Math 52H: Solutions to practice problems for the midterm

1 Consider the Euclidean space $V = \mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \dots, x_n, y_n)$ and the standard dot-product. The space $\Lambda(V^*)$ of all exterior k-forms for all $k = 0, \dots, 2n$ is also an Euclidean space with the scalar product of a k-form α and an l-form β defined by the formula

$$\langle\langle \alpha, \beta \rangle\rangle = \begin{cases} \star^{-1}(\alpha \wedge \star\beta), & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases}$$

Consider a linear operator $\Omega : \Lambda(V^*) \to \Lambda(V^*)$ defined by the formula $\Omega(\alpha) = \alpha \wedge \omega$, where $\omega = \sum\limits_1^n x_i \wedge y_1$. Find the adjoint linear operator Ω^*, i.e. the operator $\Omega^* : \Lambda(V^*) \to \Lambda(V^*)$ such that

$$\langle\langle \Omega(\alpha), \beta \rangle\rangle = \langle\langle \alpha, \Omega^*(\beta) \rangle\rangle$$

for any forms $\alpha, \beta \in \Lambda(V^*)$.

We have

$$\langle\langle \Omega(\alpha), \beta \rangle\rangle = \star^{-1}(\alpha \wedge \omega \wedge \star\beta) = \star^{-1}(\alpha \wedge \star(\star^{-1}(\omega \wedge \star\beta)) = \langle\langle \alpha, \star^{-1}(\omega \wedge \star\beta) \rangle\rangle.$$

Therefore,

$$\langle\langle \alpha, \star^{-1}(\omega \wedge \star\beta) - \Omega^*(\beta) \rangle\rangle = 0$$

for any k=forms α, β, which means that

$$\Omega^*(\beta) = \star^{-1}(\omega \wedge \star\beta)$$

2. The cylindrical coordinates

$$r \in [0, \infty),\ \varphi \in [0, 2\pi),\ z \in \mathbb{R}\,,$$

are introduced in $\mathbb{R}^3$ by the formulas

$$x = r\cos\varphi, y = r\sin\varphi, z\,,$$

where (x, y, z) are Cartesian coordinates. Consider a differential 1-form

$$\alpha = \cos r dz + \frac{r\sin r}{\pi} d\varphi\,.$$

a) Describe the plane field ξ defined by the Pfaffian equation $\alpha = 0$.

The plane field ξ is orthogonal to the z-axis at the points of the axis, because $\alpha = dz$ when $r = 0$. Elsewhere ξ is tangent to rays $z = \text{const}, \varphi = \text{const}$ because α does not have a term with dr. One also notice that ξ is symmetric with respect to rotations around z-axis and with respect to translations along this axis. When $r = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ the plane is vertical, i.e. tangent to z-axis, and when $r = k\pi$ it is horizontal. It keeps rotating as r increases.

b) Suppose that a curve $\Gamma \subset \mathbb{R}^3$ be given by the parametric equations

$$r = \frac{\pi}{4},\ z = h(t),\ \varphi = 2t,\ t \in [0, \pi]\,.$$

Find the function h such that $\alpha|_\Gamma = 0$ and $h(0) = 1$.

The restriction of α to Γ is equal $\left(\frac{\sqrt{2}}{2}h' + \frac{\sqrt{2}}{4}\right)dt$. Hence we get $h' = -\frac{1}{2}$, and thus $h(t) = -\frac{t}{2} + C$. The constant C is equal to $h(0) = 1$. Hence, $h(t) = -\frac{t}{2} + 1$.

c) Find all values of R for which the horizontal circles $\{z = \text{const}, r = R\}$ are tangent to the plane field ξ.

This was already answered in a).

3. Consider a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$. Let S_f be a surface in $\mathbb{R}^4$ given by equations

$$x_3 = \frac{\partial f}{\partial x_1}(x_1, x_2), \quad x_4 = \frac{\partial f}{\partial x_2}(x_1, x_2) \tag{1}$$

Suppose that this system of equations can is solved with respect to the coordinates x_2 and x_4, i.e. there exist smooth functions $x_2 = g(x_1, x_3)$ and $x_4 = h(x_1, x_3)$ such that

$$\begin{aligned} x_3 &\equiv \frac{\partial f}{\partial x_1}(x_1, g(x_1, x_3)), \\ h(x_1, x_3) &\equiv \frac{\partial f}{\partial x_2}(x_1, g(x_1, x_3)). \end{aligned} \tag{2}$$

Prove that the Jacobian of the map $(h, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is equal to -1, i.e. that

$$\begin{vmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial g}{\partial x_3} \end{vmatrix} = -1.$$

Hint: Examine the restriction of the form $\omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$ to the surface S_f, and then consider the pull-back of the form ω by a map $\mathbb{R}^2 \to S_f \subset \mathbb{R}^4$ given by the formulas

$$(x_1, x_3) \mapsto (x_1, g(x_1, x_3), x_3, h(x_1, x_3)).$$

One can check that $\omega|_{S_f} = 0$. Let F be the map described in the Hint. Then we have

$$0 = F^*\omega = dx_1 \wedge dx_3 + dg \wedge dh = (1 + J)dx_1 \wedge dx_3),$$

where $J = \begin{vmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial g}{\partial x_3} \end{vmatrix}$. Hence, $J = -1$.

4. Consider a differential 1-form $\alpha = dx_3 + x_2 dx_1$ on $\mathbb{R}^3$. Let $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a map such that $f^*\alpha = h\alpha$ for some positive function $h : \mathbb{R}^3 \to \mathbb{R}$. Find a function $g : \mathbb{R}^3 \to \mathbb{R}$ such that the map $F : \mathbb{R}^4 \to \mathbb{R}^4$ given by the formula

$$F(x_1, x_2, x_3, x_4) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3), x_4 g(x_1, x_2, x_3))$$

satisfies the equation $F^*(x_4\alpha) = x_4\alpha$.

$F^*(x_4\alpha) = x_4 g f^*\alpha = x_4 g h\alpha$. Hence, $g = \frac{1}{h}$.

5. Consider a smooth differential k-form

$$\alpha = \sum_{1\le i_1<\dots<i_k\le n} f_{i_1\dots i_k}(x)dx_{i_1}\wedge\dots dx_{i_k}$$

in $\mathbb{R}^n$ such that $f_{i_1\dots i_k}(0)=0$ (i.e. all coefficients of the form α are equal to 0 at the origin). Let $F:\mathbb{R}^n\to\mathbb{R}^n$ denote the dilatation $x\mapsto 2x$. Suppose that $F^*\alpha=\alpha$. Prove that $\alpha\equiv 0$.

We have for any point $a\in\mathbb{R}^n$ and vectors $X_1,\dots,X_n\in\mathbb{R}^n_a$

$$\alpha_a(X_1,\dots,X_n) = \left(F^{-1}\right)^*\alpha_a(X_1,\dots,X_k) = \alpha_{\frac{a}{2}}\left(\frac{1}{2}X_1,\dots,\frac{1}{2}X_n\right).$$

Iterating this formula, we get

$$\alpha_a(X_1,\dots,X_n) = \left(f^{-n}\right)^*\alpha_a(X_1,\dots,X_k) = \alpha_{\frac{a}{2^n}}\left(\frac{1}{2^n}X_1,\dots,\frac{1}{2^n}X_n\right)\underset{n\to\infty}{\to}0.$$

6. Given a function $f:\mathbb{R}^n\to\mathbb{R}$, consider a map $F:\mathbb{R}^n\to\mathbb{R}^{2n+1}$ defined by the formula

$$F(x_1,\dots,x_n) = \left(x_1,\dots,x_n,\frac{\partial f}{\partial x_1}(x_1,\dots,x_n),\dots,\frac{\partial f}{\partial x_n}(x_1,\dots,x_n),f(x_1,\dots,x_n)\right).$$

Compute $F^*(\alpha)$, where

$$\alpha = dx_{2n+1} - \sum_{i=1}^n x_{i+n}dx_i\,.$$

We have

$$F^*\alpha = df - \sum_1^n \frac{\partial f}{\partial x_i}dx_i = 0.$$