Math 52H: Homework N4

Due to Friday, February 4

1. Spherical coordinates $\rho \in [0, \infty)$, $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi)$, are introduced in $\mathbb{R}^3$ by the formulas:

$$x = \rho \sin\varphi \cos\theta,\ y = \rho \sin\varphi \sin\theta, z = \rho \cos\varphi.$$

Express the 1-form $dz + \frac{1}{2}(xdy - ydx)$ in spherical coordinates.

2. A map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given in polar coordinates by the formula

$$f(r, \varphi) = (r, \varphi + r^2 \sin\varphi).$$

Find $f^*(dx \wedge dy)$, where $dx \wedge dy$ is the area form.

3. Consider two differential 1-forms in $\mathbb{R}^3$:

$$\alpha = dx + ydz \quad \text{and} \quad \beta = xdy\,.$$

Prove that there is no map $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f^*(\beta) = \alpha$.

4. Consider a closed differential 1-form $\omega = F_1 dx + F_2 dy + F_3 dz$ in $\mathbb{R}^3$. Suppose that each function F_k, $k = 1, 2, 3$, satisfies the homogeneity equation

$$F_k(tx, ty, tz) = tF_k(x, y, z),$$

for any $t \in \mathbb{R}$. Prove that $\omega = df$, where

$$f(x,y,z) = \frac{1}{2}\left(xF_1(x,y,z) + yF_2(x,y,z) + zF_3(x,y,z)\right) .$$

5. Consider in $\mathbb{R}^{2n}$ differential forms

$$\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \text{ and } \theta = \sum_{n+1}^{2n}(-1)^{j-1}x_j dx_{n+1} \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{2n}.$$

Prove that the $(2n-1)$-form

$$\Omega = \frac{\omega \wedge \theta}{(\sum_1^n x_i x_{i+n})^n}$$

is closed.

Each problem is 10 points.