

# Math 52H: Homework N3

Due to Friday, January 28

1. Consider the space  $\mathbb{R}^{2n}$  with coordinates  $(x_1, y_1, \dots, x_n, y_n)$ . Denote  $\omega = \sum_1^n x_i \wedge y_i$ .  
A linear operator  $\mathcal{A} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  is called *symplectic* if  $\mathcal{A}^*\omega = \omega$ .

Denote

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ & & & & \dots & & \\ 0 & 0 & & & \dots & 0 & -1 \\ 0 & 0 & & & \dots & 1 & 0 \end{pmatrix}.$$

Note that  $J^2 = -E$ . Let  $\mathcal{J} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  be the operator with the matrix  $J$ .

An operator  $\mathcal{U} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  is called *unitary* if it is orthogonal and commutes with  $\mathcal{J}$ , i.e.  $\mathcal{U} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{U}$ . Respectively, a matrix  $U$  is called unitary, if it is orthogonal and  $UJ = JU$ .

- (a) Let  $\mathcal{A}$  be symplectic. Prove that  $\det \mathcal{A} = 1$ .
- (b) Prove that an operator  $\mathcal{A}$  is symplectic if and only if its matrix  $A$  (in the standard basis of  $\mathbb{R}^{2n}$ ) satisfies the equation  $A^T J A = J$ .
- (c) Prove that an orthogonal operator  $\mathcal{U} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  is unitary if and only if it is symplectic.

(d) An  $n$ -dimensional vector subspace  $L \subset \mathbb{R}^{2n}$  is called *Lagrangian* if  $\omega|_L = 0$ . Prove that  $L$  is Lagrangian if and only if  $\mathcal{J}(L) = L^\perp$ .

(e)\* Denote  $L_0 = \{y_1 = y_2 = \cdots = y_n = 0\}$ . Prove that a subspace  $L \subset \mathbb{R}^{2n}$  is Lagrangian if and only if there exists a unitary operator  $\mathcal{U}$  such that  $L = \mathcal{U}(L_0)$ .

2. In  $\mathbb{R}^2$  with the standard dot-product consider a basis  $v_1 = (1, 0), v_2 = (1, 1)$ . Let  $(y_1, y_2)$  be coordinates dual to this basis. Given a function  $f(y_1, y_2)$  compute its gradient in these coordinates, i.e. find functions  $g_1(y_1, y_2), g_2(y_1, y_2)$  such that  $\nabla f = g_1 \frac{\partial}{\partial y_1} + g_2 \frac{\partial}{\partial y_2}$ .

3. Let  $X = \sum_1^n f_i \frac{\partial}{\partial x_i}, Y = \sum_1^n g_i \frac{\partial}{\partial x_i}$  be two vector fields on  $\mathbb{R}^n$ . Prove that there exists a vector field  $Z = \sum_1^n h_i \frac{\partial}{\partial x_i}$  such that  $D_X D_Y - D_Y D_X = D_Z$ , and find an explicit formula for the coordinate functions  $h_i$  of  $Z$  in terms of coordinate functions of the vector fields  $X, Y$ .<sup>1</sup>

All problems and subproblems are 10 points. Problem 1(e) is not mandatory.

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<sup>1</sup>The vector field  $Z$  is called the *commutator*, or *Lie bracket* of vector fields  $X$  and  $Y$ , and usually denoted by  $[X, Y]$ .