consider the function

$$\Psi_\epsilon = \frac{1}{\int\limits_{D_\epsilon(0)} \psi_{0,\epsilon} dV} \psi_{0,\epsilon},$$

where $\psi_{0,\sigma}$ is a bump function defined above in (4.38). It is supported in the disc $D_\sigma(0)$, non-negative, and satisfies

$$\int\limits_{D_\sigma(0)} \Psi_\sigma dV = 1.$$

. Given a continuous function $f : V \to \mathbb{R}$ we define a function $f_\sigma : V \to \mathbb{R}$ by the formula

$$f_\sigma(x) = \int f(x-y)\Psi_\sigma(y)d^n y. \tag{4.39}$$

Then

**Lemma 4.12.3.** *1. The function $f_\sigma$ is $C^\infty$-smooth.*

*2. For any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in C$ we have $|f(x) - f_\sigma(x)| < \epsilon$ provided that $\sigma < \delta$.*

**Proof.** 1. By the change of variable formula we have, replacing the variable $y$ by $u = y - x$:

$$f_\sigma(x) = \int\limits_{D_\epsilon(0)} f(x-y)\Psi_\sigma(y)d^n y = \int\limits_{D_\epsilon(-x)} f(-u)\Psi_\sigma(x+u)d^n u = \int\limits_V f(-u)\Psi_\sigma(x+u)d^n u.$$

But the expression under the latter integral depends on $x$ $C^\infty$-smoothly as a parameter. Hence, by the theorem about differentiating integral over a parameter, we conclude that the function $f_\epsilon$ in $C^\infty$-smooth.

2. Fix some $\sigma_0 > 0$. The function $f$ is uniformly continuous in $\overline{U_{\sigma_0}(C)}$. Hence there exists $\delta > 0$ such that $x, x' \in \overline{U_{\sigma_0}(C)}$ and $||x - x'|| < \delta$ we have $|f(x) - f(x')| < \epsilon$. Hence, for $\sigma < \min(\sigma_0, \delta)$ and for $x \in C$ we have

$$\begin{aligned}|f_\sigma(x) - f(x)| = |\int\limits_{D_\epsilon(0)} f(x-y)\Psi_\sigma(y)d^n y - \int\limits_{D_\epsilon(0)} f(x)\Psi_\sigma(y)d^n y| \le \\ \int\limits_{D_\epsilon(0)} |f(x-y) - f(x)|\Psi_\sigma(y)d^n y \le \epsilon \int\limits_{D_\epsilon(0)} \Psi_\sigma(y)d^n y = \epsilon.\end{aligned} \tag{4.40}$$

**Proof of Theorem 4.12.1.** Lemma 4.12.3 implies that for a sufficiently small $\sigma$ the function $g = f_\sigma$ is the required $C^\infty$-smooth $\epsilon$-approximation of the continuous function $f$. To prove the second part of the theorem let us assume that $f$ is already $C^\infty$-smooth on a neighborhood $U$, $B \subset U \subset C$. Let us choose a cut-off function $\sigma_{B,U}$ constructed in Lemma 4.11.2 and define the required approximation $g$ by the formula $f_\sigma + (f - f_\sigma)\sigma_{B,U}$. ■

Theorem 4.12.1 implies a similar theorem form continuous maps $C \to \mathbb{R}^n$ by applying it to all coordinate functions.

## 4.13 Integration of differential $k$-forms over $k$-dimensional submanifolds

Let $\alpha$ be a differential $k$-form defined on an open set $U \subset V$.

Consider first a $k$-dimensional compact submanifold with boundary $A \subset U$ defined parametrically by an embedding $\phi : G \to A \hookrightarrow U$, where $G \subset \mathbb{R}^k$ is possibly with boundary. Suppose that $A$ is oriented by this embedding. Then we define

$$\int\limits_A \alpha := \int\limits_G \phi^* \alpha.$$

Note that if we define $A$ by a different embedding $\widetilde{\phi} : \widetilde{G} \to A$, then we have $\widetilde{\phi} = \phi \circ h$, where $h = \phi^{-1} \circ \widetilde{\phi} : \widetilde{G} \to G$ is a diffeomorphism. Hence, using Proposition 4.6.5 we get

$$\int\limits_{\widetilde{G}} \widetilde{\phi}^* \alpha = \int\limits_{\widetilde{G}} h^*(\phi^* \alpha) = \int\limits_G \phi^* \alpha,$$

and hence $\int\limits_A \alpha$ is independent of a choice of parameterization, provided that the orientation is preserved.

Let now $A$ be any compact oriented submanifold with boundary. Let us choose a partition of unity $1 = \sum\limits_1^K \theta_j$ in a neighborhood of $A$ such that each function is supported in some coordinate neighborhood of $A$. Denote $\alpha_j = \theta_j \alpha$. Then $\alpha = \sum\limits_1^K \alpha_j$, where each form $\alpha_j$ is supported in one of coordinate neighborhoods. Hence there exist orientation preserving embeddings $\phi_j : G_j \to A$ of domains with boundary $G_j \subset \mathbb{R}^k$, such that $\phi_j(G_j) \supset \mathrm{Supp}(\alpha_j)$, $j = 1, \dots, K$. Hence, we can define

$$\int\limits_A \alpha_j := \int\limits_{G_j} \phi_j^* \alpha_j \quad \text{and} \quad \int\limits_A \alpha := \sum_1^K \int\limits_A \alpha_j.$$

**Lemma 4.13.1.** *The above definition of* $\int\limits_A \alpha$ *is independent of a choice of a partition of unity.*

**Proof.** Consider two different partitions of unity $1 = \sum\limits_1^K \theta_j$ and $1 = \sum\limits_1^{\widetilde{K}} \widetilde{\theta}_j$ subordinated to coverings $U_1, \dots, U_K$ and $\widetilde{U}_1, \dots, \widetilde{U}_{\widetilde{K}}$, respectively. Taking the product of two partitions we get another partition $1 = \sum\limits_{i=1}^K \sum\limits_{j=1}^{\widetilde{K}} \theta_{ij}$, where $\theta_{ij} := \theta_i \theta_j$, which is subordinated to the covering by intersections $U_i \cap U_j$, $i = 1, \dots, K$, $j = 1, \dots, \widetilde{K}$. Denote $\alpha_i := \theta_i \alpha$, $\widetilde{\alpha}_j := \widetilde{\theta}_j \alpha$ and $\alpha_{ij} = \theta_{ij} \alpha$. Then $\sum\limits_{i=1}^K \alpha_{ij} = \widetilde{\alpha}_j$, $\sum\limits_{j=1}^{\widetilde{K}} \alpha_{ij} = \alpha_i$ and $\alpha = \sum\limits_1^K \alpha_i = \sum\limits_1^{\widetilde{K}} \widetilde{\alpha}_j$. Then, using the linearity of the integral we get

$$\sum_1^K \int\limits_A \alpha_i = \sum_{i=1}^K \sum_{j=1}^{\widetilde{K}} \int\limits_A \alpha_{ij} = \sum_1^{\widetilde{K}} \int\limits_A \widetilde{\alpha}_j.$$

■

When $k = 1$ the above definition of the integral coincides with the definition of the integral of a 1-form over an oriented curve which was given above in Section 4.1.

Let us extend the definition of integration of differential forms to an important case of integration of 0-form over oriented 0-dimensional submanifolds. Let us recall a compact oriented 0-dimensional submanifold of $V$ is just a finite set of points $a_1, \dots, a_m \in V$ with assigned signs to every point. So in view of the additivity of the integral it is sufficient to define integration over 1 point with a sign. On the other hand, a 0-form is just a function $f : V \to \mathbb{R}$. So we define

$$\int\limits_{\pm a} f := \pm f(a).$$

A partition of unity is a convenient tool for studying integrals, but not so convenient for practical computations. The following proposition provides a more practical method for computations.

**Proposition 4.13.2.** *Let $A$ be a compact oriented submanifold of $V$ and $\alpha$ a differential $k$-form given on a neighborhood of $A$. Suppose that $A$ presented as a union $A = \bigcup_1^N A_j$, where $A_j$ are codimension 0 submanifolds of $A$ with boundary with corners. Suppose that $A_i$ and $A_j$ for any $i \neq j$ intersect only along pieces of their boundaries. Then*

$$\int_A \alpha = \sum_1^N \int_{A_j} \alpha.$$

*In particular, if each $A_j$ is parameterized by an orientation preserving embedding $\phi_j : G_j \to A$, where $G_j \subset \mathbb{R}^k$ is a domain with boundary with corners. Then*

$$\int_A \alpha = \sum_1^N \int_{G_j} \phi_j^* \alpha.$$

We leave the proof to the reader as an exercise.

**Exercise 4.13.3.** *Compute the integral*

$$\int_S 1/3(x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2),$$

*where $S$ is the sphere*

$$\{x_1^2 + x_2^2 + x_3^2 = 1\},$$

*cooriented by its exterior normal vector.*

**Solution.** Let us present the sphere as the union of northern and southern hemispheres:

$$S = S_- \cup S_+, \quad \text{where } S_- = S \cap \{x_3 \leq 0\},\ S_+ = S \cap \{x_3 \geq 0\}.$$

Then $\int\limits_S \omega = \int\limits_{S^+} \omega + \int\limits_{S_-} \omega$. Let us first compute $\int\limits_{S^+} \omega$.

We can parametrize $S_+$ by the map $(u,v) \to (u,v,\sqrt{R^2-u^2-v^2})$, $(u,v) \in \{u^2+v^2 \leq R^2\} = \mathcal{D}_R$. One can check that this parametrization agrees with the prescribed coorientation of $S$. Thus, we have

$$\int\limits_{S_+} \omega = 1/3 \int\limits_{\mathcal{D}_R} \Big(udv \wedge d\sqrt{R^2-u^2-v^2} + vd\sqrt{R^2-u^2-v^2} \wedge du + \sqrt{R^2-u^2-v^2}du \wedge dv\Big).$$

Passing to polar coordinates $(r,\varphi)$ in the plane $(u,v)$ we get

$$\int\limits_{S_+} \omega = 1/3 \int\limits_P r\cos\varphi d(r\sin\varphi) \wedge d\sqrt{R^2-r^2} + r\sin\varphi d\sqrt{R^2-r^2} \wedge d(r\cos\varphi)$$

$$+\sqrt{R^2-r^2}d(r\cos\varphi) \wedge d(r\sin\varphi),$$

where $P = \{0 \leq r \leq R, 0 \leq \varphi \leq 2\pi\}$. Computing this integral we get

$$\begin{aligned}\int\limits_{S_+} \omega &= \frac{1}{3}\int_P -\frac{r^3\cos^2\varphi d\varphi \wedge dr}{\sqrt{R^2-r^2}} + \frac{r^3\sin^2\varphi dr \wedge d\varphi}{\sqrt{R^2-r^2}} + \sqrt{R^2-r^2}dr \wedge d\varphi \\ &= \frac{1}{3}\int_P \left(\frac{r^3}{\sqrt{R^2-r^2}} + r\sqrt{R^2-r^2}\right) dr \wedge d\varphi \\ &= \frac{2\pi}{3}\int_0^R \frac{rR^2}{\sqrt{R^2-r^2}}dr = -\frac{2\pi R^2}{3}\sqrt{R^2-r^2}\int_0^R = \frac{2\pi R^3}{3}\end{aligned}$$

Similarly, one can compute that

$$\int\limits_{S_-} \omega = \frac{2\pi R^3}{3}.$$

Computing this last integral, one should notice the fact that the parametrization

$$(u,v) \mapsto (u,v,-\sqrt{R^2-u^2-v^2})$$

defines the *wrong* orientation of $S_-$. Thus one should use instead the parametrization

$$(u,v) \mapsto (v,u,-\sqrt{R^2-u^2-v^2}),$$

and we get the answer

$$\int\limits_S \omega = \frac{4\pi R^3}{3}.$$

This is just the volume of the ball bounded by the sphere. The reason for such an answer will be clear below from Stokes' theorem.

# 5 Stokes' theorem

## 5.1 Statement of Stokes' theorem

**Theorem 5.1.1.** *Let $A \subset V$ be a compact oriented submanifold with boundary (and possibly with corners). Let $\omega$ be a $C^2$-smooth differential form defined on a neighborhood $U \supset A$. Then*

$$\int\limits_{\partial A} \omega = \int\limits_A d\omega \,.$$

Here $d\omega$ is the exterior differential of the form $\omega$ and $\partial A$ is the oriented boundary of $A$.

We will discuss below what exactly Stokes' theorem means for the case $k \leq 3$ and $n = \dim V \leq 3$.

Let us begin with the case $k = 1$, $n = 2$. Thus $V = \mathbb{R}^2$. Let $x_1, x_2$ be coordinates in $\mathbb{R}^2$ and $U$ a domain in $\mathbb{R}^2$ bounded by a smooth curve $\Gamma = \partial U$. Let us co-orient $\Gamma$ with the outward normal $\nu$ to the boundary of $U$. This defines a counter-clockwise orientation of $\Gamma$.

Let $\omega = P_1(x_1,x_2)dx_1 + P_2(x_1,x_2)dx_2$ be a differential 1-form. Then the above Stokes' formula asserts

$$\int\limits_U d\omega = \int\limits_\Gamma \omega \,,$$

or

$$\int\limits_U \left(\frac{\partial P_2}{\partial x_1} - \frac{\partial P_1}{\partial x_2}\right) dx_1 \wedge dx_2 = \int_\Gamma P_1 dx_1 + P_2 dx_2.$$

This is called *Green's formula.* In particular, when $d\omega = dx_1 \wedge dx_2$, e.g. $\omega = xdy$ or $\omega = \frac{1}{2}(xdy - ydx)$, the integral $\int\limits_\Gamma \omega$ computes the area of the domain $U$.

Consider now the case $n = 3$, $k = 2$. Thus

$$V = \mathbb{R}^3,\ \omega = P_1 dx_2 \wedge dx_3 + P_2 dx_3 \wedge dx_1 + P_3 dx_1 \wedge dx_2.$$

Let $U \subset \mathbb{R}^3$ be a domain bounded by a smooth surface $S$. We co-orient $S$ with the exterior normal $\nu$. Then

$$d\omega = \left(\frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} + \frac{\partial P_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3.$$

Thus, Stokes' formula

$$\int\limits_S \omega = \int\limits_U d\omega$$

gives in this case

$$\int\limits_S P_1 dx_2 \wedge dx_3 + P_2 dx_3 \wedge dx_1 + P_3 dx_1 \wedge dx_2 = \int\limits_U \left(\frac{\partial P_1}{\partial x_1} + \frac{\partial P_2}{\partial x_2} + \frac{\partial P_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3$$

This is called the *divergence theorem* or Gauss-Ostrogradski's formula. Consider the case

$k = 0$, $n = 1$. Thus $\omega$ is just a function $f$ on an interval $I = [a, b]$. The boundary $\partial I$ consists of 2 points: $\partial I = \{a, b\}$. One should orient the point $a$ with the sign $-$ and the point $b$ with the sign $+$.

Thus, Stokes' formula in this case gives

$$\int\limits_{[a,b]} df = \int\limits_{\{-a,+b\}} f,$$

or

$$\int_a^b f'(x)dx = f(b) - f(a).$$

This is Newton-Leibnitz' formula. More generally, for a 1-dimensional oriented connected curve $\Gamma \subset \mathbb{R}^3$ with boundary $\partial\Gamma = B \cup (-A)$ and any smooth function $f$ we get the formula

$$\int\limits_{\Gamma} df = \int\limits_{B\cup(-A)} f = f(B) - f(A),$$

which we already proved earlier, see Theorem 4.3.1 .

Consider now the case $n = 3$, $k = 1$.

Thus $V = \mathbb{R}^3$ and $\omega = P_1 dx_1 + P_2 dx_2 + P_3 dx_3$. Let $S \subset \mathbb{R}^3$ be an oriented surface with boundary $\Gamma$. We orient $\Gamma$ in the same way, as in Green's theorem. Then Stokes' formula

$$\int\limits_S d\omega = \int\limits_{\Gamma} \omega$$

gives in this case

$$\int\limits_S \left(\frac{\partial P_3}{\partial x_2} - \frac{\partial P_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial P_1}{\partial x_3} - \frac{\partial P_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial P_2}{\partial x_1} - \frac{\partial P_1}{\partial x_2}\right) dx_1 \wedge dx_2$$

$$= \int\limits_{\Gamma} P_1 dx_1 + P_2 dx_2 + P_3 dx_3.$$

This is the original Stokes' theorem.

Stokes' theorem allows one to clarify the geometric meaning of the exterior differential.

**Lemma 5.1.2.** *Let $\beta$ be a differential $k$-form in a domain $U \subset V$. Take any point $a \in U$ and vectors $X_1, \dots, X_{k+1} \in V_a$. Given $\epsilon > 0$ let us consider the parallelepiped $P(\epsilon X_1, \dots, \epsilon X_{k+1})$ as a subset of $V$ with vertices at points $a_{i_1 \dots i_{k+1}} = a + \epsilon \sum\limits_1^{k+1} i_j X_j$, where each index $i_j$ takes values $0, 1$. Then*

$$d\beta_a(X_1, \dots X_{k+1}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \int\limits_{\partial P(\epsilon X_1, \dots, \epsilon X_{k+1})} \beta.$$

**Proof.** First, it follows from the definition of integral of a differential form that

$$d\beta_a(X_1,\dots,X_{k+1}) = \lim_{\epsilon\to 0}\frac{1}{\epsilon^{k+1}}\int\limits_{P(\epsilon X_1,\dots,\epsilon X_{k+1})} d\beta. \tag{5.1}$$

Then we can continue using Stokes' formula

$$d\beta_a(X_1,\dots,X_{k+1}) = \lim_{\epsilon\to 0}\frac{1}{\epsilon^{k+1}}\int\limits_{P(\epsilon X_1,\dots,\epsilon X_{k+1})} d\beta = \lim_{\epsilon\to 0}\frac{1}{\epsilon^{k+1}}\int\limits_{\partial P(\epsilon X_1,\dots,\epsilon X_{k+1})} \beta. \tag{5.2}$$

■

## 5.2 Proof of Stokes' theorem

We prove in this section Theorem 5.1.1. We will consider only the case when $A$ is a manifold with boundary *without corners* and leave the corner case as an exercise to the reader.

Let us cover $A$ by coordinate neighborhoods such that in each neighborhood $A$ is given either by (4.34) or (4.35). First we observe that it is sufficient to prove the theorem for the case of a form supported in one of these coordinate neighborhoods. Indeed, let us choose finitely many such neighborhoods covering $A$. Let $1=\sum\limits_1^N \theta_j$ be a partition of unity subordinated to this covering. We set $\omega_j=\theta_j\omega$, so that $\omega=\sum\limits_1^N \omega_j$, and each of $\omega_j$ is supported in one of coordinate neighborhoods. Hence, if formula 5.1.1 holds for each $\omega_j$ it also holds for $\omega$.

Let us now assume that $\omega$ is supported in one of coordinate neighborhoods. Consider the corresponding parameterization $\phi: G\to U\subset V$, $G\subset\mathbb{R}^n$, introducing coordinates $u_1,\dots,u_n$. Then $A\cap U=\phi(G\cap L)$, where $L$ is equal to the subspace $\mathbb{R}^k=\{u_{k+1}=\dots u_n=0\}$ in the case (4.34) and the upper-half space $\mathbb{R}^k\cap\{u_1\geq 0\}$. By definition, we have $\int\limits_A\omega=\int\limits_U d\omega=\int\limits_{G\cap L}\phi^*d\omega=\int\limits_{G\cap L}d\phi^*\omega$.[9] Though the form $\widetilde{\omega}=\phi^*\omega|_{G\cap L}$ is defined only on $G\cap L$, it is supported in this neighborhood, and hence we can extend it to a smooth form on the whole $L$ by setting it equal to 0 outside the neighborhood. With this extension we

[9]We assume here that the coordinates $u_1,\dots,u_k$ define the given orientation of $A$.

have $\int\limits_{G\cap L} d\widetilde{\omega} = \int\limits_{L} d\widetilde{\omega}$. The $(k-1)$-form $\widetilde{\omega}$ can be written in coordinates $u_1, \ldots, u_k$ as

$$\widetilde{\omega} = \sum_1^j f_j(u) du_1 \wedge \ldots \overset{j}{\check{}} \cdots \wedge du_k.$$

Then

$$\int\limits_{G\cap L} d\widetilde{\omega} = \int\limits_{L} \left( \sum_1^k \frac{\partial f_j}{\partial u_j} \right) du_1 \wedge \cdots \wedge du_k.$$

Let us choose a sufficiently $R > 0$ so that the cube $I = \{|u_i| \leq R, i = 1, \ldots, k\}$ contains $\mathrm{Supp}(\widetilde{\omega})$. Thus in the case (4.34) we have

$$\int\limits_{G\cap L} d\widetilde{\omega} = \sum_1^k \int\limits_{\mathbb{R}^k} \frac{\partial f_j}{\partial u_j} dV = \sum_1^k \int\limits_{-R}^{R} \ldots \int\limits_{-R}^{R} \frac{\partial f_j}{\partial u_j} du_1 \ldots du_n =$$
$$\sum_1^k \int\limits_{-R}^{R} \ldots \left( \int\limits_{-R}^{R} \frac{\partial f_j}{\partial u_j} du_j \right) du_1 \ldots du_{j-1} du_{j+1} \ldots du_n = 0 \tag{5.3}$$

because

$$\int\limits_{-R}^{R} \frac{\partial f_j}{\partial u_j} du_j = f_j(u_1, \ldots, u_{i-1}, R, u_i, \ldots, u_n) - f_j(u_1, \ldots, u_{i-1}, -R, u_i, \ldots, u_n)) = 0.$$

On the other hand, in this case $\int\limits_{A} \omega = 0$, because the support of $\omega$ does not intersect the boundary of $A$. Hence, Stokes' formula holds in this case. In case (4.35) we similarly get

$$\int\limits_{G\cap L} d\widetilde{\omega} = \sum_1^k \int\limits_{\{u_1 \geq 0\}} \frac{\partial f_j}{\partial u_j} dV =$$
$$\sum_1^k \int\limits_{0}^{R} \left( \int\limits_{-R}^{R} \ldots \int\limits_{-R}^{R} \frac{\partial f_j}{\partial u_j} du_n \ldots du_2 \right) du_1 = \int\limits_{-R}^{R} \left( \int\limits_{-R}^{R} \ldots \int\limits_{0}^{R} \frac{\partial f_1}{\partial u_1} du_1 \ldots du_{n-1} \right) du_n =$$
$$- \int\limits_{-R}^{R} \ldots \int\limits_{-R}^{R} f_1(0, u_2, \ldots, u_n) du_2 \ldots du_n. \tag{5.4}$$

because all terms in the sum with $j > 1$ are equal to 0 by the same argument as in (5.3). On the other hand, in this case

$$\int\limits_{\partial A} \omega = \int\limits_{\{u_1=0\}} \phi^* \omega = \int \int\limits_{\{u_1=0\}} f_1(0, u_2, \dots, u_n) du_2 \wedge \dots \wedge du_n =$$

$$-\int\limits_{-R}^{R} \dots \int\limits_{-R}^{R} f_1(0, u_2, \dots, u_n) du_2 \dots du_n. \tag{5.5}$$

The sign minus appears in the last equality in front of the integral because the induced orientation on the space $\{u_1 = 0\}$ as the boundary of the upper-half space $\{u_1 \geq 0\}$ is opposite to the orientation defined by the volume form $du_2 \wedge \dots \wedge du_n$. Comparing the expressions (5.4) and (5.5) we conclude that $\int\limits_A d\omega = \int\limits_{\partial A} \omega$, as required. ■

## 5.3 Integration of functions over submanifolds

In order to integrate functions over a submanifold we need a notion of volume for subsets of the submanifold.

Let $A \subset V$ be an oriented $k$-dimensional submanifold, $0 \leq k \leq n$. By definition, the *volume form* $\sigma = \sigma_A$ of $A$ (or the *area form* if $k = 2$, or the *length form* if $k = 1$) is a differential $k$-form on $A$ whose value on any $k$ tangent vectors $v_1, \dots, v_k \in T_x A$ equals the oriented volume of the parallelepiped generated by these vectors.

Given a function $f : A \to \mathbb{R}$ we define its integral over $A$ by the formula

$$\int\limits_A f dV = \int\limits_A f \sigma_A \,, \tag{5.6}$$

and, in particular,

$$\operatorname{Vol} A = \int\limits_A \sigma_A \,.$$

Notice that the integral $\int\limits_A f dV$ *is independent of the orientation of* $A$. Indeed, changing the orientation we also change the sign of the form $\sigma_A$, and hence the integral remains unchanged.

This allows us to define the integral $\int\limits_A f dV$ even for a non-orientable $A$. Indeed, we can cover $A$ by coordinate charts, find a subordinated partition of unity and split correspondingly the function $f = \sum\limits_1^N f_j$ in such a way that each function $f_j$ is supported in a coordinate neighborhood. By orienting in arbitrary ways each of the coordinate neighborhoods we can compute each of the integrals $\int\limits_A f_j dV$, $j = 1, \dots, N$. It is straightforward to see that the integral $\int\limits_A f dV = \sum\limits_j \int\limits_A f_j dV$ is independent of the choice of the partition of unity.

Let us study in some examples how the form $\sigma_A$ can be effectively computed.

**Example 5.3.1.** *Volume form of a hypersurface.* Let us fix a Cartesian coordiantes in $V$. Let $A \subset V$ is given by the equation

$$A = \{F = 0\}$$

for some function $F : V \to \mathbb{R}$ which has no critical points on $A$. The vector field $\nabla F$ is orthogonal to $A$, and

$$\mathbf{n} = \frac{\nabla F}{||\nabla F||}$$

is the unit normal vector field to $A$. Assuming $A$ to be co-oriented by $\mathbf{n}$ we can write down the volume form of $A$ as the contraction of $\mathbf{n}$ with the volume form $\Omega = dx_1 \wedge \dots \wedge dx_n$ of $\mathbb{R}^n$, i.e.

$$\sigma_A = \mathbf{n} \lrcorner \Omega = \frac{1}{||\nabla F||} \sum_1^n (-1)^{i-1} \frac{\partial F}{\partial x_i} dx_1 \wedge \overset{i}{\dots\vee\dots} \wedge dx_n\,.$$

In particular, if $n = 3$ we get the following formula for the area form of an implicitely given 2-dimensional surface $A = \{F = 0\} \subset \mathbb{R}^3$:

$$\sigma_A = \frac{1}{\sqrt{\left(\frac{\partial F}{\partial x_1}\right)^2 + \left(\frac{\partial F}{\partial x_2}\right)^2 + \left(\frac{\partial F}{\partial x_3}\right)^2}} \left( \frac{\partial F}{\partial x_1} dx_2 \wedge dx_3 + \frac{\partial F}{\partial x_2} dx_3 \wedge dx_1 + \frac{\partial F}{\partial x_3} dx_1 \wedge dx_2 \right). \tag{5.7}$$

**Example 5.3.2.** *Length form of a curve.*

Let $\Gamma \subset \mathbb{R}^n$ be an oriented curve given parametrically by a map $\gamma : [a,b] \to \mathbb{R}^n$. Let $\sigma = \sigma_\Gamma$ be the length form. Let us compute the form $\gamma^*\sigma_\Gamma$. Denoting the coordinate in $[a,b]$ by $t$ and the unit vector field on $[a,b]$ by $e$ we have

$$\gamma^*\sigma_\Gamma = f(t)dt,$$

where

$$f(t) = \gamma^*\sigma_\Gamma(e) = \sigma_\Gamma\left(\gamma'(t)\right) = ||\gamma'(t)||\,.$$

In particular the length of $\Gamma$ is equal to

$$\int\limits_\Gamma \sigma_\Gamma = \int\limits_a^b ||\gamma'(t)||dt = \int\limits_a^b \sqrt{\sum_{i=1}^n (x_i'(t))^2}\, dt\,,$$

where

$$\gamma(t) = (x_1(t), \dots, x_n(t))\,.$$

Similarly, given any function $f : \Gamma \to \mathbb{R}$ we have

$$\int\limits_\Gamma f ds = \int\limits_a^b f\left(\gamma(t)\right) ||\gamma'(t)||dt\,.$$

**Example 5.3.3.** *Area form of a surface given parametrically.*

Suppose a surface $S \subset \mathbb{R}^n$ is given parametrically by a map $\Phi : U \to \mathbb{R}^n$ where $U$ in the plane $\mathbb{R}^2$ with coordinates $(u,v)$.

Let us compute the pull-back form $\Phi^*\sigma_S$. In other words, we want to express $\sigma_S$ in coordinates $u,v$. We have

$$\Phi^*\sigma_S = f(u,v)du \wedge dv\,.$$

To determine $f(u,v)$ take a point $z = (u,v) \in \mathbb{R}^2$ and the standard basis $e_1, e_2 \in \mathbb{R}^2_z$. Then

$$(\Phi^*\sigma_S)_z(e_1, e_2) = f(u,v)du \wedge dv(e_1, e_2)\,. \tag{5.8}$$

On the other hand, by the definition of the pull-back form we have

$$(\Phi^*\sigma_S)_z(e_1, e_2) = (\sigma_S)_{\Phi(z)}(d_z\Phi(e_1), d_z\Phi(e_2))\,. \tag{5.9}$$

But $d_z\Phi(e_1) = \frac{\partial\Phi}{\partial u}(z) = \Phi_u(z)$ and $d_z\Phi(e_2) = \frac{\partial\Phi}{\partial v}(z) = \Phi_v(z)$. Hence from (5.8) and (5.9) we get

$$f(u,v) = \sigma_S(\Phi_u, \Phi_v)\,. \tag{5.10}$$

The value of the form $\sigma_S$ on the vectors $\Phi_u, \Phi_v$ is equal to the area of the parallelogram generated by these vectors, because the surface is assumed to be oriented by these vectors, and hence $\sigma_S(\Phi_u, \Phi_v) > 0$. Denoting the angle between $\Phi_u$ and $\Phi_v$ by $\alpha$ we get[10] $\sigma_S(\Phi_u, \Phi_v) = ||\Phi_u||\,||\Phi_v||\sin\alpha$. Hence

$$\sigma_S(\Phi_u, \Phi_v)^2 = ||\Phi_u||^2\,||\Phi_v||^2\sin^2\alpha = ||\Phi_u||^2\,||\Phi_v||^2(1-\cos^2\alpha) = ||\Phi_u||^2\,||\Phi_v||^2 - (\Phi_u\cdot\Phi_v)^2\,,$$

and therefore,

$$f(u,v) = \sigma_S(\Phi_u, \Phi_v) = \sqrt{||\Phi_u||^2\,||\Phi_v||^2 - (\Phi_u\cdot\Phi_v)^2}\,.$$

It is traditional to introduce the notation

$$E = ||\Phi_u||^2,\ F = \Phi_u\cdot\Phi_v,\ G = ||\Phi_v||^2\,,$$

so that we get

$$\Phi^*\sigma_S = \sqrt{EG - F^2}du\wedge dv\,,$$

and hence we get for any function $f : S \to \mathbb{R}$

$$\int\limits_S f dS = \int\limits_S f\sigma_S = \int\limits_U f(\Phi(u,v))\sqrt{EG-F^2}du\wedge dv = \iint\limits_U f(\Phi(u,v))\sqrt{EG-F^2}dudv\,. \tag{5.11}$$

Consider a special case when the surface $S$ defined as a graph of a function $\phi$ over a domain $D\subset\mathbb{R}^2$. Namely, suppose

$$S = \{z = \phi(x,y),\ \ (x,y)\in D\subset\mathbb{R}^2\}\,.$$

The surface $S$ as parametrized by the map

$$(x,y)\overset{\Phi}{\mapsto}(x,y,\phi(x,y)).$$

[10]See a computation in a more general case below in Example 5.3.4.

Then

$$E = ||\Phi_x||^2 = 1 + \phi_x^2,\ G = ||\Phi_y||^2 = 1 + \phi_y^2,\ F = \Phi_x \cdot \Phi_y = \phi_x \phi_y \,,$$

and hence

$$EG - F^2 = (1 + \phi_x^2)(1 + \phi_y^2) - \phi_x^2 \phi_y^2 = 1 + \phi_x^2 + \phi_y^2 \,.$$

Therefore, the formula (5.11) takes the form

$$\int\limits_S f dS = \int\int\limits_D f(\Phi(x,y))\sqrt{EG - F^2} dx \wedge dy = \int\int\limits_D f\left(x, y, \phi(x,y)\right) \sqrt{1 + \phi_x^2 + \phi_y^2}\, dxdy \,. \tag{5.12}$$

Note that the formula (5.12) can be also deduced from (5.7). Indeed, the surface

$$S = \{z = \phi(x,y),\ \ (x,y) \in D \subset \mathbb{R}^2\} \,,$$

can also be defined implicitly by the equation

$$F(x,y,z) = z - \phi(x,y) = 0,\ \ (x,y) \in D \,.$$

We have

$$\nabla F = (-\frac{\partial \phi}{\partial x}, -\frac{\partial \phi}{\partial y}, 1),$$

and, therefore,

$$\begin{aligned}
\int\limits_S f dS &= \int\limits_S \frac{f(x,y,z)}{||\nabla F||} \left( \frac{\partial F}{\partial x} dy \wedge dz + \frac{\partial F}{\partial y} dz \wedge dx + \frac{\partial F}{\partial z} dx \wedge dy \right) \\
&= \int\int\limits_D \frac{f(x,y,\phi(x,y))}{\sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}} \left( -\frac{\partial \phi}{\partial x} dy \wedge d\phi - \frac{\partial \phi}{\partial y} d\phi \wedge dx + dx \wedge dy \right) \\
&= \int\int\limits_D \frac{f(x,y,\phi(x,y))}{\sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}} \left( 1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right) dxdy \\
&= \int\int\limits_D f(x,y,\phi(x,y)) \sqrt{1 + \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2} dxdy \,.
\end{aligned}$$

**Example 5.3.4.** *Integration over a parametrically given $k$-dimensional submanfold.*

Consider now a more general case of a parametrically given $k$-submanifold $A$ in an $n$-dimensional Euclidean space $V$. We fix a Cartersian coordinate system in $V$ and thus identify $V$ with $\mathbb{R}^n$ with the standard dot-product.

Let $U \subset \mathbb{R}^k$ be a compact domain with boundary and $\phi : U \to \mathbb{R}^n$ be an embedding. We assume that the submanifold with boundary $A = \phi(U)$ is oriented by this parameterization. Let $\sigma_A$ be the volume form of $A$. We will find an explicit expression for $\phi^*\sigma_A$. Namely. denoting coordinates in $\mathbb{R}^k$ by $(u_1, \dots, u_k)$ we have $\phi^*\sigma_A = f(u)du_1 \wedge \dots \wedge du_k$, and our goal is to compute the function $f$.

By definition, we have

$$\begin{aligned} f(u) =&\phi^*(\sigma_A)_u(e_1, \dots, e_k) = (\sigma_A)_u(d_u\phi(e_1), \dots, d_u\phi(e_k)) = \\ &(\sigma_A)_u\left(\frac{\partial\phi}{\partial u_1}(u), \dots, \frac{\partial\phi}{\partial u_k}(u)\right) = \mathrm{Vol}_k\left(P\left(\frac{\partial\phi}{\partial u_1}(u), \dots, \frac{\partial\phi}{\partial u_k}(u)\right)\right) \end{aligned} \tag{5.13}$$

In Section 1.17 we proved two formulas for the volume of a parallelepiped. Using formula (1.12) we get

$$\mathrm{Vol}_k\left(P\left(\frac{\partial\phi}{\partial u_1}(u), \dots, \frac{\partial\phi}{\partial u_k}(u)\right)\right) = \sqrt{\sum_{1\le i_1<\dots<i_k\le n} Z^2_{i_1\dots i_k}}. \tag{5.14}$$

where

$$Z_{i_1\dots i_k} = \begin{vmatrix} \frac{\partial\phi_{i_1}}{\partial u_1}(u) & \dots & \frac{\partial\phi_{i_1}}{\partial u_k}(u) \\ \dots & \dots & \dots \\ \frac{\partial\phi_{i_k}}{\partial u_1}(u) & \dots & \frac{\partial\phi_{i_k}}{\partial u_k}(u) \end{vmatrix}. \tag{5.15}$$

Thus

$$\int_A f dV = \int_A f\sigma_A = \int_U f(\phi(u))\sqrt{\sum_{1\le i_1<\dots<i_k\le n} Z^2_{i_1\dots i_k}}du_1 \wedge \dots \wedge du_k.$$

Rewriting this formula for $k = 2$ we get

$$\int_A f dV = \int_U f(\phi(u)) \sqrt{\sum_{1 \le i < j \le n} \begin{vmatrix} \frac{\partial \phi_i}{\partial u_1} & \frac{\partial \phi_i}{\partial u_2} \\ \frac{\partial \phi_j}{\partial u_1} & \frac{\partial \phi_j}{\partial u_2} \end{vmatrix}^2} du_1 \wedge du_2. \tag{5.16}$$

Alternatively we can use formula (**??**). Then we get

$$\mathrm{Vol}_k \left( P \left( \frac{\partial \phi}{\partial u_1}(u), \dots, \frac{\partial \phi}{\partial u_k}(u) \right) \right) = \sqrt{\det(D\phi)^T D\phi}, \tag{5.17}$$

where

$$D\phi = \begin{pmatrix} \frac{\partial \phi_1}{\partial u_1} & \dots & \frac{\partial \phi_1}{\partial u_k} \\ \dots & \dots & \dots \\ \frac{\partial \phi_n}{\partial u_1} & \dots & \frac{\partial \phi_n}{\partial u_k} \end{pmatrix}$$

is the Jacobi matrix of $\phi$.[11] Thus using this expression we get

$$\int_A f dV = \int_A f \sigma_A = \int_U f(\phi(u)) \sqrt{\det(D\phi)^T D\phi}\, dV. \tag{5.18}$$

**Exercise 5.3.5.** *In case $n = 3$, $k = 2$ show explicitely equivalence of formulas* (5.11)*,* (5.16) *and* (5.18)*.*

**Exercise 5.3.6.** *Integration over an implicitly defined $k$-dimensional submanfold.* Suppose that $A = \{F_1 = \dots = F_{n-k} = 0\}$ and the differentials of defining functions are linearly independent at points of $A$. Show that

$$\sigma_A = \frac{*\,(dF_1 \wedge \dots \wedge dF_{n-k})}{||\, dF_1 \wedge \dots \wedge dF_{n-k} \,||}.$$

**Example 5.3.7.** Let us compute the volume of the unit 3-sphere $S^3 = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$.

By definition, $\mathrm{Vol}(S^3) = \int\limits_{S^3} \mathbf{n} \lrcorner \Omega$, where $\Omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$, and $\mathbf{n}$ is the outward unit normal vector to the unit ball $B_4$. Here $S^3$ should be co-oriented by the vector field $\mathbf{n}$.

[11] The symmetric matrix $(D\phi)^T D\phi$ is the Gram matrix of vectors $\frac{\partial \phi}{\partial u_1}(u), \dots, \frac{\partial \phi}{\partial u_k}(u)$, and its entrees are pairwise scalar products of these vectors, see Remark **??**.

Then using Stokes' theorem we have

$$\int_{S^3} \mathbf{n} \lrcorner \Omega = \int_{S^3} (x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3) =$$
$$4 \int_{B^4} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = 4 \int_{B^4} dV. \tag{5.19}$$

Introducing polar coordinates $(r, \phi)$ and $(\rho, \theta)$ in the coordinate planes $(x_1, x_2)$ and $(x_3, x_4)$ and using Fubini's theorem we get

$$\int_{B^4} = \int_0^{2\pi} \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{1-r^2}} r \rho d\rho dr d\theta d\phi =$$
$$2\pi^2 \int_0^1 (r - r^3) dr = \frac{\pi^2}{2}. \tag{5.20}$$

Hence, $\mathrm{Vol}(S^3) = 2\pi^2$.

**Exercise 5.3.8.** Find the ratio $\frac{\mathrm{Vol}_n(B_R^n)}{\mathrm{Vol}_{n-1}(S_R^{n-1})}$.

## 5.4 Work and Flux

We introduce in this section two fundamental notions of vector analysis: a *work of a vector field along a curve*, and a *flux of a vector field through a surface.* Let $\Gamma$ be an oriented smooth curve in a Euclidean space $V$ and $T$ the unit tangent vector field to $\Gamma$. Let $\mathbf{v}$ be another vector field, defined along $\Gamma$. The function $\langle \mathbf{v}, \mathbf{T} \rangle$ equals the projection of the vector field $\mathbf{v}$ to the tangent directions to the curve. If the vector field $\mathbf{v}$ is viewed as a force field, then the integral $\int_\Gamma \langle \mathbf{v}, \mathbf{T} \rangle ds$ has the meaning of a *work* $\mathrm{Work}_\Gamma(\mathbf{v})$ performed by the field $\mathbf{v}$ to transport a particle of mass 1 along the curve $\Gamma$ in the direction determined by the orientation. If the curve $\Gamma$ is closed then this integral is sometimes called the *circulation* of the vector field $\mathbf{v}$ along $\Gamma$ and denoted by $\oint_\Gamma \langle \mathbf{v}, \mathbf{T} \rangle ds$. As we already indicated earlier, the sign $\oint$ in this case has precisely the same meaning as $\int$, and it is used only to stress the point that we are integrating along a *closed* curve.

Consider now a co-oriented hypersurface $\Sigma \subset V$ and denote by $\mathbf{n}$ the unit normal vector field to $\Sigma$ which determines the given co-orientation of $\Sigma$. Given a vector field $\mathbf{v}$ along $\Sigma$ we will view it as the velocity vector field of a flow of a fluid in the space. Then we can interpret the integral

$$\int\limits_{\Sigma} \langle \mathbf{v}, \mathbf{n} \rangle dV$$

as the *flux* $\mathrm{Flux}_{\Sigma}(\mathbf{v})$ of $\mathbf{v}$ through $\Sigma$, i.e. the volume of fluid passing through $\Sigma$ in the direction of $\mathbf{n}$ in time 1.

**Lemma 5.4.1.** *1. For any co-oriented hypersurface $\Sigma$ and a vector field $\mathbf{v}$ given in its neighborhood we have*

$$\langle \mathbf{v}, \mathbf{n} \rangle \sigma_{\Sigma} = (\mathbf{v} \lrcorner \Omega)_{\Sigma},$$

*where $\Omega$ is the volume form in $V$.*

*2. For any oriented curve $\Gamma$ and a vector field $\mathbf{v}$ near $\Gamma$ we have*

$$\langle \mathbf{v}, \mathbf{T} \rangle \sigma_{\Gamma} = \mathcal{D}(\mathbf{v})|_{\Gamma}.$$

**Proof.** 1. For any $n-1$ vectors $T_1, \dots, T_{n-1} \in T_x\Sigma$ we have

$$\mathbf{v} \lrcorner \Omega(T_1, \dots, T_{n-1}) = \Omega(\mathbf{v}, T_1, \dots, T_{n-1}) = \mathrm{Vol}P(\mathbf{v}, T_1, \dots, T_{n-1}).$$

Using (1.4) we get

$$\begin{aligned} &\mathrm{Vol}P(\mathbf{v}, T_1, \dots, T_{n-1}) = \langle \mathbf{v}, \mathbf{n} \rangle \mathrm{Vol}_{n-1} P(T_1, \dots, T_{n-1}) = \\ &\langle \mathbf{v}, \mathbf{n} \rangle \mathrm{Vol}P(\mathbf{n}, T_1, \dots, T_{n-1}) = \langle \mathbf{v}, \mathbf{n} \rangle \sigma_{\Sigma}(T_1, \dots, T_{n-1}). \end{aligned} \tag{5.21}$$

2. The tangent space $T_x\Gamma$ is generated by the vector $\mathbf{T}$, and hence we just need to check that $\langle \mathbf{v}, \mathbf{T} \rangle \sigma_{\Gamma}(\mathbf{T}) = \mathcal{D}(\mathbf{v})(\mathbf{T})$. But $\sigma_{\Gamma}(\mathbf{T}) = \mathrm{Vol}(\mathbf{n}, \mathbf{T}) = 1$, and hence

$$\mathcal{D}(\mathbf{v})(\mathbf{T}) = \langle \mathbf{v}, \mathbf{T} \rangle = \langle \mathbf{v}, \mathbf{T} \rangle \sigma_{\Gamma}(\mathbf{T}).$$

■

Note that if we are given a Cartesian coordinate system in $V$ and $\mathbf{v}=\sum\limits_1^n a_j\frac{\partial}{\partial x_j}$, then

$$\mathbf{v}\lrcorner\Omega=\sum_1^{n-1}(-1)^{i-1}a_i dx_1 \overset{i}{\dots} dx_n, \quad \mathcal{D}(\mathbf{v})=\sum_1^n a_i dx_i\,.$$

Thus, we have

**Corollary 5.4.2.**

$$\mathrm{Flux}_\Sigma(\mathbf{v})=\int\limits_\Sigma\langle\mathbf{v},\,\mathbf{n}\rangle dV=\int\limits_\Sigma(\mathbf{v}\lrcorner\Omega)=\int\limits_\Sigma\sum_1^n(-1)^{i-1}a_i dx_1 \overset{i}{\dots} dx_n;$$

$$\mathrm{Work}_\Gamma(\mathbf{v})=\int\limits_\Gamma\langle\mathbf{v},\mathbf{T}\rangle=\int\limits_\Gamma\mathcal{D}(\mathbf{v})=\int\limits_\Gamma\sum_1^n a_i dx_i.$$

In particular if $n=3$ we have

$$\mathrm{Flux}_\Sigma(\mathbf{v})=\int\limits_\Sigma a_1dx_2\wedge dx_3+a_2dx_3\wedge dx_1+a_3dx_1\wedge dx_2.$$

Let us also recall that in a Euclidean space $V$ we have $v\lrcorner\Omega=*\mathcal{D}(v)$. Hence, the equation $\omega=v\lrcorner\Omega$ is equivalent to the equation

$$v=\mathcal{D}^1\left(*^{-1}\omega\right)=(-1)^{n-1}\mathcal{D}^{-1}(*\omega).$$

In particular, when $n=3$ we get $\mathbf{v}=\mathcal{D}^{-1}(*\omega)$. Thus we get

**Corollary 5.4.3.** *For any differential $(n-1)$-form $\omega$ and an oriented compact hypersurface $\Sigma$ we have*

$$\int\limits_\Sigma\omega=\mathrm{Flux}_\Sigma\,\mathbf{v},$$

*where* $\mathbf{v}=(-1)^{n-1}\mathcal{D}^{-1}(*\omega)$.

Integration of functions along curves and surfaces can be interpreted as the work and the flux of appropriate vector fields. Indeed, suppose we need to compute an integral $\int\limits_\Gamma f ds$.

Consider the tangent vector field $\mathbf{v}(x) = f(x)\mathbf{T}(x), \quad x \in \Gamma$, along $\Gamma$. Then $\langle \mathbf{v}, \mathbf{T} \rangle = f$ and hence the integral $\int_\Gamma f ds$ can be interpreted as the work $\text{Work}_\Gamma(\mathbf{v})$. Therefore, we have

$$\int\limits_\Gamma f ds = \text{Work}_\Gamma(\mathbf{v}) = \int\limits_\Gamma \mathcal{D}(\mathbf{v})$$

.

Note that we can also express $\mathbf{v}$ through $\omega$ by the formula $\mathbf{v} = \mathcal{D}^{-1} \star \omega$, see Section 3.5.

Similarly, to compute an integral $\int\limits_\Sigma f dS$ let us co-orient the surface $\Sigma$ with a unit normal to $\Sigma$ vector field $\mathbf{n}(x)$, $x \in \Sigma$ and set $\mathbf{v}(x) = f(x)\,\mathbf{n}(x)$. Then $\langle \mathbf{v}, \mathbf{n} \rangle = f$, and hence

$$\int\limits_\Gamma f dS = \int\limits_\Gamma \langle \mathbf{v}, \mathbf{n} \rangle dS = \text{Flux}_\Sigma(\mathbf{v}) = \int\limits_\Gamma \omega,$$

where $\omega = \mathbf{v} \lrcorner \Omega$.

## 5.5 Integral formulas of vector analysis

We interpret in this section Stokes' formula in terms of integrals of functions and operations on vector fields. Let us consider again differential forms, which one can associate with a vector field $\mathbf{v}$ in an Euclidean 3-space. Namely, this is a differential 1-form $\alpha = \mathcal{D}(\mathbf{v})$ and a differential 2-form $\omega = \mathbf{v} \lrcorner \Omega$, where $\Omega = dx \wedge dy \wedge dz$ is the volume form.

Using Corollary 5.4.2 we can reformulate Stokes' theorem for domains in a $\mathbb{R}^3$ as follows.

**Theorem 5.5.1.** *Let $\mathbf{v}$ be a smooth vector field in a domain $U \subset \mathbb{R}^3$ with a smooth (or piece-wise) smooth boundary $\Sigma$. Suppose that $\Sigma$ is co-oriented by an outward normal vector field. Then we have*

$$\text{Flux}_\Sigma \mathbf{v} = \int\int\limits_U\int \text{div}\, \mathbf{v} dx dy dz\,.$$

Indeed, $\text{div}\, \mathbf{v} = *d\omega$. Hence we have

$$\int\limits_U \text{div}\, \mathbf{v} dV = \int\limits_U (\star d\omega) dx \wedge dy \wedge dz = \int\limits_U d\omega = \int\limits_\Sigma \omega = \int\limits_\Sigma \mathbf{v} \lrcorner \Omega = \text{Flux}_\Sigma \mathbf{v}\,.$$

This theorem clarifies the meaning of $\operatorname{div} \mathbf{v}$:

Let $B_r(x)$ be the ball of radius $r$ centered at a point $x \in \mathbb{R}^3$, and $S_r(x) = \partial B_r(x)$ be its boundary sphere co-oriented by the outward normal vector field. Then

$$\operatorname{div} \mathbf{v}(x) = \lim_{r \to 0} \frac{\operatorname{Flux}_{S_r(x)} \mathbf{v}}{\operatorname{Vol}(B_r(x))}.$$

**Theorem 5.5.2.** *Let $\Sigma$ be a piece-wise smooth compact oriented surface in $\mathbb{R}^3$ with a piece-wise smooth boundary $\Gamma = \partial\Sigma$ oriented respectively. Let $\mathbf{v}$ be a smooth vector field defined near $\Sigma$. Then*

$$\operatorname{Flux}_\Sigma(\operatorname{curl} \mathbf{v}) = \int\limits_\Sigma \langle \operatorname{curl} \mathbf{v},\, \mathbf{n} \rangle dV = \oint\limits_\Gamma \mathbf{v} \cdot \mathbf{T} ds = \operatorname{Work}_\Gamma \mathbf{v}.$$

To prove the theorem we again use Stokes' theorem and the connection between integrals of functions and differential forms. Set $\alpha = \mathcal{D}(\mathbf{v})$. Then $\operatorname{curl} \mathbf{v} = \mathcal{D}^{-1} \star d\alpha$. We have

$$\oint\limits_\Gamma \mathbf{v} \cdot \mathbf{T} ds = \int\limits_\Gamma \alpha = \int\limits_\Sigma d\alpha = \operatorname{Flux}_\Sigma(\mathcal{D}^{-1} \star (d\alpha)) = \operatorname{Flux}_\Sigma(\operatorname{curl} \mathbf{v}).$$

Again, similar to the previous case, this theorem clarifies the meaning of curl. Indeed, let us denote by $D_r(x, \mathbf{w})$ the 2-dimensional disc of radius $r$ in $\mathbb{R}^3$ centered at a point $x \in \mathbb{R}^3$ and orthogonal to a unit vector $\mathbf{w} \in \mathbb{R}^3_x$. Set

$$c(x, \mathbf{w}) = \lim_{r \to 0} \frac{\operatorname{Work}_{\partial D_r(x, \mathbf{w})} \mathbf{v}}{\pi r^2}.$$

Then

$$c(x, wb) = \lim_{r \to 0} \frac{\operatorname{Flux}_{D_r(x, \mathbf{w})(\operatorname{curl} \mathbf{v}}}{\pi r^2} = \lim_{r \to 0} \frac{\int\limits_{D_r(x, \mathbf{w})} \langle \operatorname{curl} \mathbf{v},\, \mathbf{w} \rangle}{\pi r^2} = \langle \operatorname{curl} \mathbf{v},\, \mathbf{w} \rangle.$$

Hence, $||\operatorname{curl} \mathbf{v}(x)|| = \max\limits_{\mathbf{w}} c(x, \mathbf{w})$ and direction of $\operatorname{curl} \mathbf{v}(x)$ coincides with the direction of the vector $\mathbf{w}$ for which the maximum value of $c(x, \mathbf{w})$ is achieved.

## 5.6 Expressing div and curl in curvilinear coordinates

Let us show how to compute $\operatorname{div} \mathbf{v}$ and $\operatorname{curl} \mathbf{v}$ of a vector field $\mathbf{v}$ in $\mathbb{R}^3$ given in a curvilinear coordinates $u_1, u_2, u_3$, i.e. expressed through the coordinate vector fields $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}$ and $\frac{\partial}{\partial u_3}$. Let

$$\Omega = f(u_1, u_2, u_3) du_1 \wedge du_2 \wedge du_3$$

be the volume form $dx_1 \wedge dx_2 \wedge dx_3$ expressed in coordinates $u_1, u_2, u_3$.

Let us first compute $\star(du_1 \wedge du_2 \wedge du_3)$. We have

$$\star(du_1 \wedge du_2 \wedge du_3) = \star\left(\frac{1}{f} dx_1 \wedge dx_2 \wedge dx_3\right) = \frac{1}{f}.$$

Let

$$\mathbf{v} = a_1 \frac{\partial}{\partial u_1} + a_2 \frac{\partial}{\partial u_2} + a_3 \frac{\partial}{\partial u_3}.$$

Then we have

$$\begin{aligned}
\operatorname{div} \mathbf{v} &= \star d(\mathbf{v} \lrcorner \Omega) \\
&= \star d\left(\left(\sum_1^3 a_i \frac{\partial}{\partial u_i}\right) \lrcorner f du_1 \wedge du_2 \wedge du_3\right) \\
&= \star d\left(f a_1 du_2 \wedge du_3 + f a_2 du_3 \wedge du_1 + f a_3 du_1 \wedge du_2\right) \\
&= \star\left(\left(\frac{\partial(f a_1)}{\partial u_1} + \frac{\partial(f a_2)}{\partial u_2} + \frac{\partial(f a_3)}{\partial u_3}\right) du_1 \wedge du_2 \wedge du_3\right) \\
&= \frac{1}{f}\left(\frac{\partial(f a_1)}{\partial u_1} + \frac{\partial(f a_2)}{\partial u_2} + \frac{\partial(f a_3)}{\partial u_3}\right) \\
&= \frac{\partial a_1}{\partial u_1} + \frac{\partial a_2}{\partial u_2} + \frac{\partial a_3}{\partial u_3} + \frac{1}{f}\left(\frac{\partial f}{\partial u_1} a_1 + \frac{\partial f}{\partial u_2} a_2 + \frac{\partial f}{\partial u_3} a_3\right).
\end{aligned}$$

In particular, we see that the divergence of a vector field is expressed by the same formulas as in the cartesian case *if and only if* the volume form is proportional to the form $du_1 \wedge du_2 \wedge du_3$ with a *constant* coefficient.

For instance, in the spherical coordinates the volume form can be written as

$$\Omega = r^2 \sin\varphi dr \wedge d\theta \wedge d\varphi, \text{[12]}$$

[12] Note that the spherical coordinates ordered as $(r, \theta, \varphi)$ determine the same orientation of $\mathbb{R}^3$ as the cartesian coordinates $(x, y, z)$.

and hence the divergence of a vector field

$$\mathbf{v} = a\frac{\partial}{\partial r} + b\frac{\partial}{\partial \theta} + c\frac{\partial}{\partial \varphi}$$

can be computed by the formula

$$\operatorname{div}\mathbf{v} = \frac{\partial a}{\partial r} + \frac{\partial b}{\partial \theta} + \frac{\partial c}{\partial \varphi} + \frac{2a}{r} + c\cot\varphi\,.$$

The general formula for $\operatorname{curl}\mathbf{v}$ in curvilinear coordinates looks pretty complicated. So instead of deriving the formula we will explain here how it can be obtained in the general case, and then illustrate this procedure for the spherical coordinates.

By the definition we have

$$\operatorname{curl}\mathbf{v} = D^{-1}\left(\star\left(d\left(D(\mathbf{v})\right)\right)\right)\,.$$

Hence we first need to compute $D(\mathbf{v})$. To do this we need to introduce a symmetric matrix

$$G = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix},$$

where

$$g_{ij} = \langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \rangle, \quad i,j = 1,2,3\,.$$

The matrix $G$ is called the *Gram matrix*.

Notice, that if $D(\mathbf{v}) = A_1 du_1 + B du_2 + C du_3$ then for any vector $h = h_1\frac{\partial}{\partial u_1} + h_2\frac{\partial}{\partial u_2} + h_3\frac{\partial}{\partial u_3}$ . we have

$$D(\mathbf{v})(h) = A_1h_1 + A_2h_2 + A_3h_3 = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \langle \mathbf{v}, h \rangle\,.$$

But

$$\langle \mathbf{v}, h \rangle = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} G \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} G\,,$$

or, equivalently, because the Gram matrix $G$ is symmetric we can write

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = G \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} ,$$

and therefore,

$$A_i = \sum_{i=1}^{n} g_{ij} a_j, \quad i = 1, 2, 3\,.$$

After computing

$$\omega = d(D(\mathbf{v})) = B_1 du_2 \wedge du_3 + B_2 du_3 \wedge du_1 + B_3 du_1 \wedge du_2$$

we compute curl $\mathbf{v}$ by the formula $=$ curl $\mathbf{v} = D^{-1}(\star\omega)$. Let us recall (see Proposition 1.18.4 above) that for any vector field $\mathbf{w}$ the equality $D\,\mathbf{w} = \star\omega$ is equivalent to the equality $\mathbf{w} \lrcorner \Omega = \omega$, where $\Omega = f du_1 \wedge du_2 \wedge du_3$ is the volume form. Hence, if

$$\operatorname{curl} \mathbf{v} = c_1 \frac{\partial}{\partial u_1} + c_2 \frac{\partial}{\partial u_2} + c_3 \frac{\partial}{\partial u_3}$$

then we have

$$\mathbf{w} \lrcorner \Omega = f c_1 du_2 \wedge du_3 + f c_2 du_3 \wedge du_1 + f c_3 du_1 \wedge du_2,$$

and therefore,

$$\operatorname{curl} \mathbf{v} = \frac{B_1}{f} \frac{\partial}{\partial u_1} + \frac{B_2}{f} \frac{\partial}{\partial u_2} + \frac{B_3}{f} \frac{\partial}{\partial u_3}\,.$$

Let us use the above procedure to compute curl $\mathbf{v}$ of the vector field

$$\mathbf{v} = a \frac{\partial}{\partial r} + b \frac{\partial}{\partial \theta} + c \frac{\partial}{\partial \varphi}$$

given in the spherical coordinates. The Gram matrix in this case is the diagonal matrix

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r \sin\varphi & 0 \\ 0 & 0 & r \end{pmatrix} .$$

Hence,

$$D\,\mathbf{v} = adr + br\sin\varphi d\theta + crd\varphi\,,$$

and

$$\begin{aligned}\omega = d(D\,\mathbf{v}) &= da\wedge dr + d(br\sin\varphi)\wedge d\theta + d(cr)\wedge d\varphi\\ &= \left(r\frac{\partial c}{\partial\theta} - rb\cos\varphi - r\sin\varphi\frac{\partial b}{\partial\varphi}\right)d\theta\wedge d\varphi\\ &+ \left(\frac{\partial a}{\partial\varphi} - r\frac{\partial c}{\partial r} - c\right)d\varphi\wedge dr + \left(b\sin\varphi + r\sin\varphi\frac{\partial b}{\partial r} - \frac{\partial a}{\partial\theta}\right)dr\wedge d\theta\,.\end{aligned}$$

Finally, we get the following expression for curl $\mathbf{v}$:

$$\operatorname{curl}\mathbf{v} = \frac{r\frac{\partial c}{\partial\theta} - rb\cos\varphi - r\sin\varphi\frac{\partial b}{\partial\varphi}}{r^2\cos\varphi}\frac{\partial}{\partial r} + \frac{\frac{\partial a}{\partial\varphi} - r\frac{\partial c}{\partial r} - c}{r^2\cos\varphi}\frac{\partial}{\partial\theta} + \frac{b\sin\varphi + r\sin\varphi\frac{\partial b}{\partial r} - \frac{\partial a}{\partial\theta}}{r^2\cos\varphi}\frac{\partial}{\partial\varphi}\,.$$

# 6 Applications of Stokes' formula

## 6.1 Integration of closed and exact forms

Let us recall that a differential $k$-form $\omega$ is called *closed* if $d\omega = 0$, and that it is called *exact* if there exists a $(k-1)$-form $\alpha$, called *primitive* of $\omega$, such that $\omega = d\alpha$.

Any exact form is closed, because $d(d\alpha) = 0$. Any $n$-form in a $n$-dimensional space is closed.

**Proposition 6.1.1.** *a) For a closed $k$-form $\omega$ defined near a $(k+1)$-dimensional submanifold $\Sigma$ with boundary $\partial\Sigma$ we have*

$$\int_{\partial\Sigma}\omega = 0\,.$$

*b) If $\omega$ is exact $k$-form defined near a closed $k$-dimensional submanifold $S$ then*

$$\int_S \omega = 0\,.$$

The proof immediately follows from Stokes' formula. Indeed, in case a) we have

$$\int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega = 0\,.$$

In case b) we have $\omega = d\alpha$ and $\partial S = \varnothing$. Thus

$$\int_{S} d\alpha = \int_{\emptyset} \alpha = 0\,.$$

Proposition 6.1.1b) gives a necessary condition for a closed form to be exact.

**Example 6.1.2.** The differential 1-form $\alpha = \frac{1}{x^2+y^2}(xdy - ydx)$ defined on the punctured plane $\mathbb{R}^2 \setminus 0$ is closed but not exact.

Indeed, it is straightforward to check that $\alpha$ is exact (one can simplify computations by passing to polar coordinates and computing that $\alpha = d\varphi$). To check that it is not exact we compute the integral $\int_S \alpha$, where $S$ in the unit circle $\{x^2 + y^2 = 1\}$. We have

$$\int_S \alpha = \int_0^{2\pi} d\varphi = 2\pi \neq 0\,.$$

More generally, an $(n-1)$-form

$$\theta_n = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{r^n} dx_1 \wedge \overset{i}{\check{\dots}} \wedge dx_n \quad (dx_i \text{ is missing})$$

is closed in $\mathbb{R}^n \setminus 0$. However, it is not exact. Indeed, let us show that $\int\limits_{S^{n-1}} \theta_n \neq 0$, where $S^{n-1}$ is the unit sphere oriented as the boundary of the unit ball. Let us recall that the volume form $\sigma_{S^{n-1}}$ on the unit sphere is defined as

$$\sigma_{S^{n-1}} = \mathbf{n} \lrcorner \Omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{r} dx_1 \wedge \overset{i}{\check{\dots}} \wedge dx_n.$$

Notice that $\theta_n|_{S^{n-1}} = \sigma_{S^{n-1}}$, and hence

$$\int_{S^{n-1}} \theta_n = \int_{S^{n-1}} \sigma_{S^{n-1}} = \int_{S^{n-1}} dV = \mathrm{Vol}(S^{n-1}) > 0.$$

## 6.2 Homotopy

Let $A, B$ be any 2 subsets of vector spaces $V$ and $W$, respectively. Two continuous maps $f_0, f_1 : A \to B$ are called *homotopic* if there exists a continuous map $F : A \times [0,1] \to B$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $t \in [0,1]$. Notice that the family $f_t : A \to B$, $t \in [0,1]$, defined by the formula $f_t(x) = F(x,t)$ is a continuous deformation connecting $f_0$ and $f_1$. Conversely, any such continuous deformation $\{f_t\}_{t\in[0,1]}$ provides a homotopy between $f_0$ and $f_1$.

Given a subset $C \subset A$, we say that a homotopy $\{f_t\}_{t\in[0,1]}$ is fixed over $C$ if $f_t(x) = f_0(x)$ for all $x \in C$ and all $t \in [0,1]$.

A set $A$ is called *contractible* if there exists a point $a \in A$ and a homotopy $f_t : A \to A$, $t \in [0,1]$, such that $f_1 = \mathrm{Id}$ and $f_0$ is a constant map, i.e. $f_1(x) = x$ for all $x \in A$ and $f_0(x) = a \in A$ for all $x \in A$.

**Example 6.2.1.** *Any star-shaped domain $A$ in $V$ is contractible. Indeed, assuming that it is star-shaped with respect to the origin, the required homotopy $f_t : A \to A$, $t \in [0,1]$, can be defined by the formula $f_t(x) = tx,\ x \in A$.*

**Remark 6.2.2.** In what follows we will always assume all homotopies to be smooth. According to Theorem 4.12.1 this is not a serious constraint. Indeed, any continuous map can be $C^0$-approximated by smooth ones, and any homotopy between smooth maps can be $C^0$-approximated by a smooth homotopy between the same maps.

**Lemma 6.2.3.** *Let $U \subset V$ be an open set, $A$ a compact oriented manifold (possibly with boundary) and $\alpha$ a smooth closed differential $k$-form on $U$. Let $f_0, f_1 : A \to U$ be two maps which are homotopic relative to the boundary $\partial A$. Then*

$$\int\limits_A f_0^*\alpha = \int\limits_A f_1^*\alpha.$$

**Proof.** Let $F : A \times [0,1] \to U$ be the homotopy map between $f_0$ and $f_1$. By assumption

$d\alpha = 0$, and hence $\int\limits_{A\times[0,1]} F^* d\alpha = 0$. Then, using Stokes' theorem we have

$$0 = \int\limits_{A\times[0,1]} F^* d\alpha = \int\limits_{A\times[0,1]} dF^*\alpha = \int\limits_{\partial(A\times[0,1])} F^*\alpha = \int\limits_{\partial A\times[0,1]} F^*\alpha + \int\limits_{A\times 1} F^*\alpha + \int\limits_{A\times 0} F^*\alpha$$

where the boundary $\partial(A \times [0,1]) = (A \times 1) \cap (A \times 0) \cap (\partial A \times [0,1])$ is oriented by an outward normal vector field $\mathbf{n}$. Note that $\mathbf{n} = \frac{\partial}{\partial t}$ on $A \times 1$ and $\mathbf{n} = -\frac{\partial}{\partial t}$ on $A \times 0$, where we denote by $t$ the coordinate corresponding to the factor $[0,1]$. First, we notice that $F^*\alpha|_{\partial A\times[0,1]} = 0$ because the map $F$ is independent of the coordinate $t$, when restricted to $\partial A \times [0,1]$. Hence $\int\limits_{\partial A\times[0,1]} F^*\alpha = 0$. Consider the inclusion maps $A \to A \times [0,1]$ defined by the formulas $j_0(x) = (x,0)$ and $j_1(x) = (x,1)$. Note that $j_0, j_1$ are diffeomorphisms $A \to A \times 0$ and $A \to A \times 1$, respectively. Note that the map $j_1$ preserves the orientation while $j_0$ reverses it. We also have $F \circ j_0 = f_0$, $F \circ j_1 = f_1$. Hence, $\int\limits_{A\times 1} F^*\alpha = \int\limits_A f_1^*\alpha$ and $\int\limits_{A\times 0} F^*\alpha = -\int\limits_A f_0^*\alpha$. Thus,

$$0 = \int\limits_{\partial(A\times[0,1])} F^*\alpha = \int\limits_{\partial A\times[0,1]} F^*\alpha + \int\limits_{A\times 1} F^*\alpha + \int\limits_{A\times 0} F^*\alpha = \int\limits_A f_1^*\alpha - \int\limits_A f_0^*\alpha.$$

■

**Lemma 6.2.4.** *Let $A$ be an oriented $m$-dimensional manifold, possibly with boundary. Let $\Omega(A)$ denote the space of differential forms on $A$ and $\Omega(A \times [0,1])$ denote the space of differential forms on the product $A \times [0,1]$. Let $j_0, j_1 : A \to A \times [0,1]$ be the inclusion maps $j_0(x) = (x,0) \in A \times [0,1]$ and $j_1(x) = (x,1) \in A \times [0,1]$. Then there exists a linear map $K : \Omega(A \times [0,1]) \to \Omega(A)$ such that*

- *If $\alpha$ is a $k$-form, $k = 1, \ldots, m$ then $K(\alpha)$ is a $(k-1)$-form;*
- *$d \circ K + K \circ d = j_1^* - j_0^*$, i.e. for each differential $k$-form $\alpha \in \Omega^k(A \times [0,1]$ one has $dK(\alpha) + K(d\alpha) = j_1^*\alpha - j_0^*\alpha$.*

**Remark 6.2.5.** Note that the first $d$ in the above formula denotes the exterior differential $\Omega^k(A) \to \Omega^k(A)$, while the second one is the exterior differential $\Omega^k(A \times [0,1]) \to \Omega^k(A \times [0,1])$.

**Proof.** Let us write a point in $A \times [0,1]$ as $(x,t)$, $x \in A, t \in [0,1]$. To construct $K(\alpha)$ for a given $\alpha \in \Omega^k(A \times [0,1]$ we first contract $\alpha$ with the vector field $\frac{\partial}{\partial t}$ and then integrate the resultant form with respect to the $t$-coordinate. More precisely, note that any $k$-form $\alpha$ on $A \times [0,1]$ can be written as $\alpha = \beta(t) + dt \wedge \gamma(t)$, $t \in [0,1]$, where for each $t \in [0,1]$

$$\beta(t) \in \Omega^k(A), \gamma(t) \in \Omega^{k-1}(A).$$

Then $\frac{\partial}{\partial t} \lrcorner \alpha = \gamma(t)$ and we define $K(\alpha) = \int_0^1 \gamma(t)dt$.

If we choose a local coordinate system $(u_1, \ldots, u_m)$ on $A$ then $\gamma(t)$ can be written as $\gamma(t) = \sum_{1 \leq i_1 < \cdots < i_{\leq} m} h_{i_1 \ldots i_k}(t) du_{i_1} \wedge \cdots \wedge du_{i_k}$, and hence

$$K(\alpha) = \int_0^1 \gamma(t)dt = \sum_{1 \leq i_1 < \cdots < i_{\leq} m} \left( \int_0^1 h_{i_1 \ldots i_k}(t)dt \right) du_{i_1} \wedge \cdots \wedge du_{i_k}.$$

Clearly, $K$ is a linear operator $\Omega^k(A \times I) \to \Omega^{k-1}(A)$.

Note that if $\alpha = \beta(t) + dt \wedge \gamma(t) \in \Omega^k(A \times [0,1])$ then

$$j_0^* \alpha = \beta(0), \quad j_1^* \alpha = \beta(1).$$

We further have

$$K(\alpha) = \int_0^1 \gamma(t)dt;$$

$$d\alpha = d_{U \times I}\alpha = d_U \beta(t) + dt \wedge \dot{\beta}(t) - dt \wedge d_U \gamma(t) = d_U \beta(t) + dt \wedge (\dot{\beta}(t) - d_U \gamma(t)),$$

where we denoted $\dot{\beta}(t) := \frac{\partial \beta(t)}{\partial t}$ and $I = [0,1]$. Here the notation $d_{U \times I}$ stands for exterior differential on $\Omega(U \times I)$ and $d_U$ denotes the exterior differential on $\Omega(U)$. In other words, when we write $d_U \beta(t)$ we view $\beta(t)$ as a form on $U$ depending on $t$ as a parameter. We do not write any subscript for $d$ when there could not be any misunderstanding.

Hence,

$$K(d\alpha) = \int_0^1 (\dot\beta(t) - d_U\gamma(t))dt = \beta(1) - \beta(0) - \int_0^1 d_U\gamma(t)dt;$$

$$d(K(\alpha)) = \int_0^1 d_U\gamma(t)dt.$$

Therefore,

$$d(K(\alpha)) + K(d(\alpha)) = \beta(1) - \beta(0) - \int_0^1 d_U\gamma(t)dt + \int_0^1 d_U\gamma(t)dt$$

$$= \beta(1) - \beta(0) = j_1^*(\alpha) - j_0^*(\alpha).$$

■

**Theorem 6.2.6.** (Poincaré's lemma) *Let $U$ be a contractible domain in $V$. Then any closed form in $U$ is exact. More precisely, let $F : U \times [0,1] \to U$ be the contraction homotopy to a point $a \in U$, i.e. $F(x,1) = x$, $F(x,0) = a$ for all $x \in U$. Then if $\omega$ a closed $k$-form in $U$ then*

$$\omega = dK(F^*\omega),$$

*where $K : \Omega^{k+1}(U \times [0,1]) \to \Omega^k(U)$ is an operator constructed in Lemma 6.2.4.*

**Proof.** Consider a contraction homotopy $F : U \times [0,1] \to U$. Then $F \circ j_0(x) = a \in U$ and $F \circ j_1(x) = x$ for all $x \in U$. Consider an operator $K : \Omega U) \to \Omega(U)$ constructed above. Thus

$$K \circ d + d \circ K = j_1^* - j_0^*.$$

Let $\omega$ be a closed $k$-form on $U$. Denote $\alpha := F^*\omega$. Thus $\alpha$ is a $k$-form on $U \times [0,1]$. Note that $d\alpha = dF^*\omega = F^*d\omega = 0$, $j_1^*\alpha = (F \circ j_1)^*\omega = \omega$ and $j_0^*\alpha = (F \circ j_0)^*\omega = 0$. Then, using Lemma 6.2.4 we have

$$K(d\alpha) + dK(\alpha) = dK(\alpha) = j_1^*\alpha - j_0^*\alpha = \omega, \tag{6.1}$$

i.e. $\omega = dK(F^*\omega)$. ■

In particular, *any closed form is locally exact.*

**Example 6.2.7.** Let us work out explicitly the formula for a primitive of a closed 1-form in a star-shaped domain $U \subset \mathbb{R}^n$. We can assume that $U$ is star-shaped with respect to the origin. Let $\alpha = \sum\limits_1^n f_i dx_i$ be a closed 1-form. Then according to Theorem 6.2.6 we have $\alpha = dF$, where $F = K(\Phi^*\alpha)$, where $\Phi : U \times [0,1] \to U$ is a contraction homotopy, i.e. $\Phi(x,1) = x, \Phi(x,1) = 0$ for $x \in U$. $\Phi$ can be defined by the formula $\Phi(x,t) = tx,\ x \in U, t \in [0,1]$. Then

$$\Phi^*\alpha = \sum_1^n f_i(tx)d(tx_i) = \sum_1^n tf_i(tx)dx_i + \sum_1^n x_i f_i(tx)dt.$$

Hence,

$$K(\alpha) = \int_0^1 \frac{\partial}{\partial t} \lrcorner \Phi^*\alpha = \int_0^1 \left(\sum_1^n x_i f_i(tx)\right) dt.$$

Note that this expression coincides with the expression in formula (4.7) in Section 4.3.

**Exercise 6.2.8.** Work out an explicit expression for a primitive of a closed 2-form $\alpha = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ on a star-shaped domain $U \subset \mathbb{R}^3$.

**Example 6.2.9.** 1. $\mathbb{R}^n_+ = \{x_1 \geq 0\} \subset \mathbb{R}^n$ *is not diffeomorphic to* $\mathbb{R}^n$. Indeed, suppose there exists such a diffeomorphism $f : \mathbb{R}^n_+ \to \mathbb{R}_n$. Denote $a := f(0)$. Without loss of generality we can assume that $a = 0$. Then $\widetilde{f} = f|_{\mathbb{R}^n_+ \setminus 0}$ is a diffeomorphism $\mathbb{R}^n_+ \setminus 0 \to \mathbb{R}^n \setminus 0$. But $\mathbb{R}^n_+ \setminus 0$ is star-shaped with respect to any point with positive coordinate $x_1$, and hence it is contractible. In particular any closed form on $\mathbb{R}^n_+ \setminus 0$ is exact. On the other hand, we exhibited above in 6.1.2 a closed $(n-1)$-form on $\mathbb{R}^n \setminus 0$ which is not exact.

2. Borsuk's theorem: *There is no continuous map $D^n \to \partial D^n$ which is the identity on $\partial D^n$.* We denote here by $D^n$ the unit disc in $\mathbb{R}^n$ and by $\partial D^n$ its boundary $(n-1)$-sphere.

**Proof.** Suppose that there is such a map $f : D^n \to \partial D^n$. One can assume that $f$ is smooth. Indeed, according to Theorem 4.12.1 one can approximate $f$ by a smooth map, keeping it fixed on the boundary where it is the identity map, and hence smooth. Take the closed non-exact form $\theta_n$ from Example 6.1.2 on $\partial D^n$. Then $\Theta_n = f^*\theta_n$ is a closed $(n-1)$-form on $D^n$ which coincides with $\theta_n$ on $\partial D^n$. $D^n$ is star-shaped, and therefore $\Theta_n$ is exact, $\Theta_n = d\omega$. But then $\theta_n = d(\omega|_{\partial D^n})$ which is a contradiction. ■

3. Brouwer's fixed point theorem: *Any continuous map $f : D^n \to D^n$ has at least 1 fixed point.*

**Proof.** Suppose $f : D^n \to D^n$ has no fixed points. Let us define a map $F : D^n \to \partial D^n$ as follows. For each $x \in D^n$ take a ray $r_x$ from the point $f(x)$ which goes through $x$ till it intersects $\partial D^n$ at a point which we will denote $F(x)$. The map is well defined because for any $x$ the points $x$ and $f(x)$ are distinct. Note also that if $x \in \partial D^n$ then the ray $r_x$ intersects $\partial D^n$ at the point $x$, and hence $F(x) = x$ in this case. But existence of such $F$ is ruled out by Borsuk' theorem. ■

### $k$-connected manifolds

A subset $A \subset V$ is called *$k$-connected*, $k = 0, 1, \ldots,$ if for any $m \leq k$ any two continuous maps of discs $f_0, f_1 : D^m \to A$ which coincide along $\partial D^m$ are homotopic relative to $\partial D^m$. Thus, 0-connectedness is equivalent to path-connectedness. 1-connected submanifolds are also called *simply connected.*

**Exercise 6.2.10.** *Prove that $k$-connectedness can be equivalently defined as follows: $A$ is $k$-connected if any map $f : S^m \to A$, $m \leq k$ is homotopic to a constant map.*

**Example 6.2.11.**

1. *If $A$ is contractible then it is $k$-connected for any $k$.* For some classes of subsets, e.g. submanifolds, the converse is also true (J.H.C Whitehead's theorem) but this is a quite deep and non-trivial fact.

2. *The $n$-sphere $S^n$ is $(n-1)$-connected but not $n$-connected.* Indeed, to prove that $S^{n-1}$ simply connected we will use the second definition. Consider a map $f : S^k \to S^n$. We first notice that according to Theorem 4.12.1 we can assume that the map $f$ is smooth. Hence, according to Corollary 4.4.8 $\mathrm{Vol}_n f(S^k) = 0$ provided that $k < n$. In

particular, there exists a point $p \in S^n \setminus f(S^k)$. But the complement of a point $p$ in $S^n$ is diffeomorphic to $S^n$ vis the stereographic projection from the point $p$. But $R^n$ is contractible, and hence $f$ is homotopic to a constant map. On the other hand, the identity map $\mathrm{Id}: S^n \to S^n$ is not homotopic to a constant map. Indeed, we know that there exists a closed $n$-form on $S^n$, say the form $\theta_n$ from Example 6.1.2, such that $\int\limits_{S^n} \theta_n \neq 0$. Hence, $\int\limits_{S^n} \mathrm{Id}^*\theta_n \neq 0$. On the other hand if Id were homotopic to a constant map this integral would vanish.

**Exercise 6.2.12.** Prove that $\mathbb{R}^{n+1} \setminus 0$ is $(n-1)-$ *connected* but not $n$-connected.

**Proposition 6.2.13.** *Let $U \subset V$ be a $m$-connected domain. Then for any $k \leq m$ any closed differential $k$-form $\alpha$ in $U$ is exact.*

**Proof.** We will prove here only the case $m = 1$. Though the general case is not difficult, it requires developing certain additional tools. Let $\alpha$ be a closed differential 1-form. Choose a reference point $b \in U$. By assumption, $U$ is path-connected. Hence, any other point $x$ can be connected to $b$ by a path $\gamma_x : [0,1] \to U$, i.e. $\gamma_x(0) = b,\ \gamma_x(1) = x$. Let us define the function $F: U \to \mathbb{R}$ by the formula $F(x) = \int\limits_{\gamma_x} \alpha$. Note that due to the simply-connectedness of the domain $U$, any $\delta : [0,1] \to U$ connecting $b$ and $x$ is homotopic to $\gamma_x$ relative its ends, and hence according to Lemma 6.2.3 we have $\int\limits_{\gamma_x} \alpha = \int\limits_{\delta} \alpha$. Thus the above definition of the function $F$ is independent of the choice of paths $\gamma_x$. We claim that the function $F$ is differentiable and $dF = \alpha$. Note that if the primitive of $\alpha$ exists than it has to be equal to $F$ up to an additive constant. But we know that in a sufficiently small ball $B_\epsilon(a)$ centered at any point $a \in U$ there exists a primitive $G$ of $\alpha$. Hence, $G(x) = F(x) + \mathrm{const}$, and the the differentiability of $G$ implies differentiablity of $F$ and we have $dF = dG = \alpha$. ■

### Winding and linking numbers

Given a loop $\gamma : S^1 \to \mathbb{R}^2 \setminus 0$ we define its *winding number* around 0 as the integral

$$w(\gamma) = \frac{1}{2\pi} \int\limits_{S}^{1} \theta_2 = \frac{1}{2\pi} \int\limits_{S^1} \frac{x dy - y dx}{x^2 + y^2},$$

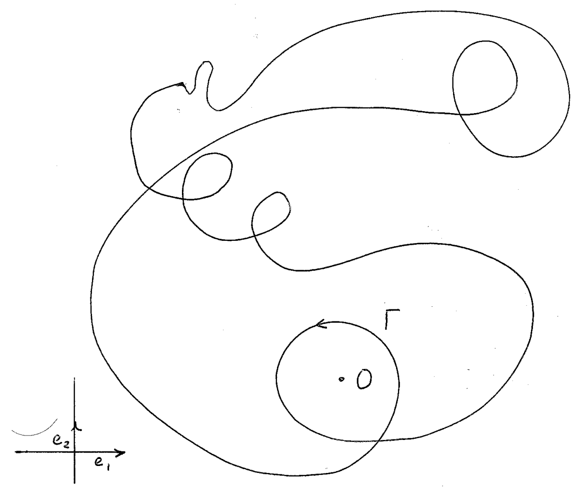


Figure 4: $w(\Gamma) = 2$.

where we orient $S^1$ as the boundary of the unit disc in $\mathbb{R}^2$. For instance, if $j : S^1 \hookrightarrow \mathbb{R}^2$ is the inclusion map then $w(j) = 1$. For the loop $\gamma_n$ parameterized by the map $t \mapsto (\cos nt, \sin nt), t \in [0, 2\pi]$ we have $w(\gamma_n) = n$.

**Proposition 6.2.14.** *1. For any loop $\gamma$ the number $w(\gamma)$ is an integer.*

2. *If loops $\gamma_0, \gamma_1 : S^1 \to \mathbb{R}^2 \setminus 0$ are homotopic in $\mathbb{R}^2 \setminus 0$ then $w(\gamma_0) = w(\gamma_1)$.*

3. *$w(\gamma) = n$ then the loop $\gamma$ is homotopic (as a loop in $\mathbb{R}^2 \setminus 0$) to the loop $\zeta_n : [0, 1] \to \mathbb{R}^2 \setminus 0$ given by the formula $\zeta_n(t) = (\cos 2\pi nt, \sin 2\pi nt)$.*

**Proof.** 1. Let us define the loop $\gamma$ parametrically in polar coordinates:

$$r = r(s), \phi = \phi(s),\ s \in [0, 1],$$

where $r(0) = r(1)$ and $\phi(1) = \phi(0) + 2n\pi$. The form $\theta_2$ in polar coordinates is equal to $d\phi$, and hence

$$\int_\gamma \alpha = \frac{1}{2\pi}\int_0^1 \phi'(s)ds = \frac{\phi(1)-\phi(0)}{2\pi} = n.$$

2. This is an immediate corollary of Proposition 6.2.3.

3. Let us write both loops $\gamma$ and $\zeta_n$ in polar coordinates. Respectively,we have $r = r(t), \phi = \phi(s)$ for $\gamma$ and $r = 1, \phi = 2\pi ns$ for $\zeta_n$, $s \in [0,1]$. The condition $w(\gamma) = n$ implies, in view of part 1, that $\phi(1) = \phi(0) + 2n\pi$. Then the required homotopy $\gamma_t$, $t \in [0,1]$, connecting the loops $\gamma_0 = \gamma$ and $\gamma_1 = \zeta_n$ can be defined by the parametric equations $r = (1-t)r(s) + t$, $\phi = \phi_t(s) = (1-t)\phi(s) + 2n\pi st$. Note that for all $t \in [0,1]$ we have $\phi_t(1) = \phi_t(0) + 2n\pi$. Therefore, $\gamma_t$ is a loop for all $t \in [0,1]$. ■

Given two disjoint loops $\gamma, \delta : S^1 \to \mathbb{R}^3$ (i.e. $\gamma(s) \neq \delta(t)$ for any $s, t \in S^1$) consider a map $F_{\gamma,\delta} : T^2 \to \mathbb{R}^3 \setminus 0$, where $T^2 = S^1 \times S^1$ is the 2-torus, defined by the formula

$$F_{\gamma,\delta}(s,t) = \gamma(s) - \delta(t).$$

Then the number

$$l(\gamma,\delta) := \frac{1}{4\pi}\int_{T}^{2} F^*_{\gamma,\delta}\theta_3 = \frac{1}{4\pi}\int_{T}^{2} F^*_{\gamma,\delta}\left(\frac{xdy\wedge dz + ydz\wedge dx + zdx\wedge dy}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$$

is called the *linking number* of loops $\gamma, \delta$.[13]

**Exercise 6.2.15.** Prove that

1. The number $l(\gamma,\delta)$ remains unchanged if one continuously deforms the loops $\gamma, \delta$ *keeping them disjoint*;

2. The number $l(\gamma,\delta)$ is an integer for any disjoint loops $\gamma, \delta$;

3. $l(\gamma,\delta) = l(\delta,\gamma)$;

4. Let $\gamma(s) = (\cos s, \sin s, 0), s \in [0, 2\pi]$ and $\delta(t) = (-1 + \frac{1}{2}\cos t, 0, \frac{1}{2}\sin t)$, $t \in [0, 2\pi]$. Then $l(\gamma,\delta) = 1$.

[13]This definition of the linking number is due to Carl Friedrich Gauss.

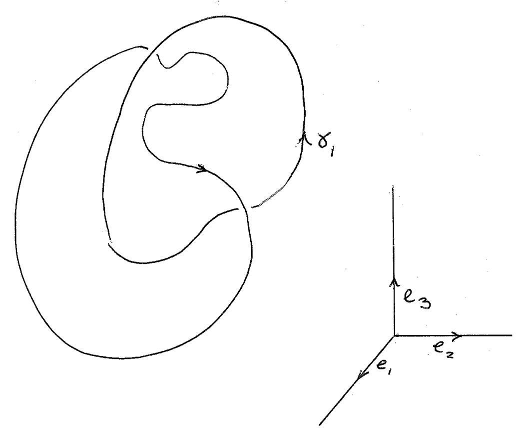


Figure 5: $l(\gamma_1, \gamma_2) = 1$.

## 6.3 Properties of $k$-forms on $k$-dimensional manifolds

A $k$-form $\alpha$ on $k$-dimensional submanifold is always closed. Indeed, $d\alpha$ is a $(k+1)$-form and hence it is identically 0 on a $k$-dimensional manifold.

**Remark 6.3.1.** Given a $k$-dimensional submanifold $A \subset V$, and a $k$-form $\alpha$ on $V$, the differential $d_x\alpha$ does not need to vanish at a point $x \in A$. However, $d\alpha_x|_{T_x(A)}$ does vanish.

The following theorem is the main result of this section.

**Theorem 6.3.2.** *Let $A \subset V$ be an orientable compact connected $k$-dimensional submanifold, possibly with boundary, and $\alpha$ a differential $k$-form on $A$.*

1. *Suppose that $\partial A \neq \varnothing$. Then $\alpha$ is exact, i.e. there exists a $(k-1)$-form $\beta$ on $A$ such that $d\beta = \alpha$.*

2. *Suppose that $A$ is closed, i.e. $\partial A = \varnothing$. Then $\alpha$ is exact if and only if $\int\limits_A \alpha = 0$.*

To prove Theorem 6.3.2 we will need a few lemmas.

**Lemma 6.3.3.** *Let $I^k$ be the $k$-dimensional cube* $\{-1 \le x_j \le 1,\ j = 1, \dots, k\}$.

1. *Let $\alpha$ be a differential $k$-form on $I^k$ such that*

$$\operatorname{Supp}(\alpha) \cap \left(0 \times I^{k-1} \cup [0,1] \times \partial I^{k-1}\right) = \varnothing.$$

   *Then there exists a $(k-1)$-form $\beta$ such that $d\beta = \alpha$ and such that* $\operatorname{Supp}(\beta) \cap \left(-1 \times I^{k-1} \cup [-1,1] \times \partial I^{k-1}\right) = \varnothing$.

2. *Let $\alpha$ be a differential $k$-form on $I^k$ such that* $\operatorname{Supp}(\alpha) \subset \operatorname{Int} I^k$ *and* $\int\limits_{I^k} = 0$. *Then there exists a $(k-1)$-form $\beta$ such that $d\beta = \alpha$ and* $\operatorname{Supp}(\beta) \subset \operatorname{Int} I^k$.

**Proof.** We have

$$\alpha = f(x_1, \dots, x_k) dx_1 \wedge \dots \wedge dx_k,$$

In the first case of the lemma the function $f$ vanishes on $0 \times I^{k-1} \cup [-1,1] \times \partial I^{k-1}$. We will look for $\beta$ in the form

$$\beta = g(x_1, \dots, x_k) dx_2 \wedge \dots \wedge dx_k.$$

Then

$$d\beta = \frac{\partial g}{\partial x_1}(x_1, \dots, x_k) dx_1 \wedge dx_2 \wedge \dots \wedge dx_k.$$

and hence the equation $d\beta = \alpha$ is equivalent to

$$\frac{\partial g}{\partial x_1}(x_1, \dots, x_k) = f(x_1, \dots, x_k).$$

Hence, if we define

$$g(x_1, \dots, x_k) := \int\limits_{-1}^{x_1} f(u, x_2, \dots, x_k) du,$$

then the form $\beta = g(x_1, \dots, x_k) dx_2 \wedge \dots \wedge dx_k$ has the required properties.

The second part of the lemma we will prove here only for the case $k = 2$. The general case can be handled similarly by induction over $k$. We have in this case $\operatorname{Supp}(f) \subset \operatorname{Int} I^2$ and

$\int\limits_{I^2} f dS = 0$. Let us denote $h(x_2) := \int\limits_{-1}^{1} f(x_1, x_2)dx_1$. Note that $h(u) = 0$ if $u$ is sufficiently close to $-1$ or $1$. According to Fubini's theorem, $\int\limits_{-1}^{1} h(x_2)dx_2 = 0$. We can assume that $f(x_1, x_2) = 0$ for $x_1 \geq 1 - \epsilon$, and hence $\int\limits_{-1}^{u} f(x_1, \dots, x_{k-1}, t)dt = h(x_1, \dots, x_{k-1})$ for $u \in [1-\epsilon, 1]$. Consider any non-negative $C^\infty$-function $\theta : [1-\epsilon, 1] \to \mathbb{R}$ such that $\theta(u) = 1$ for $u \in [1-\epsilon, 1-\frac{2\epsilon}{3}]$ and $\theta(u) = 0$ for $u \in [1 - \frac{\epsilon}{3}, 1]$. Define a function $g_1 : I^2 \to \mathbb{R}$ by the formula

$$g_1(x_1 x_2) = \begin{cases} \int\limits_{-1}^{x_1} f(u, x_2)du, & x_1 \in [-1, 1-\epsilon], \\ h(x_2)\theta(x_1), & x_1 \in (1-\epsilon, 1]. \end{cases}$$

Denote $\beta_1 = g_1(x_1, x_2)dx_2$. Then $d\beta = \alpha$ on $[-1, 1-\epsilon] \times [0, 1]$ and $d\beta_1 = h(x_2)\theta'(x_1)dx_1 \wedge dx_2$ on $[1-\epsilon, 1] \times [0, 1]$. Note that $\mathrm{Supp}(\beta_1) \subset \mathrm{Int}\, I^2$.

Let us define

$$g_2(x_1, x_2) := \begin{cases} 0, & x_1 \in [-1, 1-\epsilon], \\ \theta'(x_1) \int\limits_{-1}^{x_2} h(u)du, & x_1 \in (1-\epsilon, 1] \end{cases}$$

and denote $\beta_2 = g_2(x_1, x_2)dx_1$. Then $d\beta_2 = 0$ on $[-1, 1-\epsilon] \times [-1, 1]$ and

$$d\beta_2 = -h(x_2)\theta'(x_1)dx_1 \wedge dx_2$$

on $[1-\epsilon, 1] \times [-1, 1]$. Note that $g_2(x_1, 1) = -\theta'(x_1) \int\limits_{-1}^{1} h(u)du = 0$. Taking into account that $h(u) = 0$ when $u$ is sufficiently close to $-1$ or $1$ we conclude that $h(x_1, x_2) = 0$ near $\partial I^2$, i.e. $\mathrm{Supp}(\beta_2) \subset \mathrm{Int}\, I^2$. Finally, if we define $\beta = \beta_1 + \beta_2$ then we have $d\beta = \alpha$ and $\mathrm{Supp}(\beta) \subset \mathrm{Int}\, I^2$. ■

The following lemma is a special case of the, so-called, *tubular neighborhood theorem.*

**Lemma 6.3.4.** *Let $A \subset V$ be a compact $k$-dimensional submanifold with boundary. Let $\phi : [-1, 1] \to A$ be an embedding such that $\phi(1) \in \partial A$, $\phi'(1) \perp T_{\phi(1)}(\partial A)$ and $\phi([0, 1)) \subset \mathrm{Int}\, A$. Then the embedding $\phi$ can be extended to an embedding $\Phi : [-1, 1] \times I^{k-1} \to A$ such that*

- $\Phi(t, 0) = \phi(t)$, *for $t \in [-1, 1]$, $0 \in I^{k-1}$;*

- $\Phi(1 \times I^{k-1}) \subset \partial A$, $\Phi([-1,1) \times I^{k-1}) \subset \mathrm{Int}\, A$;
- $\frac{\partial \Phi}{\partial t}(1, x) \notin T(\partial A)$ *for all* $x \in I^{k-1}$.

There are many ways to prove this lemma. We will explain below one of the arguments.

**Proof.** Step 1. We first construct $k-1$ ortonormal vector fields $\nu_1, \dots, \nu_k$ along $\Gamma = \phi([-1,1])$ which are tangent to $A$ and normal to $\Gamma$. To do that let us denote by $N_u$ the normal $(k-1)$-dimensional space $N_u$ to $T_u\Gamma$ in $T_uA$. Let us observe that in view of compactness of $\Gamma$ there is an $\epsilon > 0$ with the following property: for any two points $u = \phi(t), u' = \phi(t') \in \Gamma$, $t, t' \in [-1,1]$, such that $|t - t'| \leq \epsilon$ the orthogonal projection $N_u \to N_{u'}$ is non-degenerate (i.e. is an isomorphism). Choose $N < \frac{1}{2\epsilon}$ and consider points $u_j = \phi(t_j)$, where $t_j = -1 + \frac{2j}{N}, j = 1, \dots N$. Choose any orthonormal basis $\nu_1(0), \dots, \nu_k(0) \in N_{u_0}$, parallel transport these vectors to all points of the arc $\Gamma_1 = \phi([-1, t_1])$, project them orthogonally to the normal spaces $N_u$ in these points, and then orthonormalize the resulted bases via the Gram-Schmidt process. Thus we constructed orthonormal vector fields $\nu_1(t), \dots \nu_k(t) \in N_{\phi(t)}$, $t \in [-1, t_1]$. Now we repeat this procedure beginning with the basis $\nu_1(t_1), \dots \nu_k(t_1) \in N_{\phi(t_1)} = N_{u_1}$ and extend the vector fields $\nu_1, \dots, \nu_k$ to $\Gamma_2 = \phi([t_1, t_2])$. Continuing this process we will construct the orhonormal vector fields $\nu_1, \dots, \nu_k$ along the whole curve $\Gamma$.[14]

Step 2. Consider a map $\Psi : [-1,1] \times I^{k-1} \to V$ given by the formula

$$\Psi(t, x_1, \dots, x_{k-1}) = \phi(t) + \sigma \sum_1^{k-1} x_j \nu_j(t),\ t, x_1, \dots, x_{k-1} \in [-1,1],$$

where a small positive number $\sigma$ will be chosen later. The map $\Psi$ is an embedding if $\sigma$ is chosen small enough.[15] Unfortunately the image $\Psi([-1,1] \times I^{k-1})$ is not contained in $A$. We will correct this in the next step.

Step 3. Take any point $a \in A$ and denote by $\pi_a$ the orthogonal projection $V \to T_aA$. Let us make the following additional assumption (in the next step we will show how to get rid of

[14]Strictly speaking, the constructed vector fields only piece-wise smooth, because we did not make any special precautions to ensure smoothness at the points $u_j, j = 1, \dots, N-1$. This could be corrected via a standard smoothing procedure.

[15]Exercise: prove it!

it): there exists a neighborhood $U \ni a = \phi(1)$ in $\partial A$ such that $\pi_a(U) \subset N_a \subset T_aA$. Given $\epsilon > 0$ let us denote by $B_\epsilon(a)$ the $(k-1)$-dimensional ball of radius $\epsilon$ in the space $N_\alpha \subset T_aA$. In view of compactness of $A$ one can choose an $\epsilon > 0$ such that for all points $a \in \Gamma$ there exists an embedding $e_a : B_\epsilon(a) \to A$ such that $\pi_a \circ e_a = \mathrm{Id}$. Then for a sufficiently small $\sigma < \frac{\epsilon}{\sqrt{k-1}}$ the map $\widetilde{\Psi} : [-1,1] \times I^{k-1} \to A$ defined by the formula

$$\widetilde{\Psi}(t,x) = e_{\phi(t)} \circ \Psi(t,x),\ t \in [-1,1], x \in I^{k-1}$$

is an embedding with the required properties.

STEP 4 It remains to show how to satisfy the additional condition at the boundary point $\phi(1) \in \Gamma \cap \partial A$ which were imposed above in Step 3. Take the point $a = \phi(1) \in \Gamma \cap \partial A$. Without loss of generality we can assume that $a = 0 \in V$. Choose an orthonormal basis $v_1, \dots, v_n$ of $V$ such that $v_1, \dots, v_k \in N_a$ and $v_k$ is tangent to $\Gamma$ and pointing inward $\Gamma$. Let $(y_1, \dots, y_n)$ be the corresponding cartesian coordinates in $V$. Then there exists a neighborhood $U \ni a$ in $A$ which is graphical in these coordinates and can be given by

$$y_j = \theta_j(y_1, \dots, y_k),\ j = k+1, \dots, n,\ \epsilon \geq y_k \geq \theta_k(y_1, \dots, y_{k-1}), |y_i| \leq \epsilon, i = 1, \dots, k-1,$$

where all the first partial derivatives of the functions $\theta_k, \dots, \theta_n$ vanish at the origin. Take a $C^\infty$ cut-off function $\sigma : [0,\infty)] \to \mathbb{R}$ which is equal to 1 on $[0, \frac{1}{2}]$ and which is supported in $[0,1]$ (see Lemma 4.11.2). Consider a map $F$ given by the formula

$$F(y_1, \dots, y_n) = (y_1, \dots, y_{k-1}, y_k - \theta_k(y_1, \dots, y_{k-1})\sigma\left(\frac{||y||}{\epsilon}\right), y_{k+1}, \dots, y_n).$$

For a sufficiently small $\epsilon > 0$ this is a diffeomorphism supported in an $\epsilon$-ball in $V$ centered in the origin. On the other hand, the manifold $\widetilde{A} = F(A)$ satisfies the extra condition of Step 3. ■

**Lemma 6.3.5.** *Let $A \subset V$ be a (path)-connected submanifold with a non-empty boundary. Then for any point $a \in A$ there exists an embedding $\phi_a : [-1,1] \to A$ such that $\phi_a(0) = a$, $\phi_a(1) \in \partial A$ and $\phi_a'(1) \perp T_{\phi_a(1)}(\partial A)$.*

**Sketch of the proof.** Because $A$ is path-connected with non-empty boundary, any interior point can be connected by a path with a boundary point. However, this path need not be an embedding. First, we perturb this path to make it an immersion $\psi : [-1,1] \to A$, i.e. a map with non-vanising derivative. This can be done as follows. As in the proof of the previous lemma we consider a sufficiently small partition of the path, so that two neighboring subdivision points lie in a coordinate neighborhood. Then we can connect these points by a straight segment in these coordinate neighborhoods. Finally we can smooth the corners via the standard smoothing procedure. Unfortunately the constructed immersed path $\psi$ may have self-intersection points. First, one can arrange that there are only finitely many intersections, and then "cut-out the loops", i.e. if $\psi(t_1) = \psi(t_2)$ for $t_1 < t_2$ we can consider a new piece-wise smooth path which consists of $\psi|_{[-1,t_1]}$ and $\psi|_{[t_2,1]}$ The new path has less self-intersection points, and thus continuing by induction we will end with a piece-wise smooth embedding. It remains to smooth again the corners. ■

**Proof of Theorem 6.3.2.** 1. For every point $a \in A$ choose an embedding $\phi_a : [-1,1] \to A$, as in Lemma 6.3.5, and using Lemma 6.3.4 extend $\phi_a$ to an embedding $\Phi_a : [-1,1] \times I^{k-1} \to A$ such that

- $\Phi_a(t,0) = \phi(t)$, for $t \in [-1,1]$, $0 \in I^{k-1}$;

- $\Phi_a(1 \times I^{k-1}) \subset \partial A$, $\Phi_a([-1,1) \times I^{k-1}) \subset \operatorname{Int} A$;

- $\frac{\partial \Phi_a}{\partial t}(1,x) \notin T(\partial A)$ for all $x \in I^{k-1}$.

Due to compactness of $A$ we can choose finitely many such embeddings $\Phi_j = \Phi_{a_j}$, $j = 1,\dots,N$, such that $\bigcup\limits_1^N \Phi_j((-1,1] \times \operatorname{Int}(I^{k-1}) = A$. Choose a partition of unity subordinated to this covering and split the $k$-form $\alpha$ as a sum $\alpha = \sum\limits_1^K \alpha_j$, where each $\alpha_i$ is supported in $\Phi_j((-1,1] \times \operatorname{Int}(I^{k-1}))$ for some $j = 1,\dots,N$. To simplify the notation we will assume that $N = K$ and each $\alpha_j$ is supported in $\Phi_j((-1,1] \times \operatorname{Int}(I^{k-1}))$, $j = 1,\dots,N$. Consider the pull-back form $\widetilde{\alpha}_j = \Phi_j^* \alpha_j$ on $I^k = [-1,1] \times I^{k-1}$. According to Lemma 6.3.3.1 there exists a $(k-1)$-form $\widetilde{\beta}_j$ such that $\operatorname{Supp}(\widetilde{\beta}_j) \subset (-1,1] \times \operatorname{Int}(I^{k-1})$ and $d\widetilde{\beta}_j = \widetilde{\alpha}_j$. Let us transport the form $\widetilde{\beta}_j$ back to $A$. Namely, set $\beta_j$ equal to $(\Phi_j^{-1})^*\widetilde{\beta}_j$ on $\Phi_j((-1,1]\times \mathrm{Int}(I^{k-1}))\subset A$ and extend it as 0 elsewhere on $A$. Then $d\beta_j=\alpha_j$, and hence $d(\sum\limits_1^N \beta_j)=\sum\limits_1^N \alpha_j=\alpha$.

2. Choose a point $a\in A$ and parameterize a coordinate neighborhood $U\subset A$ by an embedding $\Phi: I^k\to A$ such that $\Phi(0)=a$. Take a small closed ball $D_\epsilon(0)\subset I^k\subset \mathbb{R}^k$ and denote $\widetilde{D}=\Phi(D_\epsilon(0))$. Then $\widetilde{A}=A\setminus \mathrm{Int}\,\widetilde{D}$ is a submanifold with non-empty boundary, and $\partial\widetilde{A}=\partial\widetilde{D}$. Let us use part 1 of the theorem to construct a form $\widetilde{\beta}$ on $\widetilde{A}$ such that $d\widetilde{\beta}=\alpha|_{\widetilde{A}}$. Let us extent the form $\widetilde{\beta}$ in any way to a form, still denoted by $\widetilde{\beta}$ on the whole submanifold $A$. Then $d\widetilde{\beta}=\alpha+\eta$ where $\mathrm{Supp}(\eta)\subset\widetilde{D}\subset \mathrm{Int}\,\Phi(I^k)$. Note that

$$\int\limits_{\Phi(I^k)}\eta=\int\limits_A\eta=\int\limits_A\alpha-\int\limits_A d\widetilde{\beta}=0$$

because $\int\limits_A\alpha=0$ by our assumption, and $\int_A d\widetilde{\beta}=0$ by Stokes' theorem. Thus, $\int\limits_I^k\Phi^*\eta=0$, and hence, we can apply Lemma 6.3.3.2 to the form $\Phi^*\eta$ on $I^k$ and construct a $(k-1)$-form $\lambda$ on $I^{k-1}$ such that $d\lambda=\Phi^*\eta$ and $\mathrm{Supp}(\lambda)\subset \mathrm{Int}\,I^k$. Now we push-forward the form $\lambda$ to $A$, i.e. take the form $\widetilde{\lambda}$ on $A$ which is equal to $(\Phi^{-1})^*\lambda$ on $\Phi(I_k)$ and equal to 0 elsewhere. Finally, we have $d(\widetilde{\beta}+\widetilde{\lambda})=d\widetilde{\beta}+\eta=\alpha$, and hence $\beta=\widetilde{\beta}+\widetilde{\lambda}$ is the required primitive of $\alpha$ on $A$. ∎

**Corollary 6.3.6.** *Let $A$ be an oriented compact connected $k$-dimensional submanifold with non-empty boundary and $\alpha$ a differential $k$-form on $A$ from Theorem 6.3.2. Then for any smooth map $f: A\to A$ such that $f|_{\partial A}=\mathrm{Id}$ we have*

$$\int\limits_A f^*\alpha=\int\limits_A\alpha.$$

**Proof.** According to Theorem 6.3.2.1 there exists a form $\beta$ such that $\alpha=d\beta$. Then

$$\int\limits_A f^*\alpha=\int\limits_A f^*d\beta=\int\limits_A df^*\beta=\int\limits_{\partial A}f^*\beta=\int\limits_{\partial A}\beta=\int\limits_A\alpha.$$

∎

**Degree of a map**

Consider two closed connected oriented submanifolds $A \subset V$, $B \subset W$ of the same dimension $k$. Let $\omega$ be an $n$-form on $B$ such that $\int\limits_B \omega = 1$. Given a smooth map $f : A \to B$ the integer $\deg(f) := \int\limits_A f^*\omega$ is called the *degree* of the map $f$.

**Proposition 6.3.7.** *1. Given any two $k$-forms on $B$ such $\int\limits_B \omega = \int\limits_B \widetilde{\omega}$ we have $\int\limits_A f^*\omega = \int\limits_A f^*\widetilde{\omega}$, for any smooth map $f : A \to B$, and thus* $\deg(f)$ *is independent of the choice of the form $\omega$ on $B$ with the property $\int\limits_A \omega = 1$.*

*2. If the maps $f, g : A \to B$ are homotopic then* $\deg(f) = \deg(g)$.

*3. Let $b \in B$ be a regular value of the map $f$. Let $f^{-1}(b) = \{a_1, \dots, a_d\}$. Then*

$$\deg(f) = \sum_1^d \mathrm{sign}(\det Df(a_j)).$$

*In particular,* $\deg(f)$ *is an integer number.*

**Proof.** The second part follows from Lemma 6.2.3. To prove the first part, let us write $\widetilde{\omega} = \omega + \eta$, where $\int\limits_B \eta = 0$. Using Theorem 6.3.2.2 we conclude that $\eta = d\beta$ for some $(k-1)$-form $\beta$ on $B$. Then

$$\int\limits_A f^*\widetilde{\omega} = \int\limits_A f^*\omega + \int\limits_A f^*\eta = \int\limits_A f^*\omega + \int\limits_A df^*\beta = \int\limits_A f^*\omega.$$

Let us prove the last statement of the theorem. By the inverse function theorem there exists a neighborhood $U \ni b$ in $B$ and neighborhoods $U_1 \ni a_1, \dots, U_d \ni a_d$ in $A$ such that the restrictions of the map $f$ to the neighborhoods $U_1, \dots, U_d$ are diffeomorphisms $f|_{U_j} : U_j \to U$, $j = 1, \dots, d$. Let us consider a form $\omega$ on $B$ such that $\mathrm{Supp}\omega \subset U$ and $\int\limits_B \omega = \int\limits_U \omega = 1$. Then

$$\deg(f) = \int\limits_A f^*\omega = \sum_1^d \int\limits_{U_j} f^*\omega = \sum_1^d \mathrm{sign}(\det Df(a_j)),$$

because according to Theorem 4.6.1 we have

$$\int_{U_j} f^*\omega = \operatorname{sign}(\det Df(a_j)) \int_U \omega = \operatorname{sign}(\det Df(a_j)).$$

for each $j = 1, \ldots, d$. ■

**Remark 6.3.8.** Any continuous map $f : A \to B$ can be approximated by a homotopic to $f$ smooth map $A \to B$, and any two such smooth approximations of $f$ are homotopic. Hence this allows us to define the degree of any *continuous* map $f : A \to B$.

**Exercise 6.3.9.** 1. Let us view $\mathbb{R}^2$ as $\mathbb{C}$. In particular, we view the unit sphere $S^1 = S^1_1(0)$ as the set of complex numbers of modulus 1:

$$S^1 = \{z \in \mathbb{C}; |z| = 1\}.$$

Consider a map $h_n : S^1 \to S^1$ given by the formula $h_n(z) = z^n, z \in S^1$. Then $\deg(h_n) = n$.
2. Let $f : S^{n-1} \to S^{n-1}$ be a map of degree $d$. Let $p_\pm$ be the north and south poles of $S^{n+1}$, i.e. $p_\pm = (0, \ldots, 0, \pm 1)$. Given any point $x = (x_1, \ldots, x_{n+1}) \in S^n \setminus \{p_+, p_-\}$ we denote by $\pi(x)$ the point

$$\frac{1}{\sqrt{\sum\limits_1^n x_j^2}}(x_1, \ldots, x_n) \in S^{n-1}$$

and define a map $\Sigma f : S^n \to S^n$ by the formula

$$\Sigma f(x) = \begin{cases} p_\pm, & \text{if } x = p_\pm, \\ \left(\sqrt{\sum\limits_1^n x_j^2} f(\pi(x)), x_{n+1}\right), & \text{if } x \neq p_\pm. \end{cases}$$

Prove that $\deg(\Sigma(f)) = d$.[16]
3. Prove that two maps $f, g : S^n \to S^n$ are homotopic if and only if they have the same degree. In particular, any map of degree $n$ is homotopic to the map $h_n$. (*Hint: For n=1 this follows from Proposition 6.2.14. For $n > 1$ first prove that any map is homotopic to a suspension.*)

[16] The map $\Sigma f$ is called the *suspension* of the map $f$.

4. Give an example of two non-homotopic orientation preserving diffeomorphisms $T^2 \to T^2$. Note that the degree of both these maps is 1. Hence, for manifolds, other than spheres, having the same degree is not sufficient for their homotopy.

5. Let $\gamma, \delta : S^1 \to \mathbb{R}^3$ be two disjoint loops in $\mathbb{R}^3$. Consider a map $\widetilde{F}_{\gamma,\delta} : T^2 \to S^2$ defined by the formula

$$\widetilde{F}_{\gamma,\delta}(s,t) = \frac{\gamma(s) - \delta(t)}{||\gamma(s) - \delta(t)||}, \ s, t \in S^1.$$

Prove that $l(\gamma, \delta) = \deg(\widetilde{F}_{\gamma,\delta})$. Use this to solve Exercise 6.2.15.4 above.