Math 52H: Homework N9

Due to **Thursday**, March 10

1. Solve Exercise 5.3.6 from the Online Text.

2. Solve Exercise 6.2.7 in the Online Text.

3. Prove that the 2-sphere S^2 and the 2-torus $T^2 = S^1 \times S^1$ are not diffeomorphic. (Here one can view S^2 as the unit sphere $\{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3$ and T^2 as the "flat torus" $\{x_1^2 + x_2^2 = 1, x_3^2 + x_4^2 = 1\} \subset \mathbb{R}^4$.)

4. Let C be the intersection of the sphere $S = \{x^2 + y^2 + z^2 = 1\}$ and the plane $P = \{x + y + z = 0\}$. We orient C counter-clockwise when looking from the point $(0, 0, 100)$.

   Compute $\int\limits_C z^3 dx$.

5. Let M be an oriented closed n-dimensional manifold, and ω be a differential $(n-1)$-form on M. Prove that there exists a point $a \in M$ such that $(d\omega)_a = 0$.

6. Let $f : S^{2k} \to S^{2k}$ be a smooth map. Prove that there exists a point $x \in S^{2k}$ such that $f(x) = -x$ or $f(x) = x$.

7*. Let us consider the space $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$. Consider the matrix

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & & \dots & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & & & \dots & & 0 & -1 \\ 0 & 0 & & & \dots & & 1 & 0 \end{pmatrix}$$

and denote by $\mathcal{J}$ the linear operator $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with this matrix. A subspace $C \subset \mathbb{R}^{2n}$ is called *complex* if it is invariant with respect to $\mathcal{J}$, i.e. $\mathcal{J}(X) \in L$ for any vector $X \in L$. Given a point $a \in \mathbb{R}^{2n}$ a subspace $L_a \subset \mathbb{R}^{2n}_a$ is called complex if it is a parallel transport of a complex subspace $L \subset \mathbb{R}^{2n}$. Any complex line L has a canonical orientation given by a basis Z, iZ for any non-zero vector $Z \in L$. This orientation is called the *complex orientation* of L.

Consider a differential 2-form $\omega = dx_1 \wedge dy_1 + \dots dx_n \wedge dy_n$ on $\mathbb{R}^{2n}$.

a) Prove the *Wirtinger inequality*: For any 2 vectors $X, Y \in \mathbb{R}^{2n}_a$ we have

$$|\omega(X,Y)| \leq \mathrm{A}rea P(X,Y), \tag{1}$$

where $\mathrm{A}rea P(X,Y)$ is the (un-signed) area of the parallelogram spanned by X, Y. Show that the equality in (1) is achieved if and only if $\mathrm{S}pan(X,Y)$ is a complex subspace. Moreover, if X, Y are linearly independent, then the equality $\omega(X,Y) = \mathrm{A}rea(X,Y)$ is achieved iff in addition the orientation of the plane $\mathrm{S}pan(X,Y)$ defined by the basis X, Y coincides with its orientation by the basis $X, \mathcal{J}(X)$.

b) Let $A \subset \mathbb{R}^{2n}$ be a 2-dimensional compact submanifold with boundary. Suppose that each tangent plane T_aA is a complex subspace in $\mathbb{R}^{2n}_a$. Prove that for any other 2-dimensional orientable[1] submanifold $B \subset \mathbb{R}^{2n}$ with $\partial B = \partial A$ we have

$$\mathrm{A}rea(B) \geq \mathrm{A}rea(A).$$

[1]The statement holds even without the assumption of orientability.

Hint: Compare $\mathrm{Area}(A), \mathrm{Area}(B)$ *with integrals* $\int\limits_A \omega, \int\limits_B \omega$ *and use the Wirtinger inequality.*

Each problem and subproblem is 10 points. Problem 7 is an extra-credit.