Math 52H: Final Exam

March 14, 2011

1. Let $T \subset \mathbb{R}^3$ be the torus defined by the parametric equations

$$x = (a + R\cos\theta)\cos\phi$$
$$y = (a + R\cos\theta)\sin\phi$$
$$z = R\sin\theta,$$

where $0 < R < a$ are constants and $0 \leq \phi, \theta \leq 2\pi$.

a) Compute the area of T;

b) Compute the volume of the domain bounded by the torus T.

a) Let us compute the pull-back of the area form σ of the torus T by the parametrization map

$$\Phi(\theta, \phi) = (a + R\cos\theta)\cos\phi, (a + R\cos\theta)\sin\phi, R\sin\theta).$$

We have

$$\frac{\partial\Phi}{\partial\theta} = (-R\sin\theta\cos\phi, -R\sin\theta\sin\phi, R\cos\theta),$$
$$\frac{\partial\Phi}{\partial\phi} = (-(a + R\cos\theta)\sin\phi, (a + R\cos\theta)\cos\phi, 0).$$

Then

$$E = \left\| \frac{\partial \Phi}{\partial \theta} \right\|^2 = R^2,$$
$$E = \left\langle \frac{\partial \Phi}{\partial \theta}, \frac{\partial \Phi}{\partial \phi} \right\rangle = 0,$$
$$G = \left\| \frac{\partial \Phi}{\partial \phi} \right\|^2 = (a + R\cos\theta)^2.$$

Hence,

$$\sqrt{EG - F^2} = R(a + R\cos\theta),$$

and therefore,

$$\Phi^* \sigma = R(a + R\cos\theta) d\theta \wedge d\phi$$

and

$$\text{Area}(T) = \int_T \sigma = \int_{0 \le \phi, \theta \le 2\pi} \Phi^* \sigma = \int_0^{2\pi} \int_0^{2\pi} R(a + R\cos\theta) d\theta d\phi = 4\pi^2 aR.$$

b) Denote by U the domain bounded by T. By Stokes' theorem we have

$$\text{Vol}(U) = \int_U dx \wedge dy \wedge dz = \int_T z dx \wedge dy.$$

Using the parameterization Φ we get

$$\int_T z dx \wedge dy = \int_{0 \le \phi, \theta \le 2\pi} \Phi^*(z dx \wedge dy)$$
$$= R\sin\theta \left(-R\sin\theta\cos\phi d\theta - (a + R\cos\theta)\sin\phi d\phi\right) \wedge \left(-R\sin\theta\sin\phi d\theta + (a + r\cos\theta)\cos\phi) d\phi\right)$$
$$= R^2(a + R\cos\theta)\sin^2\theta d\phi \wedge d\theta.$$

Thus,

$$\text{Vol}(U) = \int_{0 \le \phi, \theta \le 2\pi} R^2(a + R\cos\theta)\sin^2\theta d\phi \wedge d\theta = 2\pi^2 aR^2.$$

2. In $\mathbb{R}^3$ consider the vector field

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

Compute $\text{Flux}_E X$, where E is the ellipsoid

$$E = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

co-oriented by the outward normal vector field. Let W be the solid body bounded by E. Then

$\text{Flux}_E X = \int\limits_W \text{div} X dV = 3\text{Vol}(W)$. But $\text{Vol}(W) = abc\text{Vol}(B_1)$, where B_1 is the unit ball. Indeed, W can be obtained from the ball B_1 by a linear transformation $(x, y, z) \mapsto (ax, by, cz)$ whic scaled the volume by its determinant abc. Thus, $\text{Flux}_E X = 4\pi abc$.

3.Let u, v be two smooth functions on the unit disc $D = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$. Suppose that

- everywhere in D one has

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \quad (2)$$

- $u = x, v = y$ when $x^2 + y^2 \leq \frac{1}{2}$;
- $u^2 + v^2 \neq 0$ in $D \setminus 0$.

Prove that the differential 1-form

$$\alpha = \frac{udy - vdx}{u^2 + v^2}$$

is closed in D and compute $\int\limits_{\partial D} \alpha$. Here ∂D is oriented counter-clockwise.

We will denote partial derivatives by u_x, u_y, v_x, v_y. Thus we have $u_x = v_y, u_y = -v_x$. We compute

$$
\begin{aligned}
d\alpha &= \frac{(du \wedge dy - dv \wedge dx)}{u^2+v^2} - 2\frac{(udu + vdv) \wedge (udy - vdx)}{(u^2+v^2)^2} \\
&= \frac{1}{(u^2+v^2)^2}\big((u^2+v^2)(u_x+v_y)dx \wedge dy - 2(uu_xdx + uu_ydy + vv_xdx + vv_ydy) \wedge (udy - vdx)\big) \\
&= \frac{1}{(u^2+v^2)^2}\big(u^2u_x + v^2u_x + u^2v_y + v^2v_y - 2(u^2u_x + uvv_x + uvu_y + uvu_y + v^2v_y)\big)\, dx \wedge dy \\
&== \frac{1}{(u^2+v^2)^2}\big(u^2(-u_x+v_y) + v^2(u_x - v_y) - 2uv(v_x+u_y)\big)\, dx \wedge dy = 0
\end{aligned}
$$

in view of the equations for the partial derivatives.

Thus the form α is closed in $\mathbb{R}^2 \setminus 0$. Denote $D' = \{x^2+y^2 \leq \frac{1}{2}\}$. Then

$$\int_{\partial D} \alpha = \int_{\partial D'} \alpha = \int_{\partial D'} \frac{xdy - ydx}{x^2+y^2} = 2\pi.$$

4. Prove that for any smooth map $f : S^k \to \mathbb{R}^k$ there exists a point $x \in S^k$ such that

$$\operatorname{rank} d_x f < k,$$

where $d_x f$ is the differential $d_x f : T_x S^k \to \mathbb{R}^k$.

Denote $\Omega = dx_1 \wedge \cdots \wedge dx_k$. Note that the form Ω is exact, Hence, we have $\int_{S^k} f^*\Omega = 0$. Then by a problem from a homework we know that there exists a point $x \in S^k$ such that $(f^*\Omega)_x = 0$. Take tangent vectors $T_1, \ldots, T_k \in T_x(S^k)$ which form a basis of $T_x S^k$. Then the value $(f^*\Omega)_x(T_1, \ldots, T_k) = \Omega(d_xf(T_1), \ldots, d_xf(T_k)) = \mathrm{Vol}_k P(d_xf(T_1), \ldots, d_xf(T_k))$. Hence, $\mathrm{Vol}_k P(d_xf(T_1), \ldots, d_xf(T_k)) = 0$, and therefore vectors $d_xf(T_1), \ldots, d_xf(T_k)$ are linearly dependent. But this means that $\operatorname{rank} d_x f < k$.

An alternative solution: Let f_1 be the first coordinate function of the map f. Then f achieves its maximum at some point $x \in S^k$. Hence, at this point the first row of the Jacobi matrix of the map f (written with respect to any local coordinates) is equal to 0, and hence its rank, and therefore the rank of $d_x f$ is $< k$.

5.* Consider the following two subsets of $\mathbb{R}^2$:

$$A = \{xy(x+y)(x-y) = 0\};$$
$$B = \{xy(x+2y)(x-y) = 0\}.$$

Prove that there is no diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(A) = B$.

Let us denote $D := d_0 f$. The diffeomorphism f maps 4 lines $l_1 = \{y = 0\}, l_2 = \{y = x\}, l_3 = \{x = 0\}$ and $l_4 = \{y = -x\}$ to the 4 lines $l_1, l_2, l_3 . l_4' = \{y = -\frac{x}{2}\}$. Hence, its differential of f at the origin also has this property. Thus the problem is reduced to the case when f is a linear map. The map f also preserves the cyclic ordering of the lines. One needs to consider the 4 cases, depending to which line is mapped the line l_1. We will analyze here only the case when l_1 is mapped onto l_1. All the other cases can be analyzed in a similar way.

In this case l_2 is mapped to l_2, l_3 to l_3 and l_4 to l_4'. Consider vectors $e_1 = (1,0) \in l_1$, $e_2 = (1,1) \in l_2$, $e_3 = (0,1) \in l_3$ and $e_4 = (1,-1) \in l_4$. Then we have $f(e_1) = (a,0), f(e_3) = (0,b), f(e_2) = (c,c), f(e_4) = (d,-2d)$ for some numbers $a,b,c,d \neq 0$. But $e_2 = e_1 + e_3$ and hence $f(e_2) = f(e_1) + f(e_3)$. Therefore, $(c,c) = (a,b)$, and thus $a = b = c$. Then $(d,-2d) = f(e_4) = f(e_1) - f(e_3) = (c,-c)$, which is impossible.