Math 52H: Solutions to Midterm Exam

February 2, 2011

1. Let $k \in \mathbb{R}$ be any real number. Consider on $\mathbb{R}^n \setminus 0$ a differential $(n-1)$-form

$$\theta_k = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{r^k} dx_1 \wedge \overset{i}{\check{\dots}} \wedge dx_n \quad (dx_i \text{ is missing}),$$

where $r = \sqrt{\sum_1^n x_j^2}$. For which values of the parameter k the form θ_k is closed? (Recall that a form θ is called *closed* if $d\theta = 0$.)

We have

$$d\left(\frac{x_i}{r^k}\right) = \frac{dx_i}{r^k} - kx_i \frac{\sum_1^n x_j dx_j}{r^{k+2}}.$$

Hence

$$d\theta_k = \sum_{i=1}^{n} (-1)^{i-1} d\left(\frac{x_i}{r^k}\right) dx_1 \wedge \overset{i}{\check{\dots}} \wedge dx_n =$$
$$\left(\frac{n}{r^k} - k\frac{\sum_1^n x_i^2}{r^{k+2}}\right) dx_1 \wedge \dots \wedge dx_n$$
$$= \frac{n-k}{r^k} dx_1 \wedge \dots \wedge dx_n.$$

Hence, θ_k is closed if (and only if) $k = n$.

2. In $\mathbb{R}^{2n}$ with coordinates $x_1, x_2, \ldots, x_{2n}$ consider an exterior 2-form

$$\eta = \sum_{k=1}^{n} x_{2k-1} \wedge x_{2k}.$$

Given a 1-form $\alpha = \sum\limits_{1}^{2n} a_i x_i$ find the 1-form

$$\beta = \star \left(\alpha \wedge \underbrace{\eta \wedge \cdots \wedge \eta}_{n-1} \right).$$

We have

$$\eta^{n-1} = (n-1)! \sum_{1}^{n} x_1 \wedge \overset{2j-1}{\check{\ldots}} \; \overset{2j}{\check{\ldots}} \wedge x_{2n} \quad (x_{2j-1} \wedge x_{2j} \ \text{ is missing}).$$

Then

$$x_{2j-1} \wedge \eta^{n-1} = (n-1)! \sum_{1}^{n} x_1 \wedge \overset{2j}{\check{\ldots}} \wedge x_{2n} \quad (x_{2j} \ \text{ is missing})$$

and

$$x_{2j} \wedge \eta^{n-1} = (n-1)! \sum_{1}^{n} x_1 \wedge \overset{2j-1}{\check{\ldots}} \wedge x_{2n} \quad (x_{2j-1} \ \text{ is missing}).$$

Hence, $\star\,(x_{2j-1} \wedge \eta^{n-1}) = (n-1)! x_{2j}$ and $\star\,(x_{2j} \wedge \eta^{n-1}) = -(n-1)! x_{2j-1}$.

Therefore

$$\beta = \star(\alpha \wedge \eta^{n-1}) = \sum_{1}^{n} \big(a_{2j-1} \star (x_{2j-1} \wedge \eta^{n-1}) + a_{2j} \star (x_{2j} \wedge \eta^{n-1}) \big)$$

$$= (n-1)! \sum_{1}^{n} (-a_{2j} x_{2j-1} + a_{2j-1} x_{2j}).$$

Note that these formulas also holds for $n = 1$. In this case, $\star(a_1 x_1 + a_2 x_2) = a_2 x_1 - a_1 x_2$.

3. Consider a differential 1-form β which in cylindrical coordinates (r, ϕ, z) has the form

$$\beta = f(r) dz + g(r) d\phi, \quad \text{where } \ g'(0) = 0.$$

Find a condition when $\beta \wedge d\beta$ is a volume form, i.e. it does not vanish anywhere. Interpret this condition *geometrically* in terms of the properties of the curve given in $\mathbb{R}^2$ with Cartesian coordinates (u, v) by parametric equations

$$u = f(r), v = g(r) \text{ for } r \in [0, \infty).$$

We have

$$d\beta = f'(r)dr \wedge dz + g'(r)dr \wedge d\theta,$$

and

$$\beta \wedge d\beta = (f'g - g'f)dr \wedge dz \wedge d\theta.$$

Hence the required condition reads

$$f'(r)g(r) - g'(r)f(r) \neq 0$$

for all $r \neq 0$.

Remark. *Note that if $r = 0$ we cannot make computations in cylindrical coordinates. The condition $g'(0) = g(0) = 0$ together with the condition $f'(0) = 0$ allows us to extend the form smoothly to $r = 0$. The condition that $\beta \wedge d\beta \neq 0$ along z-axis then reads: $f(0) \neq 0$, $g''(0) \neq 0$.*

The condition $f'g - gf' \neq 0$ means that the velocity vector (f', g') of the curve

$$u = f(r), v = g(r) \text{ for } r \in [0, \infty)$$

is never collinear with the radius-vector (f, g) of the curve. If we re-express this condition in polar coordinates (ρ, ϕ) in the (u, v)-plane, it then reads that $\phi' \neq 0$, i.e. when $r \to \infty$ the point $(f(r), g(r))$ keeps rotating around the origin in the same direction.

4. In $\mathbb{R}^{2n}$ with coordinates $x_1, y_1, \ldots, x_n, y_n$ consider a differential 2-form $\omega = \sum_1^n dx_i \wedge dy_i$.

a) Let $F : \mathbb{R}^{2n} \to \mathbb{R}$ be a C^1-function. Find the vector field X_F such that $X_F \lrcorner \omega = dF$.

b) Prove that for any two C^1-functions $F, G : \mathbb{R}^{2n} \to \mathbb{R}$ one has

$$D_{X_F} G = -D_{X_G} F.$$

c)* Denote $\{F, G\} = D_{X_G} F$. [1] Prove that for any two C^2-functions $F, G : \mathbb{R}^{2n} \to \mathbb{R}$ the following identity holds

$$[X_F, X_G] = -X_{\{F,G\}}.$$

Here the notation $[X, Y]$ stands for the Lie bracket of two vector fields.

a) Let us write $X_F = \sum\limits_1^n a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}$. Then the equation $X_F \lrcorner \omega = dF$ takes the form

$$\left(\sum_1^n a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i} \right) \lrcorner \omega = \sum_1^n \frac{\partial F}{\partial x_i} dx_i + \sum_1^n \frac{\partial F}{\partial y_i} dy_i.$$

We have

$$\frac{\partial}{\partial x_i} \lrcorner \omega = dy_i, \quad \frac{\partial}{\partial y_i} \lrcorner \omega = -dx_i.$$

Hence,

$$\left(\sum_1^n a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i} \right) \lrcorner \omega = \sum_1^n -b_i dx_i + a_i dy_i.$$

. Hence, we get

$$a_i = \frac{\partial F}{\partial y_i}, \quad b_i = -\frac{\partial F}{\partial x_i}, \quad i = 1, \dots, n.$$

Thus,

$$X_F = \sum_1^n \frac{\partial F}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i} \frac{\partial}{\partial y_i}.$$

b) We have

$$X_F = \sum_1^n \frac{\partial F}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i} \frac{\partial}{\partial y_i},$$

$$X_G = \sum_1^n \frac{\partial G}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial G}{\partial x_i} \frac{\partial}{\partial y_i}. \tag{1}$$

[1] $\{F, G\}$ is called the *Poisson bracket* of functions F and G.

$$D_{X_F}G = dG(X_F) = \sum_1^n \frac{\partial F}{\partial y_i}\frac{\partial G}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial y_i}$$

and

$$D_{X_G}F = dF(X_G) = \sum_1^n \frac{\partial G}{\partial y_i}\frac{\partial F}{\partial x_i} - \frac{\partial G}{\partial x_i}\frac{\partial F}{\partial y_i} = -D_{X_F}G. \tag{2}$$

c) As it was computed in one of the homeworks, the Lie bracket $[X, Y]$ of two vector fields $X = \sum_1^n a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}$ and $Y = \sum_1^n c_i \frac{\partial}{\partial x_i} + d_i \frac{\partial}{\partial y_i}$ is the vector field

$Z = \sum_1^n f_i \frac{\partial}{\partial x_i} + g_i \frac{\partial}{\partial y_i}$, where

$$\begin{aligned} f_i &= \sum_{j=1}^n a_j \frac{\partial c_i}{\partial x_j} - c_j \frac{\partial a_i}{\partial x_j} + b_j \frac{\partial c_i}{\partial y_j} - d_j \frac{\partial a_i}{\partial y_j}; \\ g_i &= \sum_{j=1}^n a_j \frac{\partial d_i}{\partial x_j} - c_j \frac{\partial b_i}{\partial x_j} + b_j \frac{\partial d_i}{\partial y_j} - d_j \frac{\partial b_i}{\partial y_j}. \end{aligned} \tag{3}$$

Let us compute $[X_F, X_G]$. Plugging in (3)

$$a_i = \frac{\partial F}{\partial y_i},\ b_i = -\frac{\partial F}{\partial x_i},\ c_i = \frac{\partial G}{\partial y_i},\ d_i = -\frac{\partial G}{\partial x_i}$$

we compute

$$\begin{aligned} f_i &= \sum_{j=1}^n a_j \frac{\partial c_i}{\partial x_j} - c_j \frac{\partial a_i}{\partial x_j} + b_j \frac{\partial c_i}{\partial y_j} - d_j \frac{\partial a_i}{\partial y_j} \\ &= \sum_{j=1}^n \frac{\partial F}{\partial y_j}\frac{\partial^2 G}{\partial x_j y_i} - \frac{\partial G}{\partial y_j}\frac{\partial^2 F}{\partial x_j \partial y_i} - \frac{\partial F}{\partial x_j}\frac{\partial^2 G}{\partial y_i \partial y_j} + \frac{\partial G}{\partial x_j}\frac{\partial^2 F}{\partial y_i \partial y_j}; \\ g_i &= \sum_{j=1}^n a_j \frac{\partial d_i}{\partial x_j} - c_j \frac{\partial b_i}{\partial x_j} + b_j \frac{\partial d_i}{\partial y_j} - d_j \frac{\partial b_i}{\partial y_j} \\ &= \sum_{j=1}^n -\frac{\partial F}{\partial y_j}\frac{\partial^2 G}{\partial x_i \partial x_j} + \frac{\partial G}{\partial y_j}\frac{\partial^2 F}{\partial x_i \partial x_j} + \frac{\partial F}{\partial x_j}\frac{\partial^2 G}{\partial x_i \partial y_j} - \frac{\partial G}{\partial x_j}\frac{\partial^2 F}{\partial x_i \partial y_j}; \end{aligned} \tag{4}$$

Let us now compute $X_{\{F,G\}}$. From formula (2) in b) we have

$$\{F, G\} = D_{X_G}F = \sum_1^n \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial y_i} - \frac{\partial F}{\partial y_i}\frac{\partial G}{\partial x_i}.$$

Using formula (2) we compute

$$\begin{aligned}
X_{\{F,G\}} &= \sum_{i=1}^{n} \frac{\partial\{F,G\}}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial\{F,G\}}{\partial x_i}\frac{\partial}{\partial y_i} \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial y_i}\left(\sum_{j=1}^{n} \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial y_j} - \frac{\partial F}{\partial y_j}\frac{\partial G}{\partial x_j}\right)\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial y_j} - \frac{\partial F}{\partial y_j}\frac{\partial G}{\partial x_j}\right)\frac{\partial}{\partial y_i} \\
&= \sum_{i=1}^{n} \left(\frac{\partial^2 F}{\partial x_j \partial y_i}\frac{\partial G}{\partial y_j} + \frac{\partial F}{\partial x_j}\frac{\partial^2 G}{\partial y_j \partial y_i} - \frac{\partial^2 F}{\partial y_j \partial y_i}\frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial y_j}\frac{\partial^2 G}{\partial x_j \partial y_i}\right)\frac{\partial}{\partial x_i} \\
&- \left(\frac{\partial^2 F}{\partial x_j \partial x_i}\frac{\partial G}{\partial y_j} + \frac{\partial F}{\partial x_j}\frac{\partial^2 G}{\partial y_j \partial x_i} - \frac{\partial^2 F}{\partial y_j \partial x_i}\frac{\partial G}{\partial x_j} - \frac{\partial F}{\partial y_j}\frac{\partial^2 G}{\partial x_j \partial x_i}\right)\frac{\partial}{\partial y_i}. \qquad (5)
\end{aligned}$$

Comparing with expressions for f_i and g_i from (4) we get that

$$X_{\{F,G\}} = -\sum_{1}^{n} f_i \frac{\partial}{\partial x_i} - g_i \frac{\partial}{\partial y_i} = -[X_F, X_G].$$

Remark. There is another, more conceptual and less computational proof of the formula $[X_F, X_G] = -X_{\{F,G\}}$. If you are interested, please come and I'll explain it.