Math 52H: Final Exam

March 15, 2010

1. Consider a differential form α on $\mathbb{R}^3$ such that $\alpha\wedge d\alpha = dx_1\wedge dx_2\wedge dx_3$, and which outside the unit ball $B_1(0)\subset\mathbb{R}^3$ is equal to $dx_1 + x_2 dx_3$.

a) Show that there exists a unique vector field $\mathbf{v}$ on $\mathbb{R}^3$ which satisfies two conditions

$$\mathbf{v}\,\lrcorner\, d\alpha = 0 \quad \text{and} \quad \alpha(\mathbf{v}) \equiv 1.$$

b) Compute $\operatorname{div}\mathbf{v}$ and $\operatorname{Flux}_D \mathbf{v}$, where $D = \{x_1 = 0;\ x_2^2 + x_3^2 \le 1\}$. We assume here that D is co-oriented by the vector field $\frac{\partial}{\partial x_1}$.

2. Consider a vector field

$$\mathbf{v} = \frac{1}{r^3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right\},$$

where $r = \sqrt{x^2+y^2+z^2}$. Compute $\operatorname{Flux}_S \mathbf{v}$, where S is the upper-half ellipsoid

$$S = \left\{(x,y,z)\in\mathbb{R}^3;\ \frac{x^2+y^2}{4} + \frac{z^2}{9} = 1,\ z\ge 0\right\}$$

co-oriented by an outward normal vector field to the boundary of the solid ellipsoid

$$\left\{\frac{x^2+y^2}{4} + \frac{z^2}{9} \le 1\right\}.$$

3. Consider a vector field $\mathbf{v}$ in $\mathbb{R}^3$ with coordinate functions (P, Q, R). Assuming that $\operatorname{div} \mathbf{v} = 0$, find an explicit expression for a vector field $\mathbf{w}$ such that $\mathbf{v} = \operatorname{curl} \mathbf{w}$.

4. Given vectors $v_1, \ldots, v_k \in \mathbb{R}^n$, let us denote by α the exterior k-form $\mathcal{D}(v_1) \wedge \cdots \wedge \mathcal{D}(v_k)$. Prove that

$$|\mathrm{Vol}_k P(v_1, \ldots, v_k)| = \| \alpha \| = \sqrt{*(\alpha \wedge *\alpha)}.$$

5. Let us define the n-torus $T^n(r_1, \ldots, r_n) \subset \mathbb{R}^{2n}$ by the equations

$$T^n(r_1, \ldots, r_n) = \{(x_1, y_1, \ldots, x_n, y_n) \subset \mathbb{R}^{2n}; x_1^2 + y_1^2 = r_1^2, \ldots, x_n^2 + y_n^2 = r_n^2\}.$$

Compute $\mathrm{Vol}_n(T^n)$.

Each subproblem is 10 points.