Math 52H: Homework N5

Due to Friday, February 11

1. Consider an operator $\partial = (-1)^k \star^{-1} d\star : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$. Note that we equivalently can write $\partial = (-1)^{nk+n+1} \star d\star$. Denote $\Delta := (\partial + d)^2$ (this is *Laplace-de Rham operator*).

a) Verify that Δ is an operator $\Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$.

b) Compute $\Delta(f\alpha)$ for a function $f : \mathbb{R}^n \to \mathbb{R}$ and a differential 1-form α.

c) Find explicit formulas for $\Delta\alpha$ for the cases when $n = 3$ and $k = 0, 1, 2, 3$.

d) For $n = 2$ find explicit formulas for $\Delta\alpha$ in polar coordinates for $k = 0, 1, 2$.

2. Given a function f and a vector field v on $\mathbb{R}^3$ compute $\operatorname{div}(\nabla f)$, $\operatorname{div}(\operatorname{curl} v)$, $\operatorname{curl}(\nabla f)$, $\operatorname{curl}(fv)$ and $\operatorname{div}(fv)$.

3. Let V be a Euclidean 3-dimensional space. Give a coordinate-free proof that for any two C^1-functions $f, g : \mathbb{R}^3 \to \mathbb{R}$ we have

$$\operatorname{div}(\nabla f \times \nabla g) = 0.$$

Use the fact that the cross-product $v \times w$ of two vectors can be defined by the formula $v \times w = \mathcal{D}^{-1} \star (\mathcal{D}(v) \wedge \mathcal{D}(w))$.

Subproblem 1a) is 5 points. All other subproblems and problems are 10 point each.