Math 52H: 2010 Final Exam

Solutions

1. Consider a differential form α on $\mathbb{R}^3$ such that $\alpha \wedge d\alpha = dx_1 \wedge dx_2 \wedge dx_3$, and which outside the unit ball $B_1(0) \subset \mathbb{R}^3$ is equal to $dx_1 + x_2 dx_3$.

a) Show that there exists a unique vector field $\mathbf{v}$ on $\mathbb{R}^3$ which satisfies two conditions

$$\mathbf{v} \lrcorner\, d\alpha = 0 \quad \text{and} \quad \alpha(\mathbf{v}) \equiv 1.$$

b) Compute $\operatorname{div} \mathbf{v}$ and $\operatorname{Flux}_D \mathbf{v}$, where $D = \{x_1 = 0;\ x_2^2 + x_3^2 \leq 1\}$. We assume here that D is co-oriented by the vector field $\frac{\partial}{\partial x_1}$.

a) Note that the form α nowhere vanishes, i.e. for every $x \in \mathbb{R}^3$ the linear function $\alpha_x : \mathbb{R}^3 \to \mathbb{R}$ is not equal to 0. Denote by ξ the plane field $\xi_x = \{h \in \mathbb{R}^3_x; \alpha_x(h) = 0\}$, $x \in \mathbb{R}^3$. We also note that for every $x \in \mathbb{R}^3$ the exterior 2-form $d\alpha_x$ on $\mathbb{R}^3_x$ has rank 2. Indeed, the rank of a skew-symmetric matrix is always even. On the other hand, it is not 0 according to the condition $\alpha \wedge d\alpha = dx_1 \wedge dx_2 \wedge dx_3$. Moreover, the same condition implies that the restriction of the form $d\alpha_x$ to ξ_x is non-degenerate. Hence, the homogeneous linear system of equations $h \lrcorner\, d\alpha_x = 0$ on $\mathbb{R}^3_x$ has a 1-dimensional space $l_x \subset \mathbb{R}^3_x$ of solutions, and the line l_x is transversal to the plane ξ_x. The condition $\mathbf{v} \lrcorner\, d\alpha = 0$ is equivalent to $\mathbf{v}(x) \in l_x$ for all $x \in \mathbb{R}^3$, and the condition $\alpha(\mathbf{v}) \equiv 1$ determines a unique vector $\mathbf{v}(x) \in l_x$ for each $x \in \mathbb{R}^3$. Outside $B_1(0)$ we have $\mathbf{v} = \frac{\partial}{\partial x_1}$.

b) Denote $\Omega := dx_1 \wedge dx_2 \wedge dx_3$. Then

$$\operatorname{div} \mathbf{v} = *d(\,\mathbf{v} \lrcorner \Omega) = *d(\alpha(\,\mathbf{v})d\alpha - \alpha \wedge (\,\mathbf{v} \lrcorner d\alpha)) = *d(d\alpha) = 0.$$

Hence, the divergence theorem (or Stokes' theorem) implies that $\mathrm{Flux}_D\, \mathbf{v} = \mathrm{Flux}_S\, \mathbf{v}$ for any surface S with $\partial S = \partial D$ oriented in such a way that the induced orientation of ∂S coincides with the orientation of ∂D. Choose as S the hemisphere $\{x_1^2 + x_2^2 + x_3^2 = 1;\ x_1 \geq 0\}$. Then $\mathbf{v}|_S = \frac{\partial}{\partial x_1}$. Thus

$$\mathrm{Flux}_D(\,\mathbf{v}) = \mathrm{Flux}_S(\,\mathbf{v}) = \mathrm{Flux}_S \left(\frac{\partial}{\partial x_1}\right) = \mathrm{Flux}_D \left(\frac{\partial}{\partial x_1}\right) = \pi, \tag{1}$$

because $\frac{\partial}{\partial x_1}$ is also divergence free.

2. Consider a vector field

$$\mathbf{v} = \frac{1}{r^3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right\},$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Compute $\mathrm{Flux}_S\, \mathbf{v}$, where S is the upper-half ellipsoid

$$S = \left\{(x, y, z) \in \mathbb{R}^3;\ \ \frac{x^2 + y^2}{4} + \frac{z^2}{9} = 1,\ z \geq 0\right\}$$

co-oriented by an outward normal vector field to the boundary of the solid ellipsoid

$$\left\{\frac{x^2 + y^2}{4} + \frac{z^2}{9} \leq 1\right\}.$$

The vector field $\mathbf{v}$ is divergence free in the complement of 0. In particular, similar to the previous problem we have $\mathrm{Flux}_S\, \mathbf{v} = \mathrm{Flux}_\Sigma\, \mathbf{v}$, where

$$\Sigma = \left\{(x, y, z) \in \mathbb{R}^3;\quad x^2 + y^2 + z^2 = 4,\ z \geq 0\right\}.$$

Hence $\mathrm{Flux}_S\, \mathbf{v} = \frac{1}{4}\mathrm{Area}(\Sigma) = 2\pi$.

3. Consider a vector field $\mathbf{v}$ in $\mathbb{R}^3$ with coordinate functions (P, Q, R). Assuming that $\operatorname{div} \mathbf{v} = 0$, find an explicit expression for a vector field $\mathbf{w}$ such that $\mathbf{v} = \operatorname{curl} \mathbf{w}$.

Denote $\alpha = \mathbf{v} \lrcorner \Omega$, where $\Omega := dx \wedge dy \wedge dz$. Equivalently, $\alpha = *\mathcal{D}(\mathbf{v})$. Then $\operatorname{div} \mathbf{v} = 0$ is equivalent to $d\alpha = 0$. On $\mathbb{R}^3$ this implies that the differential 2-form α is exact. Let us consider a homotopy $F : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$, defined by the formula $F(t,u) = tu$. Then $F(1,u) = u$ and $F(0,u) = 0$ for all $u = (x,y,z) \in \mathbb{R}^3$. Consider the homotopy operator $K : \Omega^2([0,1]\times\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$. It is defined by the formula $K(\omega) = \int_0^1 \frac{\partial}{\partial t} \lrcorner \omega dt$. As it was proven in the class, $\alpha = d(K(F^*\alpha))$. If $\mathbf{v}$ has coordinate functions (P,Q,R) then $\alpha = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$. We have

$$\frac{\partial}{\partial t} \lrcorner F^*\alpha = P(tu)(ydz - zdy) + Q(tu)(zdx - xdz) + R(tu)(xdy - ydx).$$

Denote

$$\beta := KF^*\alpha = \int_0^1 P(tu)dt(ydz - zdy) + \int_0^1 Q(tu)dt(zdx - xdz) + \int_0^1 R(tu)dt(xdy - ydx) =$$
$$\left(\int_0^1 (zQ(tu) - yR(tu))dt\right) dx + \left(\int_0^1 (xR(tu) - zP(tu))dt\right) dy +$$
$$\left(\int_0^1 (yP(tu) - xQ(tu))dt\right) dz \tag{2}$$

and define $\mathbf{w} = \mathcal{D}^{-1}\beta$. Then

$$\operatorname{curl} \mathbf{w} = \mathcal{D}^{-1} * d\mathcal{D}(\mathbf{w}) = \mathcal{D}^{-1} * d\mathcal{D}\mathcal{D}^{-1}\beta = \mathcal{D}^{-1} * d\beta = \mathcal{D}^{-1} * \alpha = \mathbf{v}.$$

Thus $\mathbf{w}$ has coordinate functions

$$\left(\int_0^1 (zQ(tu) - yR(tu))dt, \int_0^1 (xR(tu) - zP(tu))dt, \int_0^1 (yP(tu) - xQ(tu))dt\right).$$

4. Given vectors $v_1, \ldots, v_k \in \mathbb{R}^n$, let us denote by α the exterior k-form $\mathcal{D}(v_1) \wedge \cdots \wedge \mathcal{D}(v_k)$. Prove that

$$|\mathrm{Vol}_k P(v_1, \ldots, v_k)| = \| \alpha \| = \sqrt{*(\alpha \wedge *\alpha)}.$$

Let us choose an orthonormal basis $e_1, \ldots, e_k, \ldots, e_n$ in such a way that $\mathrm{Span}(v_1, \ldots, v_k) = \mathrm{Span}(e_1, \ldots, e_k)$, and let $x_1, \ldots, x_n$ be the corresponding Cartesian coordinates. Then $\mathrm{Vol}_k(v_1, \ldots, v_k)$

is the determinant of the square matrix formed by the first k coordinates of vectors $v_1, \dots, v_k$. On the other hand, $\alpha := \mathcal{D}(v_1) \wedge \dots \wedge \mathcal{D}(v_k) = C x_1 \wedge \dots \wedge x_k$, where $C = \alpha(e_1, \dots, e_k) = \det A$, where $A = (a_{ij})$, $a_{ij} = \mathcal{D}(v_i)(e_j) = \langle v_i, e_j \rangle$. Thus A is the matrix whose rows are coordinates of vectors v_i in the basis $e_1, \dots, e_k$. Hence, $C = \mathrm{Vol}_k(v_1, \dots, v_k)$. On the other hand, $*\alpha = C x_{k+1} \wedge \dots \wedge x_n$, and hence $*(\alpha \wedge *\alpha) = C^2$, and thus

$$|\mathrm{Vol}_k P(v_1, \dots, v_k)| = \| \alpha \| = \sqrt{*(\alpha \wedge *\alpha)}.$$

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5. Let us define the n-torus $T^n(r_1, \dots, r_n) \subset \mathbb{R}^{2n}$ by the equations

$$T^n(r_1, \dots, r_n) = \{(x_1, y_1, \dots, x_n, y_n) \subset \mathbb{R}^{2n}; x_1^2 + y_1^2 = r_1^2, \dots, x_n^2 + y_n^2 = r_n^2\}.$$

Compute $\mathrm{Vol}_n(T^n)$.

Let us parameterize the torus by the map

$$\Phi(\theta_1, \dots, \theta_n) = (r_1 \cos\theta_1, r_1 \sin\theta_1, \dots, r_n \cos\theta_n, r_n \sin\theta_n),$$

$\theta_1, \dots, \theta_n \in [0, 2\pi]$. Then

$$\frac{\partial \Phi}{\partial \theta_j}(u) = -r_j \sin\theta_j \frac{\partial}{\partial x_j} + r_j \cos\theta_j \frac{\partial}{\partial y_j}, \quad j = 1, \dots .$$

Note that all these vectors are pairwise orthogonal and

$$\left\| \frac{\partial \Phi}{\partial \theta_j}(u) \right\| = r_j.$$

Hence the Gram matrix $(D\Phi)^T D\Phi$ of the vectors $\frac{\partial \Phi}{\partial \theta_j}(u)$ is diagonal and has $r_1^2, \dots, r_n^2$ on the diagonal. Thus, $\det((D\Phi)^T D\Phi) = \prod\limits_1^n r_j^2$.

Hence,

$$\mathrm{Vol}_n(T^n) = \int\limits_0^{2\pi} \dots \int\limits_0^{2\pi} \sqrt{\det((D\Phi)^T D\Phi)} d\theta_1 \dots d\theta_n j = 2^n \pi^n \prod_1^n r_j.$$