Math 52H: Homework N1

Due on Friday, January 14

1. As it was shown in class, any bilinear function Φ can be presented in a unique way as a sum

$$\Phi = S + A,$$

where S is a symmetric and A is skew-symmetric bilinear functions. Show that a similar statement is not true for 3-linear functions on $\mathbb{R}^3$, and give an example of a 3-linear function which cannot be written as a sum of a symmetric an anti-symmetric functions.

2. Recall that the *trace* $\mathrm{Tr}A$ of a square matrix A is the sum of its diagonal elements. Let M_2 denote the vector space of symmetric 2×2 matrices with real entries. Show that the formula

$$\langle X, Y \rangle = \mathrm{Tr}(XY)$$

defines an inner product on M_2 and find the matrix of this bilinear form in the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

3. Let f and g be bilinear functions on $\mathbb{R}^n$ with matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$, respectively. Find the expression of the 4-tensor $f \otimes g$ in the basis

$$x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes x_{i_4}, \quad 1 \le i_1, i_2, i_3, i_4 \le n\,.$$

4. Let α be an exterior 2-form, and β is a 1-form on a 3-dimensional space. Suppose that $\alpha \wedge \beta = 0$. Prove that there exists a 1-form γ such that $\alpha = \beta \wedge \gamma$.

5. Let

$$\theta = \sum_{i=1}^{n-1} x_i \wedge x_{i+1}$$

be an exterior 2-form on $\mathbb{R}^n$, and $A, B \in \mathbb{R}^n$ are vectors

$$A = (1, 1, 1, \ldots, 1),\ B = (-1, 1, -1, \ldots, (-1)^n)\,.$$

Compute $\theta(A, B)$.

6. Denote coordinates in the space $\mathbb{R}^{2n}$ by $(x_1, y_1, \ldots, x_n, y_n)$. Consider a 2-form

$$\omega = \sum_{i=1}^{n} x_i \wedge y_i.$$

a) Compute

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{n}.$$

b) Let $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear map, such that $F^*\omega = \omega$. Prove that $\det F = 1$.

Each problem (including subproblems 6a) and 6b)) is 10 points.