

# Math 52H: Homework N8

Due to Friday, March 4

1. a) Give an example of two diffeomorphic submanifolds, such that one is bounded and the other is not.

b)\* Prove that any closed connected 1-dimensional submanifold is diffeomorphic to the unit circle  $S^1 = \{x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$ .

c) Prove that the submanifold  $T = \{x_1^2 + x_2^2 = 1, x_3^2 + x_4^2 = 1\} \subset \mathbb{R}^4$  and the submanifold  $U \subset \mathbb{R}^3$  given in cylindrical coordinates  $(r, \phi, z)$  by the equation  $(r - 2)^2 + z^2 = 1$ , are diffeomorphic.

2. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth function. Consider a graphical  $n$ -dimensional submanifold with boundary

$$L_f = \left\{ (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n}; \sum_1^n x_j^2 \leq 1; y_j = \frac{\partial f}{\partial x_j}(x_1, \dots, x_n), j = 1, \dots, n \right\} \subset \mathbb{R}^{2n}.$$

Suppose that  $L_f$  is oriented by the parameterization

$$(x_1, \dots, x_n) \mapsto \left( x_1, \dots, x_n, \frac{\partial f}{\partial x_1}(x_1, \dots, x_n), \frac{\partial f}{\partial x_n}(x_1, \dots, x_n) \right).$$

Suppose that  $n = 2k$ . Compute

$$\int_{L_f} (dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n)^{\wedge k}.$$

3. Let  $A \subset V$  be a closed  $n$ -dimensional,  $n = k + l + 1$ , submanifold,  $\omega$  a differential  $k$ -form and  $\eta$  a differential  $l$ -form defined on a neighborhood of  $A$  in  $V$ . Prove that

$$\int_A \omega \wedge d\eta = C \int_A \eta \wedge d\omega$$

for some constant  $C$ , and find  $C$ .

4. Consider the torus

$$T = \{x_1^2 + x_2^2 = 1, x_3^2 + x_4^2 = 1\} \subset \mathbb{R}^4.$$

This is a 2-dimensional submanifold of  $\mathbb{R}^4$ . Orient  $T$  in such a way that the tangent plane at the point  $a = (1, 0, 1, 0)$  is oriented by the vectors  $e = (0, 1, 0, 0)$ ,  $f = (0, 0, 0, 1) \in T_a T \subset \mathbb{R}_a^4$ .

Compute  $\int_T \omega$  for the following differential 2-forms:

(i)  $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ ;

(ii)  $\omega = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$ ;

(iii)  $\omega = x_2 x_4 dx_1 \wedge dx_3$ .

5. Let  $S = \{x^2 + y^2 + z^2 = 1; z \geq 0\} \subset \mathbb{R}^3$ . We co-orient the sphere by the upward pointing vector at the point  $(0, 0, 1)$ . Compute  $\int_S dx \wedge dy + 2z dz \wedge dx$  directly and using Stokes' theorem.

6\*. Let  $\alpha$  be a differential 1-form and  $\omega$  a differential 2-form on  $\mathbb{R}^5$ . Suppose that

- $d\omega = 0$ ;
- $\alpha \wedge \omega \wedge \omega \neq 0$ ;
- for any point  $a \in \mathbb{R}^5$  and any two vectors  $X, Y \in \mathbb{R}_a^5$  such that  $\alpha_a(X) = \alpha_a(Y) = 0$  we have  $\omega_a(X, Y) = d\alpha_a(X, Y)$ .

Prove that  $\omega = d\alpha$ .

Problems 1b) and 6 are extra-credits. Problems 1a), 1b), 1c) 2, 3, 5, 6 are 10 points each. Each subproblem of 4 is 5 points.