Math 52H: More practice problems for the Final Exam

1. Prove that if S is a closed surface in $\mathbb{R}^3$, $\mathbf{n}$ its unit normal vector field and $\mathbf{l}$ any fixed vector then

$$\iint_S \langle \mathbf{n}, \mathbf{l} \rangle dS = 0.$$

2. Compute the integral

$$\iint_S (x^2 + y^2) dS,$$

where S is the sphere $\{x^2 + y^2 + z^2 = R^2\}$.

3. Use Stokes' theorem to compute

$$\int_C -y^3 dx + x^3 dy - z^3 dz,$$

where C is the intersection of the cylinder $\{x^2 + y^2 = 1\}$ and the plane $\{x + y + z = 1\}$. The orientation of C corresponds to the counter-clockwise motion in the xy-plane.

4. Consider a differential form $\omega = \sum_1^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$.

a) Find a vector field $\mathbf{v}$ on $\mathbb{R}^{2n}$ such that

$$d(\mathbf{v} \lrcorner \omega) = \omega.$$

(This problem has infinitely many solutions. Find any of them.)

b) Compute $\mathrm{Flux}_S\,\mathbf{v}$, where S is an ellipsoid

$$\left\{\sum_1^n \frac{x_i^2+y_i^2}{a_i^2}=1\right\}$$

cooriented by the outward normal vector field. Explain why the answer is independent of the choice of $\mathbf{v}$ in Part a).

5. Consider a 4-dimensional submanifold with boundary in $\mathbb{R}^8$:

$$\begin{aligned}\Gamma =&\big\{(x_1,\dots,x_8)\in\mathbb{R}^8;\ x_5=x_1\cos\alpha+x_2\sin\alpha, x_6=-x_1\sin\alpha+x_2\cos\alpha,\\ &x_7=2x_3-x_4, x_8=-x_3+x_4,\ x_1^2+x_2^2+x_3^2+x_4^4\le 1\big\}.\end{aligned}$$

Suppose that Γ is oriented by its parameterization by coordinates (x_1,x_2,x_3,x_4). Compute

$$\int_\Gamma dx_5\wedge dx_6\wedge dx_7\wedge dx_8.$$

6. Consider a vector field

$$\mathbf{v}=\frac{1}{r^3}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right),$$

$r=\sqrt{x^2+y^2+z^2}$ in $\mathbb{R}^3\setminus 0$. Let us denote

$$S:=\left\{(x,y,z)\in\mathbb{R}^3;\ z=e^{x^2+y^2-\frac{1}{2}},\ x^2+y^2+z^2\le\frac{3}{2}\right\}$$

and co-orient this surface by a normal vector field which is equal to $(0,0,1)$ at the point $(0,0,\frac{1}{\sqrt{e}})\in S$. Compute $\mathrm{Flux}_S\,\mathbf{v}$.

7. Suppose that a vector field $\mathbf{v}$ in $\mathbb{R}^3$ with coordinate functions (P,Q,R) satisfies $\mathrm{curl}\,\mathbf{v}=0$. Find an explicit expression for a function F such that $\mathbf{v}=\nabla F$.

8. Let us view the space $\mathbb{R}^4$ with coordinates (x_1,y_1,x_2,y_2) as a complex vector space $\mathbb{C}^2$ with coordinates $(z_1=x_1+iy_1, z_2=x_2+iy_2)$. Consider a surface

$$S=\{(z_1,z_2)\in\mathbb{C}^2;\ z_2=z_1^2, |z_1|\le 1.\}$$

Compute $\mathrm{Area}(S)$.