Math 52H: Homework N2

Due to Friday, January 21

1. In the space $V = \mathbb{R}^{2n}$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ consider the 2-form $\omega = \sum\limits_{i=1}^{n} x_i \wedge y_i$.

a) Let A be a $n \times n$ matrix. Consider a map $L_A : \mathbb{R}^n \to \mathbb{R}^{2n}$, given by the formula $L_A(x) = (x, Ax)$, where we view the vector $x \in \mathbb{R}^n$ as a column-matrix. Compute $(L_A)^*\omega$ and show that $(L_A)^*\omega = 0$ if and only if the matrix A is symmetric.

b) Consider the map $\mathcal{C}_\omega$ of the space $V = \mathbb{R}^{2n}$ to the dual space V^* given by the formula

$$\mathcal{C}_\omega(v)(Z) = \omega(v, Z)\,, \quad v, Z \in V\,.$$

Find the matrix C of the map $\mathcal{C}_\omega$ with respect to the standard basis in $V = \mathbb{R}^{2n}$ an its dual basis in V^*.

c) Given an integer k, such that $0 \le k \le n$, consider the map $\Omega_k : \Lambda^{n-k}(V^*) \to \Lambda^{n+k}(V^*)$ given by the formula

$$\Omega_k(\alpha) = \alpha \wedge \omega^k, \quad \alpha \in \Lambda^k(V^*).$$

Prove that Ω_k is an isomorphism.

2. A 2-form β on $\mathbb{R}^4$ is called self-dual if $*\beta = \beta$. What is the dimension of the space of self-dual 2-forms on $\mathbb{R}^4$. Find a basis of this space.

3. Apply the equality

$$\mathcal{A}^*(x_1 \wedge \dots \wedge x_k \wedge x_{k+1} \wedge \dots \wedge x_n) = \mathcal{A}^*(x_1 \wedge \dots \wedge x_k) \wedge \mathcal{A}^*(x_{k+1} \wedge \dots \wedge x_n)$$

for a map $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ to deduce the formula for expansion of a determinant according to its first k rows:

$$\det A = \sum_{i_1<\cdots<i_k,\ j_1<\cdots<j_{n-k};\ i_m\neq j_l \text{ for } k\neq l} (-1)^{\mathrm{inv}(i_1,\ldots,j_{n-k})} \begin{vmatrix} a_{1,i_1} & \ldots & a_{1,i_k} \\ \vdots & \vdots & \vdots \\ a_{k,i_1} & \ldots & a_{k,i_k} \end{vmatrix} \begin{vmatrix} a_{k+1,j_1} & \ldots & a_{k+1,j_{n-k}} \\ \vdots & \vdots & \vdots \\ a_{n,j_1} & \ldots & a_{n,j_{n-k}} \end{vmatrix}.$$

4. Let $V = \mathbb{R}^3$ with the dot-product. Let $\mathcal{D}$ denote the isomorphism $V \to V^*$ defined by the formula $\mathcal{D}(v)(X) = \langle v, X\rangle,\ \ v, X \in V$. Show that the formula

$$X \times Y = \mathcal{D}^{-1}\left(\star(\mathcal{D}(X) \wedge \mathcal{D}(Y))\right)$$

defines the cross-product on $\mathbb{R}^3$.

Recall that the cross-product $X \times Y$ is defined as the vector Z orthogonal to $\mathrm{Span}(X, Y)$ which has length equal to the area of the parallelogram $P(X, Y)$ and (assuming that X, Y are linearly independent) directed in such a way that the basis (X, Y, Z) defines the standard orientation of $\mathbb{R}^3$.

5. Define

$$\exp(A) = E + A + \frac{1}{2}t^2A^2 + \frac{1}{3!}A^3 + \ldots; \quad \text{here } E \text{ denotes the unit matrix.}$$

Let A be an skew-symmetric $n \times n$ matrix, i.e. $A^T = -A$ Prove that the matrix e^A is orthogonal. Conversely, if $\exp(tA)$ is orthogonal for all t then A is skew-symmetric.

All problems and their subproblems are 10 points each. Problems 1c) and 5 are not mandatory.