Symplectic Geometry of Stein Manifolds

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Contents

Chapter 1

Introduction

To be rewritten.

A *Stein manifold* is a properly embedded complex submanifold of some $\mathbb{C}^N$. Hence Stein manifolds are necessarily noncompact, and properly embedded complex submanifolds of Stein manifolds are again Stein. Stein manifolds arise, e.g., from closed complex projective manifolds $X \subset \mathbb{C}P^N$: If $H \subset \mathbb{C}P^N$ is any hyperplane, then $X \setminus H$ is Stein.

Using this construction, it is not hard to see that every closed Riemann surface with at least one point removed is Stein. In fact, as we will see below, any open Riemann surface is Stein. Already this example shows that the class of Stein manifolds is much larger than the class of affine algebraic manifolds.

Stein manifolds can also be described intrinsically. The most important for us characterization id due to Grauert (see [28]). Let (V, J) be a complex manifold, where J denotes the complex multiplication on tangent spaces. A smooth function $\phi : V \to \mathbb{R}$ is called *exhausting* if it is proper (i.e., preimages of compact sets are compact) and bounded from below. To a function ϕ we can associate the 1-form $d^{\mathbb{C}}\phi := d\phi \circ J$ and the 2-form

$$\omega_\phi := -dd^{\mathbb{C}}\phi.$$

The function is called *J-convex* or *strictly plurisubharmonic* if $\omega_\phi(v, Jv) > 0$ for every nonzero tangent vector v. This is equivalent to saying that ω_ϕ is a symplectic (i.e., closed and nondegenerate) form compatible with J.

Since the function $\phi_{\text{st}}(z) := |z|^2$ on $\mathbb{C}^N$ is exhausting and i-convex with respect to the standard complex structure i on $\mathbb{C}^N$, every Stein manifold admits an exhausting J-convex function (namely the restriction of ϕ_{st}). The following theorem asserts that the converse is also true.

Theorem 1.1 (Grauert [28]). *A complex manifold which admits an exhausting J-convex function is Stein.*

J-convexity is an open property, and hence the exhausting J-convex function in Grauert's theorem can assumed to be Morse, i.e. having non-degenerate critical points.

We now turn to one of the main problems addressed in this book: Given a smooth manifold V^{2n} of real dimension $2n$, when does it admit a Stein structure? Clearly, a necessary condition is the existence of an *almost complex structure* J, i.e., an endomorphism of the tangent bundle with $J^2 = -1\!\!1$.

A second necessary condition arises from the following property of J-convex functions (see Chapter 2): If p is a nondegenerate critical point of a J-convex function on a complex manifold of real dimension $2n$, then its Morse index satisfies $\mathrm{ind}(p) \leq n$. Since J-convexity is a C^2-open condition, every J-convex function can be perturbed to a J-convex Morse function. In particular, every Stein manifold (V^{2n}, J) admits an exhausting Morse function with $\mathrm{ind}(p) \leq n$ at all critical points. By Morse theory, this implies that V has a handlebody decomposition using only handles of index at most n. The following theorem asserts that these two necessary conditions are also sufficient in real dimension $2n > 4$.

Theorem 1.2 (Eliashberg [14])**.** *Let V^{2n} be an open smooth manifold of dimension $2n > 4$ with an almost complex structure J and an exhausting Morse function ϕ without critical points of index $> n$.*

(i) Then V admits a Stein structure. More precisely, J is homotopic through almost complex structures to an integrable complex structure $\tilde{J}$ such that ϕ is $\tilde{J}$-convex.

(ii) If in addition J is integrable, then there exists an isotopy $h_t : V \to V$, with $h_0 = \mathrm{Id}$ such $\phi \circ h_1$ is J-convex, and, in particular, $h_1(V) \subset V$ is Stein with the induced complex structure J.

More precisely, as we explain in this book, existence of a Stein structure on a given smooth manifold is equivalent to existence of a certain symplectic geometric analogue of it, which we call *Weinstein* structure. We will show that without any dimensional constraints, a Weinstein structure can be upgraded to a Stein one, while the situation with the existence of Weinstein structure is is drastically different in dimension 4. For instance, $S^2 \times \mathbb{R}^2$ does not admit any Stein (and Weinstein). complex structure (see [LiMa]). However, Theorem 1.2 has the following topological analogue.

Theorem 1.3 (Gompf [25])**.** *Let V^4 be an oriented open topological 4-manifold which admits a (possibly infinite) handlebody decomposition without handles of index > 2. Then V is homeomorphic to a Stein surface. Moreover, any homotopy class of almost complex structures on V is induced by an orientation preserving homeomorphism from a Stein surface.*

One could ask whether the above h-principle type results can be expanded to prove an analogues of Smale's h-cobordism theorem of J-convex functions, as well as its parametric versions in the spirit of pseudoisotopy theory. In particular,

(i) Suppose a Stein manifold (V, J) *is diffeomorphic to* $\mathbb{R}^{2n}$*, and which is* J*-convex at infinity. Does it admit an exhausting* J*-convex function with only one critical point, the minimum?*

(ii) Suppose $\varphi_0, \varphi_1 : V :\to \mathbb{R}$ *be two exhausting Morse functions which are* J_0- *and* J_1*-convex, respectively for two Stein structures* J_0 *and* J_1 *on* V *Suppose that* φ_0 *and* φ_1 *have no critical points at infinity and can be connected by a path* φ_t*,* $t \in [0, 1]$ *of smooth functions without critical points of index* $> n$*, and without critical points outside a compact subset of* V*. Is there a homotopy* (J_t, φ_t)*,* $t \in [0, 1]$*, such that* φ_t *is* J_t*-convex, and all functions* φ_t*,* $t \in [0, 1]$ *have no critical points outside a compact set?*

As it was shown recently P. Seidel and I. Smith, [56] and M. McLean, [48], the answer to Question (i) is negative. On the other hand, the answer is positive in dimension 4, see [15]. We will provide in this book some partial answers to Question (ii), which in general is widely open.

This book is organized as follows. In Chapters 2 and 3 we explore basic properties of J-convex functions and hypersurfaces. Chapter 4 we construct special hypersurfaces that play a crucial role in extending J-convex functions over handles. The next two chapters contain background material which is standard but sometimes not easy to find in the literature. In Section 7.2 we derive a general result on real analytic approximations from standard results in complex analysis. Chapter 5 collects some facts about smooth embeddings and immersions, and more specifically Legendrian and isotropic embeddings in contact manifolds. In Chapter 11 we describe the attaching of handles in the almost complex and in the holomorphic category, and how to pass from one to the other. Theorem 1.2 is proved at the end of this chapter.

The last two chapters contain results whose proofs have not appeared in the literature. The main result of Chapter **??** reduces the deformation theory of Stein structures to the deformation theory of Weinstein structures. In Chapter 15 we show that in the subcritical case this deformation theory reduces to pure Morse theory, which leads to a version of the h-cobordism theorem for Stein manifolds.

Part I

J-convexity

Chapter 2

J-convex functions and hypersurfaces

2.1 Linear algebra

A *complex vector space* (V, J) is a real vector space V of dimension $2n$ with an endomorphism J satisfying $J^2 = -\mathbb{1}$. A *Hermitian form* on (V, J) is an $\mathbb{R}$-bilinear map $H : V \times V \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first variable and satisfies $H(X,Y) = \overline{H(Y,X)}$. If H is, moreover, positive definite it is called *Hermitian metric*. We can write a Hermitian form H uniquely as

$$H = g - i\omega,$$

where g is a symmetric and ω a skew-symmetric bilinear form on the real vector space V. The forms g and ω determine each other:

$$g(X,Y) = \omega(X, JY), \qquad \omega(X,Y) = g(JX,Y)$$

for $X, Y \in V$. Moreover, the forms ω and g are invariant under J, which can be equivalently expressed by the equation

$$\omega(JX,Y) + \omega(X, JY) = 0.$$

Conversely, given a skew-symmetric J-invariant form ω, we can uniquely reconstruct the corresponding Hermitian form H:

$$H(X,Y) := \omega(X, JY) - i\omega(X,Y). \tag{2.1}$$

For example, consider the complex vector space $(\mathbb{C}^n, i)$ with coordinates $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$. It carries the standard Hermitian metric

$$(v, w) := \sum_{j=1}^{n} v_j \bar{w}_j = \langle v, w\rangle - i\omega_0(v, w),$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean metric and $\omega_0 = \sum_j dx_j \wedge dy_j$ the standard symplectic form on $\mathbb{C}^n$.

2.2 J-convex functions

An *almost complex structure* on a smooth manifold V of real dimension $2n$ is an automorphism $J : TV \to TV$ satisfying $J^2 = -\mathbb{1}$ on each fiber. The pair (V, J) is called *almost complex manifold*. It is called *complex manifold* if J is *integrable*, i.e. J is induced by complex coordinates on V. By the theorem of Newlander and Nirenberg [51], a (sufficiently smooth) almost complex structure J is integrable if and only if its *Nijenhuis tensor*

$$N(X, Y) := [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y], \qquad X, Y \in TV,$$

vanishes identically.

In the following let (V, J) be an almost complex manifold. To a smooth function $\phi : V \to \mathbb{R}$ we associate the 2-form

$$\omega_\phi := -dd^{\mathbb{C}}\phi,$$

where the differential operator $d^{\mathbb{C}}$ is defined by

$$d^{\mathbb{C}}\phi(X) := d\phi(JX)$$

for $X \in TV$.[1] The form ω_ϕ is in general not J-invariant. However, it is J-invariant if J is integrable. To see this, consider the complex vector space $(\mathbb{C}^n, i)$. Given a function $\phi : \mathbb{C}^n \to \mathbb{R}$, define the complex valued $(1,1)$-form

$$\partial\bar{\partial}\phi := \sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

Using the identities

$$dz_j \circ i = i\, dz_j, \qquad d\bar{z}_j \circ i = -i\, d\bar{z}_j$$

we compute

$$d^{\mathbb{C}}\phi = \sum_j \frac{\partial \phi}{\partial z_j} dz_j \circ i + \frac{\partial \phi}{\partial \bar{z}_j} d\bar{z}_j \circ i = \sum_j i \frac{\partial \phi}{\partial z_j} dz_j - i \frac{\partial \phi}{\partial \bar{z}_j} d\bar{z}_j,$$

$$dd^{\mathbb{C}}\phi = -2i \sum_{i,j} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

[1] Sometimes it will be important for us to to reflect in the notation dependence of the operator $d^{\mathbb{C}}$ and the form ω_ϕ on J. In this case we will write d^J and $\omega_{J,\phi}$ instead of $d^{\mathbb{C}}$ and $\omega\phi$

Hence

$$\omega_\phi = 2i\partial\bar{\partial}\phi \tag{2.2}$$

and the i-invariance of ω_ϕ follows from the invariance of $\partial\bar{\partial}\phi$.

A function ϕ is called *J-convex*[2] if $\omega_\phi(X, JX) > 0$ for all nonzero tangent vectors X. If ω_ϕ is J-invariant it defines by (2.1) a unique Hermitian form

$$H_\phi := g_\phi - i\omega_\phi,$$

and ϕ is J-convex iff the Hermitian form H_ϕ is positive definite.

From (2.2) we can derive a simple expression for the form H_ϕ associated to a function $\phi : \mathbb{C}^n \to \mathbb{R}$ in terms of the matrix $a_{ij} := \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}$. For $v, w \in \mathbb{C}^n$ we have

$$\begin{aligned}\omega_\phi(v, w) &= 2i\sum_{ij} a_{ij} dz_i \wedge d\bar{z}_j(v, w) = 2i\sum_{ij} a_{ij}(v_i\bar{w}_j - w_i\bar{v}_j)\\ &= 2i\sum_{ij}(a_{ij}v_i\bar{w}_j - \bar{a}_{ij}\bar{v}_i w_j) = -4\mathrm{Im}\left(\sum_{ij} a_{ij}v_i\bar{w}_j\right),\end{aligned}$$

hence

$$H_\phi(v, w) = 4\sum_{i,j=1}^{n} \frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j} v_i\bar{w}_j. \tag{2.3}$$

Example 2.1. The function $\phi(z) := \sum_j |z_j|^2$ on $\mathbb{C}^n$ is i-convex with respect to the standard complex structure i. The corresponding form H_ϕ equals $4(\ ,\ )$, where $(\ ,\ )$ is the standard Hermitian metric on $\mathbb{C}^n$.

2.3 The Levi form of a hypersurface

Let Σ be a smooth (real) hypersurface in an almost complex manifold (V, J). Each tangent space $T_p\Sigma \subset T_pV$, $p \in \Sigma$, contains a unique maximal complex subspace $\xi_p \subset T_p\Sigma$ which is given by

$$\xi_p = T_p\Sigma \cap JT_p\Sigma.$$

Suppose that Σ is cooriented by a transverse vector field ν to Σ in V such that $J\nu$ is tangent to Σ. The hyperplane field ξ can be defined by a Pfaffian equation $\{\alpha = 0\}$, where the sign of the 1-form α is fixed by the condition $\alpha(J\nu) > 0$. The 2-form

$$\omega_\Sigma := d\alpha|_\xi$$

is then defined uniquely up to multiplication by a positive function. As in the previous section we may ask whether ω_Σ is J-invariant. The following lemma gives a necessary and sufficient condition in terms of the Nijenhuis tensor.

[2]Throughout this book, by convexity and J-convexity we will always mean *strict* convexity and J-convexity. Non-strict (J-)convexity will be referred to as *weak* (J-)convexity.

Lemma 2.2. *The form ω_Σ is J-invariant for a hypersurface Σ if and only if $N|_{\xi\times\xi}$ takes values in ξ. The form ω_Σ is J-invariant for* every *hypersurface Σ if and only if for all $X, Y \in TV$, $N(X,Y)$ lies in the complex plane spanned by X and Y. In particular, this is the case if J is integrable or if V has complex dimension 2.*

Proof. Let $\Sigma \subset V$ be a hypersurface and α a defining 1-form for ξ. Extend α to a neighborhood of Σ such that $\alpha(\nu) = 0$. For $X, Y \in \xi$ we have $[X,Y] \in T\Sigma$ and therefore $J[X,Y] = a\nu + Z$ for some $a \in \mathbb{R}$ and $Z \in \xi$. This shows that

$$\alpha(J[X,Y]) = 0$$

for all $X, Y \in \xi$. Applying this to various combinations of X, Y, JX and JY we obtain

$$\alpha\Big(N(X,Y)\Big) = \alpha([JX,JY]) - \alpha([X,Y]),$$
$$\alpha\Big(JN(X,Y)\Big) = \alpha([X,JY]) + \alpha([JX,Y]).$$

The form ω_Σ is given by

$$\omega_\Sigma(X,Y) = \frac{1}{2}\Big(X\cdot\alpha(Y) - Y\cdot\alpha(X) - \alpha([X,Y]\Big) = -\frac{1}{2}\alpha([X,Y]).$$

Inserting this in the formulae above yields

$$-\frac{1}{2}\alpha\Big(N(X,Y)\Big) = \omega_\Sigma(JX,JY) - \omega_\Sigma(X,Y),$$
$$-\frac{1}{2}\alpha\Big(JN(X,Y)\Big) = \omega_\Sigma(X,JY) + \omega_\Sigma(JX,Y).$$

Hence the J-invariance of ω_Σ is equivalent to

$$\alpha\Big(N(X,Y)\Big) = \alpha\Big(JN(X,Y)\Big) = 0,$$

i.e. $N(X,Y) \in \xi$ for all $X, Y \in \xi$. This proves the first statement and the 'if' in the second statement. For the 'only if' it suffices to note that if $N(X,Y)$ does not lie in the complex plane spanned by X and Y for some $X, Y \in TV$, then we find a hypersurface Σ such that $X, Y \in \xi$ and $N(X,Y) \notin \xi$. □

Remark 2.3. Given any hypersurface Σ, and any almost complex structure J it is always possible to find another almost complex structure $\widetilde{J}$ such that $\xi_J = \xi_{\widetilde{J}} = \xi$, and the form $d\alpha_\xi$ for a 1-form α defining ξ, is $\widetilde{J}$-invariant. Moreover, if ξ is non-integrable. i.e. if $d\alpha|_\xi$ is non-degenerate, the space of almost complex sructures $\widetilde{J}$ with these properties is contractible. See discussion of this in Section **??** below.

A hypersurface Σ is called *Levi-flat* if $\omega_\Sigma \equiv 0$. This is exactly the Frobenius integrability condition for the hyperplane field ξ on Σ. Hence, on a Levi-flat hypersurface ξ integrates to a real codimension 1 holomorphic foliation.

It is called *J-convex* (or *strictly pseudoconvex*) if $\omega_\Sigma(X, JX) > 0$ for all nonzero $X \in \xi$. If ω_Σ is J-invariant it defines a Hermitian form L_Σ on ξ by the formula

$$L_\Sigma(X,Y) := \omega_\Sigma(X, JY) - i\omega_\Sigma(X,Y)$$

for $X, Y \in \xi$. The Hermitian form L_Σ is called the *Levi form* of the (cooriented) hypersurface Σ. We will use the notation $L_\Sigma(X)$ for the quadratic form $L_\Sigma(X,X)$. Note that Σ is Levi-flat iff $L_\Sigma \equiv 0$, and J-convex iff L_Σ is positive definite. We will sometimes also refer to ω_Σ as the Levi form.

As pointed out above, the Levi form is defined uniquely up to multiplication by a positive function. Hence, in the computation of L_Σ we will sometimes use the notation $\doteq$ instead of $=$, indicating that some positive coefficients could be dropped in the computation.

If the hypersurface Σ is given by an equation $\{\phi = 0\}$ for a function $\phi : V \to \mathbb{R}$, then we can choose $\alpha = -d^{\mathbb{C}}\phi$ as the 1-form defining ξ (with the coorientation of Σ given by $d\phi$). Thus the Levi form can be defined as

$$\omega_\Sigma(X,Y) = -dd^{\mathbb{C}}\phi(X,Y).$$

This shows that regular level sets of a J-convex function ϕ are J-convex (being cooriented by $d\phi$). It turns out that the converse is also almost true (similarly to the situation for convex functions and hypersurfaces).

Lemma 2.4. *Let $\phi : V \to \mathbb{R}$ be a smooth function on an almost complex manifold without critical points such that all its level sets are compact and J-convex. Then there exists a convex increasing function $f : \mathbb{R} \to \mathbb{R}$ such that the composition $f \circ \phi$ is J-convex.*

Proof. Consider a regular level set Σ of ϕ. For a function $f : \mathbb{R} \to \mathbb{R}$ we have

$$\begin{aligned} d^{\mathbb{C}}(f\circ\phi) &= f'\circ\phi\, d^{\mathbb{C}}\phi, \\ -dd^{\mathbb{C}}(f\circ\phi) &= -f''\circ\phi\, d\phi\wedge d^{\mathbb{C}}\phi - f'\circ\phi\, dd^{\mathbb{C}}\phi. \end{aligned}$$

By the J-convexity of Σ, the term $-f'\circ\phi\ dd^{\mathbb{C}}\phi$ is positive definite on the maximal complex subspace $\xi \subset \Sigma$ if $f' > 0$. The form $\omega_phi|_V$ has the rank $= \dim_{\mathbb{R}} V - 1$, and hence there exists a unique vector field $X \in TV$ which satisfies the conditions $i(X)(\omega_\phi|_V) = 0$ and $dd^C\phi(X) = 1$. It is sufficient for us to ensure the inequality $\omega_{f\circ\phi}(X, JX) > 0$. We have $-d\phi\wedge d^{\mathbb{C}}\phi(X, J\underline{X}) = 1$, and by compactness of the level sets,

$$\omega_{f\circ\phi}(X,JX) = -dd^{\mathbb{C}}(f\circ\phi)(X,JX) > f''\circ\phi - h\circ\phi\ f'\circ\phi$$

for some smooth function $h : \mathbb{R} \to (0,\infty)$. Now solve the differential equation $f''(y) = h(y)f'(y)$ with initial condition $f'(y_0) > 0$. The solution exists for all $y \in \mathbb{R}$ and satisfies $f' > 0$, so $f\circ\phi$ is J-convex. $\square$

Remark 2.5. The proof of the preceding lemma also shows: If $\phi : V \to \mathbb{R}$ is J-convex, then $f \circ \phi$ is J-convex for any function $f : \mathbb{R} \to \mathbb{R}$ with $f' > 0$ and $f'' \geq 0$.

A vector field is called *complete* if its flow exists for all forward and backward times. For a J-convex function ϕ, let $\nabla_\phi \phi$ be the gradient of ϕ with respect to the metric $g_\phi = \omega_\phi(\cdot, J\cdot)$. In general, $\nabla_\phi \phi$ need not be complete:

Example 2.6. The function $\phi(z) := \sqrt{1+|z|^2}$ on $\mathbb{C}$ satisfies

$$\frac{\partial^2 \phi}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z} \frac{z}{\sqrt{1+|z|^2}} = \frac{1}{\sqrt{1+|z|^2}^3},$$

so $g_\phi = 4(1+|z|^2)^{-3/2}\langle\ ,\ \rangle$, where $\langle\ ,\ \rangle$ is the standard metric. In particular, ϕ is i-convex. Its gradient is determined from

$$d\phi = \frac{x\,dx + y\,dy}{\sqrt{1+|z|^2}} = \frac{4}{\sqrt{1+|z|^2}^3}\langle \nabla_\phi \phi, \cdot \rangle,$$

thus $\nabla_\phi \phi = \frac{1+|z|^2}{4}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$. A gradient line $\gamma(t)$ with $|\gamma(0)| = 1$ is given by $\gamma(t) = h(t)\gamma(0)$, where $h(t)$ satisfies $h' = \frac{1+h^2}{4}h$. This shows that $\gamma(t)$ tends to infinity in finite time, hence the gradient field $\nabla_\phi \phi$ is not complete.

However, the gradient field $\nabla_\phi \phi$ can always be made complete by composing ϕ with a sufficiently convex function:

Proposition 2.7. *Let $\phi : V \to [a, \infty)$ be an exhausting J-convex function on an almost complex manifold. Then for any diffeomorphism $f : [a,\infty) \to [b,\infty)$ such that $f'' > 0$ and $\lim_{y\to\infty} f'(y) = \infty$, the function $f \circ \phi$ is J-convex and its gradient vector field is complete.*

Proof. The function $\psi := f \circ \phi$ satisfies

$$dd^{\mathbb{C}}\psi = f'' \circ \phi\, d\phi \wedge d^{\mathbb{C}}\phi + f' \circ \phi\, dd^{C}\phi\,.$$

In particular, ψ is J-convex if $f' > 0$ and $f'' > 0$. The metric associated to ψ is given by

$$\begin{aligned} g_\psi(X,Y) &= -dd^{\mathbb{C}}\psi(X, JY) \\ &= +f'' \circ \phi\, [d\phi(X)d\phi(Y) + d^{\mathbb{C}}\phi(X)d^{C}\phi(Y)] + f' \circ \phi\, g_\phi(X,Y)\,. \end{aligned}$$

Let us compute the gradient $\nabla_\psi \psi$. We will find it in the form

$$\nabla_\psi \psi = \lambda \nabla_\phi \phi$$

for a function $\lambda : V \to \mathbb{R}$. The gradient is determined by

$$g_\psi(\nabla_\psi \psi, Y) = d\psi(Y) = f' \circ \phi\, d\phi(Y)$$

for any vector $Y \in TV$. Using $d\phi(\nabla_\phi \phi) = g_\phi(\nabla_\phi \phi, \nabla_\phi \phi) =: |\nabla_\phi \phi|^2$ and $d^{\mathbb{C}}\phi(\nabla_\phi \phi) = g_\phi(\nabla_\phi \phi, J\nabla_\phi \phi) = 0$, we compute the left hand side as

$$\begin{aligned}
&g_\psi(\nabla_\psi \psi, Y) \\
&= \lambda\Big\{f'' \circ \phi\ [d\phi(\nabla_\phi \phi) d\phi(Y) + d^{\mathbb{C}}\phi(\nabla_\phi \phi) d^C\phi(Y)] + f' \circ \phi\ g_\phi(\nabla_\phi \phi, Y)\Big\} \\
&= \lambda\{f'' \circ \phi\ |\nabla_\phi \phi|^2 d\phi(Y) + f' \circ \phi\ d\phi(Y)\}.
\end{aligned}$$

Comparing with the right side, we find

$$\lambda = \frac{f' \circ \phi}{f'' \circ \phi\ |\nabla_\phi \phi|^2 + f' \circ \phi}.$$

Since ϕ is proper, we only need to check completeness of the gradient flow for positive times. Consider an unbounded gradient trajectory $\gamma : [0, T) \to V$, i.e., a solution of

$$\frac{d\gamma}{dt}(t) = \nabla_\phi \phi\big(\gamma(t)\big), \qquad \lim_{t \to T} \phi\big(\gamma(t)\big) = \infty.$$

Here T can be finite or $+\infty$. The function ϕ maps the image of γ diffeomorphically onto some interval $[c, \infty)$. It pushes forward the vector field $\nabla_\phi \phi$ (which is tangent to the image of γ) to the vector field

$$\phi_*(\nabla_\phi \phi) = h(y)\frac{\partial}{\partial y},$$

where t and y are the coordinates on $[0, T)$ and $[c, \infty)$, respectively, and

$$h(y) := |\nabla_\phi \phi|^2 \left(\phi^{-1}(y)\right) > 0.$$

Similarly, ϕ pushes forward $\nabla_\psi \psi = \lambda \nabla_\phi \phi$ to the vector field

$$\phi_*(\nabla_\psi \psi) = \lambda\left(\phi^{-1}(y)\right) h(y)\frac{\partial}{\partial y} = \frac{f'(y)h(y)}{f''(y)h(y) + f'(y)}\frac{\partial}{\partial y} =: v(y).$$

Hence completeness of the vector field $\nabla_\psi \psi$ on the trajectory γ is equivalent to the completeness of the vector field v on $[c, \infty)$. An integral curve of v satisfies $\frac{dy}{ds} = v(y)$, or equivalently,

$$ds = \frac{f''(y)h(y) + f'(y)}{f'(y)h(y)}dy.$$

Thus completeness of the vector field v is equivalent to

$$+\infty = \int_c^\infty \frac{f''(y)h(y) + f'(y)}{f'(y)h(y)}dy = \int_c^\infty \frac{f''(y)dy}{f'(y)} + \int_c^\infty \frac{dy}{h(y)}.$$

The first integral on the right hand side is equal to $\int_c^\infty d\big(\ln f'(y)\big)$, so it diverges if and only if $\lim_{y \to \infty} f'(y) = \infty$. $\square$

We will call an exhausting J-convex function *completely exhausting* if its gradient vector field $\nabla_\phi \phi$ is complete.

2.4 J-convexity and geometric convexity

Next we investigate the relation between i-convexity and geometric convexity. Consider $\mathbb{C}^n = \mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{R}$ with coordinates $(z_1,\dots,z_{n-1},u+iv)$. Let $\Sigma\subset\mathbb{C}^n$ be a hypersurface which is given as a graph $\{u = f(z,v)\}$ for some smooth function $f:\mathbb{C}^{n-1}\times\mathbb{R}\to\mathbb{R}$. Assume that $f(0,0)=0$ and $df(0,0)=0$. Every hypersurface in a complex manifold can be locally written in this form.

The Taylor polynomial of second order of f around $(0,0)$ can be written as

$$T_2 f(z,v) = \sum_{i,j} a_{ij} z_i\bar z_j + 2Re\sum_{i,j} b_{ij} z_i z_j + v\,l(z,\bar z) + cv^2, \tag{2.4}$$

where l is some linear function of z and $\bar z$, and $a_{ij} = \frac{\partial^2 f}{\partial z_i\partial\bar z_j}(0,0)$. Let Σ be cooriented by the gradient of the function $f(z,v)-u$. Then the 2-form ω_Σ at the point 0 is given on $X,Y\in\xi_0=\mathbb{C}^{n-1}$ by

$$\begin{aligned}\omega_\Sigma(X,Y) &= 2i\partial\bar\partial f(X,Y) = 2i\sum_{i,j} a_{ij}dz_i\wedge d\bar z_j(X,Y)\\ &= -4\mathrm{Im}\,(AX,Y),\end{aligned}$$

where A is the complex $(n-1)\times(n-1)$ matrix with entries a_{ij}. Hence the Levi form at 0 is

$$L_\Sigma = 4\langle A\cdot,\cdot\rangle.$$

If the function f is (strictly) convex, then

$$T_2 f(z,0) + T_2 f(iz,0) = 2\sum_{ij} a_{ij} z_i\bar z_j > 0$$

for all $z\neq 0$, so the Levi form is positive definite. This shows that convexity of Σ implies i-convexity. The converse is not true, see the examples below. It is true, however, locally after a biholomorphic change of coordinates.

Proposition 2.8 (R.Narasimhan)**.** *A hypersurface $\Sigma\subset\mathbb{C}^n$ is i-convex if and only if it can be made (strictly) convex in a neighborhood of each of its points by a biholomorphic change of coordinates.*

Proof. The 'if' follows from the discussion above and the invariance of J-convexity under biholomorphic maps. For the converse write Σ in local coordinates as a graph $\{u=f(z,v)\}$ as above and consider its second Taylor polynomial (2.4). Let $w=u+iv$, and perform in a neighborhood of 0 the holomorphic change of coordinates $\tilde w := w - 2\sum_{ij} b_{ij} z_i z_j$. Then

$$\tilde u = \sum a_{ij} z_i\bar z_j + \tilde v\,l(z,\bar z) + c\tilde v^2 + O(3).$$

After another local change of coordinates $w' := \tilde w - \lambda\tilde w^2$, $\lambda\in\mathbb{R}$, we have

$$u' = \tilde u + \lambda(v')^2 + O(3) = \sum a_{ij} z_i\bar z_j + v' l(z,\bar z) + (c+\lambda)(v')^2 + O(3).$$

For λ sufficiently large the quadratic form on the right hand side is positive definite, so the hypersurface Σ is convex in the coordinates (z, w'). □

Consider for a moment a cooriented hypersurface in $\mathbb{R}^n$ with the Euclidean metric $\langle\ ,\ \rangle$. Its *second fundamental form*

$$II : T\Sigma \times T\Sigma \to \mathbb{R}$$

can be defined as follows. For $X \in T_x\Sigma$ let $\gamma : (-\epsilon, \epsilon) \to \Sigma$ be a curve with $\gamma(0) = x$ and $\dot\gamma(0) = X$. Then

$$II(X, X) := -\langle \ddot\gamma(0), \nu \rangle,$$

where ν is the unit normal vector to Σ in x defining the coorientation. The matrix representing the second fundamental form equals the differential of the Gauss map which associates to every point its unit normal vector. Our sign convention is chosen in such a way that the unit sphere in $\mathbb{R}^n$ has positive principal curvatures if it is cooriented by the *outward* pointing normal vector field. The *mean curvature* along a k-dimensional subspace $S \subset T_x\Sigma$ is defined as

$$\frac{1}{k}\sum_{i=1}^{k} II(v_i, v_i)$$

for some orthonormal basis $v_1, \dots, v_k$ of S. If Σ is given as a graph $\{x_n = f(x_1, \dots, x_{n-1})\}$ with $f(0) = 0$ and $df(0) = 0$, then for $X \in \mathbb{R}^{n-1}$ we can choose the curve

$$\gamma(t) := \Big(tX, f(tX)\Big)$$

in Σ. Taking the second derivative we obtain

$$II(X, X) = \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) X_i X_j, \tag{2.5}$$

if Σ is cooriented by the gradient of the function $f - x_n$. This leads to the following geometric characterization of i-convexity:

Proposition 2.9. *The Levi form of a cooriented hypersurface $\Sigma \subset \mathbb{C}^n$ with respect to the standard complex structure i is given at a point $z \in \Sigma$ by*

$$L_\Sigma(X) = \frac{1}{2}\Big(II(X, X) + II(iX, iX)\Big) \tag{2.6}$$

for $X \in T_z\Sigma$. Thus Σ is i-convex if and only if at every point $z \in \Sigma$ the mean curvature along any complex line in $T_z\Sigma$ is positive.

Proof. Write Σ locally as a graph $\{u = f(z, v)\}$ with $f(0, 0) = 0$ and $df(0, 0) = 0$, and such that the gradient of $f - u$ defines the coorientation of Σ. Consider the second Taylor polynomial (2.4) of f in $(0,0)$. In view of (2.5), the mean curvature along the complex line generated by $X \in \mathbb{C}^{n-1}$ is given by

$$\begin{aligned}\frac{1}{2}\Big(II(X,X)+II(iX,iX)\Big) &= \frac{1}{2}\Big(T_2 f(X)+T_2 f(iX)\Big)\\ &= \sum_{ij} a_{ij} X_i \bar{X}_j = L_\Sigma(X,X),\end{aligned}$$

and the proposition follows. □

As we already mentioned above the Levi form L_Σ is invariantly defined only up to multiplication by a positive function. However, for a hypersurface of $\mathbb{C}^n$ (or more generally of any Kähler manifold) we will call the form $\mathbb{L}_\Sigma$ given by the equation (2.6) the *normalized* Levi form. Furthermore, we denote in this case

$$\begin{aligned} m(\Sigma) &= \min_{X\in\xi, ||X||=1} \mathbb{L}_\Sigma(X),\\ M(\Sigma) &= \max(1, -II(X,X)), \quad X \in TV, ||X|| = 1,\\ \mu(\Sigma) &= \frac{m(\Sigma)}{M(\Sigma)}. \end{aligned}$$

The quantity $\mu(\Sigma)$ is called the *modulus of i-convexity* of the hypersurface Σ.

Lemma 2.10. *Suppose that for $\Sigma \subset \mathbb{C}^n$ we have $\mu(\Sigma) > \varepsilon$. Then there exists a positive $\delta = \delta(\varepsilon)$ such that if a complex structure J on $\mathcal{O}p\,\Sigma$ is δ-close in C^2-metric to the standard complex structure i, then Σ is J-convex.*

Proof. The condition $\mu(\Sigma) > \varepsilon$ ensure that there exists $\sigma(\varepsilon) > 0$ such that the mean normal curvature is positive along any plane which no more than by an angle σ from a complex line tangent to Σ. In a neighborhood of any point $p \in \Sigma$ there exists a δ-C^2-small biholomorphism $h : (\mathcal{O}p\,p, J) \to (\mathcal{O}p\,p, i)$. It changes the second fundamental form of Σ no more than by δ and preserves the direction of complex tangent hyperplanes to Σ up to an error of order δ. Hence, for δ small compared to σ the hypersurface $h(\Sigma)$ is i-convex, and hence Σ is J-convex. □

A family of hypersurfaces $\Sigma_t \subset \mathbb{C}^n$, parameterized by an open interval Δ, is called *uniformly* i-convex if there exists $c > 0$ such that $\mu(\Sigma_t) > c$ for all $t \in \Delta$.

Example 2.11. Let $L \subset \mathbb{C}^n$ be a compact totally real submanifold. Set $\Sigma_t = \{x \in \mathbb{C}^n; \mathrm{dist}_L(x) = c\}$. Then for a sufficiaently small $\varepsilon > 0$ the family $\{\Sigma_t\}|_{t\in(0,\varepsilon)}$ is uniformly i-convex.

Example 2.11 and Lemma 2.10 imply

Corollary 2.12. *Let $L \subset \mathbb{C}^n$ and $\Sigma_t = \{x \in \mathbb{C}^n; \mathrm{dist}_L(x) = c\}$ be as in Example 2.11. Then for J which is sufficiently C^2-close to i on $\mathcal{O}p\,L$ the hypersurfaces Σ_t are J-convex for t close to 0.*

2.5 Examples of J-convex functions and hypersurfaces

Quadratic functions. For the function

$$\phi(z) := \sum_{k=1}^{n} \lambda_k x_k^2 + \mu_k y_k^2$$

on $\mathbb{C}^n$ we have

$$\omega_\phi = 2\sum_k (\lambda_k + \mu_k) dx_k \wedge dy_k.$$

So ϕ is J-convex if and only if

$$\lambda_k + \mu_k > 0 \text{ for all } k = 1, \dots, n. \tag{2.7}$$

Consider a level set Σ of ϕ in the case $\lambda_k > 0$ and $\mu_k < 0$. The intersection of Σ with any plane in the x-coordinates is a curve with positive curvature determined by the λ_k. The intersection with a plane in the y-coordinates has negative curvature determined by the y_k. The condition (2.7) assures that along any complex line these curvatures add up to a positive mean curvature.

Totally real submanifolds. A submanifold L of an almost complex manifold (V, J) is called *totally real* if it has no complex tangent lines, i.e. $J(TL) \cap TL = \{0\}$ at every point. This condition implies $\dim_{\mathbb{R}} L \leq \dim_{\mathbb{C}} V$. For example, the linear subspaces $\mathbb{R}^k := \{(x_1, \dots, x_k, 0, \dots, 0) \mid x_i \in \mathbb{R}\} \subset \mathbb{C}^n$ are totally real for all $k = 0, \dots, n$. If we have an Hermitian metric on (V, J) we can define the distance function $dist_L : V \to \mathbb{R}$,

$$dist_L(x) := \inf\{dist(x, y) \mid y \in L\}.$$

Proposition 2.13. *Let L be a totally real submanifold of an almost complex manifold (V, J). Then the squared distance function $dist_L^2$ with respect to any Hermitian metric on V is J-convex in a neighborhood of L. In particular, if L is compact, then $\{\mathrm{dist}_L \leq \varepsilon\}$ is a tubular neighborhood of L with J-convex boundary for each sufficiently small $\varepsilon > 0$.*

???

Proof. Let $Q : T_pV \to \mathbb{R}$ be the Hessian quadratic form of $dist_L^2$ at a point $p \in L$. Its value $Q(z)$ equals the squared distance of $z \in T_pV$ from the linear subspace $T_pL \subset T_pV$. Choose an orthonormal basis $e_1, Je_1, \dots, e_n, Je_n$ of T_pV such that $e_1, \dots, e_k$ is a basis of T_pL. In this basis,

$$Q\Big(\sum_{i=1}^{n}(x_i e_i + y_i J e_i)\Big) = \sum_{j>k} x_j^2 + \sum_{i=1}^{n} y_i^2,$$

which is J-convex by Example 1. So $dist_L^2$ is J-convex on L and therefore by continuity in a neighborhood of L. $\square$

Remark 2.14. (1) The last statement of Proposition 2.13 extends to the non-compact case: *Every properly embedded totally real submanifold of an almost complex manifold has an arbitrarily small tubular neighborhoods with J-convex boundary.* Of course the radius of the neighborhood may go to 0 at infinity

(2) Proposition 2.13 can be generalized as follows. Let W be *any* compact submanifold of an almost complex manifold (V, J), and suppose that a function $\phi : V \to \mathbb{R}$ satisfies the following J-convexity condition on W: *The form* $-dd^{\mathbb{C}}\phi$ *is positive on any complex line tangent to* W. Note that this condition is vacuously satisfied for any function on a totally real manifold. Choose any Hermitian metric on V. Then the function $\phi + \lambda dist_W^2$ is J-convex in a neighborhood of W for a sufficiently large positive λ. If W is non-compact (but properly embedded), then we need to choose as λ not a constant but a positive function $\lambda : W \to \mathbb{R}$ which may grow at infinity.

Holomorphic line bundles. A complex line bundle $\pi : E \to V$ over a complex manifold V is called *holomorphic line bundle* if the total space E is a complex manifold and the bundle possesses holomorphic local trivializations. For a Hermitian metric on $E \to V$ consider the hypersurface

$$\Sigma := \{z \in E \;\Big|\; |z| = 1\} \subset E.$$

Complex multiplication $U(1) \times \Sigma \to \Sigma$, $(e^{i\theta}, z) \mapsto e^{i\theta} \cdot z$ provides Σ with the structure of a $U(1)$ principal bundle over V. Let α be the 1-form on Σ defined by

$$\alpha\Big(\frac{d}{d\theta}\Big|_0 e^{i\theta} \cdot z\Big) = 1, \qquad \alpha|_\xi = 0,$$

where ξ is the distribution of maximal complex subspaces of $T\Sigma$. The imaginary valued 1-form $i\alpha$ defines the unique connection on the $U(1)$ principal bundle $\Sigma \to V$ for which all horizontal subspaces are J-invariant. Its curvature is the imaginary valued (1,1)-form Ω on V satisfying $\pi^*\Omega = d(i\alpha)$. On the other hand, α is a defining 1-form for the hyperplane distribution $\xi \subset T\Sigma$, so $\omega_\Sigma = d\alpha|_{\xi\times\xi}$ defines the Levi form of Σ. Thus ω_Σ and the curvature form Ω are related by the equation

$$i\omega_\Sigma(X, Y) = \Omega(\pi_*X, \pi_*Y) \tag{2.8}$$

for $X, Y \in \xi$. The line bundle $E \to V$ is called *positive* (*resp. negative*) if it admits a Hermitian metric such that the corresponding curvature form Ω satisfies

$$\frac{i}{2\pi}\Omega(X, JX) > 0 \;\big(\text{resp. } < 0\big)$$

for all $0 \neq X \in TV$. Since π is holomorphic, equation (2.8) implies

Proposition 2.15. *Let* $E \to V$ *be a holomorphic line bundle over a complex manifold. There exists a Hermitian metric on* $E \to V$ *such that the hypersurface* $\{z \in E \;\big|\; |z| = 1\}$ *is J-convex if and only if* E *is a negative line bundle.*

If V is compact, then the closed 2-form $\frac{i}{2\pi}\Omega$ represents the first Chern class $c_1(E)$,

$$\left[\frac{i}{2\pi}\Omega\right] = c_1(E)$$

(see [41], Chapter 12). Conversely, for every closed (1,1)-form $\frac{i}{2\pi}\Omega$ representing $c_1(E)$, Ω is the curvature of some Hermitian connection $i\alpha$ as above ([30], Chapter 1, Section 2). So a line bundle over V is positive/negative if and only if its first Chern class can be represented by a positive/negative (1,1)-form. If V has complex dimension 1 we get a very simple criterion.

Corollary 2.16. *Let V be a compact Riemann surface and $[V] \in H_2(V, \mathbb{R})$ its fundamental class. A holomorphic line bundle $E \to V$ admits a Hermitian metric such that the hypersurface $\{z \in E \mid |z| = 1\}$ is J-convex if and only if $c_1(E) \cdot [V] < 0$.*

For example, the corollary applies to the tangent bundle of a Riemann surface of genus ≥ 2.

Proof. Since $H^2(V, \mathbb{R})$ is 1-dimensional, $c_1(E) \cdot [V] < 0$ if and only if $c_1(E)$ can be represented by a negatively oriented area form. But any negatively oriented area form on V is a negative (1,1)-form. □

Remark 2.17. If $E \to V$ is just a complex line bundle (i.e. not holomorphic), then the total space E does not carry a natural almost complex structure. Such a structure can be obtained by choosing a Hermitian connection on $E \to V$ and taking the horizontal spaces as complex subspaces with the complex multiplication induced from V via the projection. If we fix an almost complex structure on the total space E such that the projection π is J-holomorphic, then Proposition 2.13 remains valid.

2.6 J-convex functions and hypersurfaces in $\mathbb{C}^n$

Let a hypersurface $\Sigma \subset \mathbb{C}^n$ is given by an implicit equation $\Psi(x) = 0$ with $\nabla\Psi = \left(\frac{\partial\Psi}{\partial\bar{z}_1}, \dots, \frac{\partial\Psi}{\partial\bar{z}_1}\right) \neq 0$ on Σ. Let $H_p^{\mathbb{C}}(T) := \sum\limits_{i,j=1}^{N} \frac{\partial\Psi}{\partial z_i \partial\bar{z}_j}(p) T_i \bar{T}_j$, $T = (T_1, \dots, T_n)$, $p \in \mathbb{C}^n$, be the complex Hessian form of Ψ. We begin with the following expression of the normalized Levi form $\mathbb{L}_\Sigma$.

Lemma 2.18. *The normalized Levi form of Σ can be given by an expression*

$$\mathbb{L}_\Sigma(T) = \frac{H_p^{\mathbb{C}}(T)}{|\nabla H^{\mathbb{C}}(p)|},\ p \in \Sigma,\ T \in T_p\Sigma. \tag{2.9}$$

Proof. The second fundamental form II_Σ of Σ can be written as

$$II_\Sigma(T, T) = \frac{H_p(T)}{|\nabla H(p)|},\ p \in \Sigma,\ T \in T_p\Sigma,$$

Reference? or add a computation

where $H_p(T)$ is the real Hessian form of Ψ. By definition,

$$\mathbb{L}_\Sigma(T) = \frac{1}{2}\left(II_\Sigma(T,T) + II_\Sigma(iT,iT)\right)), \quad T \in \xi.$$

On the other hand, $H_p^{\mathbb{C}}(T) = \frac{1}{2}(H_p(T) + H_p(iT))$, and (2.9) follows. □

Corollary 2.19. *Let $\Sigma \subset \Omega \subset \mathbb{C}^n$ be a J-convex compact hypersurface and $f : \Omega \to \widetilde{\Omega} \subset \mathbb{C}^n$ a biholomorphism. Denote $\widetilde{\Sigma} := f(\Sigma)$. Then there exists a positive constant c which depends only on the C^2-norm of f along Σ such that $\mu(\widetilde{\Sigma}) > c\mu(\Sigma)$.*

Consider the case $n = 2$ and denote coordinates (ζ, w) instead of (z_1, z_2). We have $\dim_{\mathbb{C}} \xi = 1$ and thus $\mathbb{L}_\Sigma(T)$ is independent of $T \in \S_1\xi = \{T \in \xi, |T| = 1$. The complex line ξ is generated by the vector $T = \frac{1}{|\nabla\Psi|}\left(-\frac{\partial\Psi}{\partial w}, \frac{\partial\Psi}{\partial\zeta}\right)$. Hence,

$$\mathbb{L}_0 = \mathbb{L}(T) = \frac{1}{|\nabla\Psi|^3}\Big(\Psi_{\zeta\bar\zeta}|\Psi_w|^2 - 2\mathrm{Re}\,(\Psi_{\zeta\bar w}\Psi_w\Psi_{\bar\zeta}) + \Psi_{w\bar w}|\Psi_\zeta|^2\Big).$$

Thus we get the following criterion for an i-convexity of a hypersurface $\Sigma = \{\Psi = 0\} \subset \mathbb{C}^2\}$:

Criterion 2.20. *A hypersurface $\Sigma = \{\Psi = 0\ \} \subset \mathbb{C}^2\}$ is i-convex if and only if*

$$\Psi_{\zeta\bar\zeta}|\Psi_w|^2 - 2\mathrm{Re}\,(\Psi_{\zeta\bar w}\Psi_w\Psi_{\bar\zeta}) + \Psi_{w\bar w}|\Psi_\zeta|^2 > 0.$$

Let $\zeta = s + it$, $w = u + iv$. Suppose that a hypersurface $\Sigma \subset \mathbb{C}^2$ is given as a graph

$$\Psi(\zeta, w) := \psi(\zeta, u) - v = 0.$$

Then

$$\begin{gathered}
2\Psi_{\bar\zeta} = \psi_s + i\psi_t, \qquad 4\Psi_{\zeta\bar\zeta} = \psi_{ss} + \psi_{tt}, \qquad 4|\Psi_\zeta|^2 = \psi_s^2 + \psi_t^2, \\
2\Psi_w = \psi_u + i, \qquad 4\Psi_{w\bar w} = \psi_{uu}, \qquad 4|\Psi_w|^2 = 1 + \psi_u^2, \\
4\Psi_{\zeta\bar w} = \psi_{su} - i\psi_{tu}, \qquad 4\Psi_w\Psi_{\bar\zeta} = (\psi_u\psi_s - \psi_t) + i(\psi_s + \psi_u\psi_t), \\
16Re\Psi_{\zeta\bar w}\Psi_w\Psi_{\bar\zeta} = \psi_{su}(\psi_u\psi_s - \psi_t) + \psi_{tu}(\psi_s + \psi_u\psi_t),
\end{gathered}$$

thus we have proved

Lemma 2.21. *The normalized Levi form of the hypersurface $\Sigma = \{v = \psi(s,t,u)\} \subset \mathbb{C}^2$, cooriented by the gradient of the function $\psi(s,t,u) - v$, is given by*

$$\begin{aligned}
\mathbb{L}_0 = \frac{1}{(1+\psi_s^2+\psi_t^2)^{\frac{3}{2}}}\big(&(\psi_{ss}+\psi_{tt})(1+\psi_u^2) + \psi_{uu}(\psi_s^2+\psi_t^2) \\
&+ 2\psi_{su}(\psi_t - \psi_u\psi_s) - 2\psi_{tu}(\psi_s + \psi_u\psi_t)\big). \qquad (2.10)
\end{aligned}$$

In particular, the surface Σ is i-convex iff

$$(\psi_{ss}+\psi_{tt})(1+\psi_u^2) + \psi_{uu}(\psi_s^2+\psi_t^2) + 2\psi_{su}(\psi_t - \psi_u\psi_s) - 2\psi_{tu}(\psi_s + \psi_u\psi_t) > 0.$$

Chapter 3

Smoothing

3.1 J-convexity and plurisubharmonicity

A C^2-function $\phi : U \to \mathbb{R}$ on an open domain $U \subset \mathbb{C}$ is i-convex if and only if it is *(strictly) subharmonic*, i.e.,

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 4\frac{\partial\phi}{\partial z\partial\bar{z}} > 0.$$

Note. By "subharmonic" we will always mean "strictly subharmonic". Non-strict subharmonicity will be referred to as "weak subharmonicity". The same applies to plurisubharmonicity discussed below.

A continuous function $\phi : U \to \mathbb{R}$ is called (strictly) subharmonic if it satisfies

$$\Delta\phi \geq h$$

for a positive continuous function $h : U \to \mathbb{R}$, where the Laplacian and the inequality are understood in the distributional sense, i.e.,

$$\int_U \phi\,\Delta\delta\; dx\,dy \geq \int_U h\,\delta\; dx\,dy \tag{3.1}$$

for any nonnegative smooth function $\delta : U \to \mathbb{R}$ with compact support. Note that if ϕ is a C^2-function satisfying (3.1), then integration by parts and choice of a sequence of functions δ_n converging to the Dirac measure of a point $p \in U$ shows $\Delta\phi(p) \geq h(p)$, so the two definitions agree for C^2-functions.

If $z = x + iy \to w = u + iv$ is a biholomorphic change of coordinates on U, then

$$\Delta_z\delta\; dx\,dy = 2i\frac{\partial^2\delta}{\partial z\partial\bar{z}}dz \wedge d\bar{z} = -dd^{\mathbb{C}}\delta = \Delta_w\delta\; du\,dv, \tag{3.2}$$

so inequality (3.1) transforms into

$$\int_U \phi(w)\Delta\delta(w)du\,dv \geq \int_U h(w)\delta(w)\Big|\frac{dz}{dw}\Big|^2 du\,dv.$$

This shows that subharmonicity is invariant under biholomorphic coordinate changes and therefore can be defined for continuous functions on Riemann surfaces. The following lemma gives a useful criterion for subharmonicity of continuous functions.

Lemma 3.1. *A continuous function $\phi : U \to \mathbb{R}$ on a domain $U \subset \mathbb{C}$ is subharmonic if and only if there exists a positive continuous function $h : U \to \mathbb{R}$ such that*

$$\phi(z) + h(z)r^2 \leq \frac{1}{2\pi}\int_0^{2\pi} \phi(z + re^{i\theta})d\theta \tag{3.3}$$

for all $z \in U$ and $r > 0$ for which the disk of radius r around z in contained in U.

Proof. In a neighborhood of a point $z_0 \in U$ inequality (3.3) holds with h replaced by some constant $\lambda > 0$. Consider the function

$$\psi(z) := \phi(z) - \lambda|z - z_0|^2.$$

For $r > 0$ sufficiently small, (3.3) is equivalent to

$$\psi(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} \psi(z_0 + re^{i\theta})d\theta.$$

By a standard result (see e.g. [39]), this inequality is equivalent to $\Delta\psi(z_0) \geq 0$ in the distributional sense, and therefore to $\Delta\phi(z_0) \geq 2\lambda$.

□

Remark 3.2. The preceding proof shows: If ϕ in Lemma 3.1 is C^2, then inequality (3.3) holds with $h(z) := \frac{1}{4}\min_{D(z)}\Delta\phi$, where D is the maximal disk around z contained in U.

Now let (V, J) be an almost complex manifold. A *(nonsingular) J-holomorphic curve* is a 1-dimensional complex submanifold of (V, J). Note that the restriction of the almost complex structure J to a J-holomorphic curve is always integrable.

Lemma 3.3. *A C^2-function ϕ on an almost complex manifold (V, J) is J-convex if and only if its restriction to every J-holomorphic curve is subharmonic.*

Proof. By definition, ϕ is J-convex iff $-dd^{\mathbb{C}}\phi(X, JX) > 0$ for all $0 \neq X \in T_xV$, $x \in V$. Now for every such $X \neq 0$ there exists a J-holomorphic curve $C \subset V$ passing through x with $T_xC = span_{\mathbb{R}}\{X, JX\}$ ([52]). By formula (3.2) above, $-dd^{\mathbb{C}}\phi(X, JX) > 0$ precisely if $\phi|_C$ is subharmonic in x. □

Remark 3.4. In the proof we have used the fact that the differential operator $dd^{\mathbb{C}}$ commutes with restrictions to complex submanifolds. This is true because the exterior derivative and the composition with J both commute with restrictions to complex submanifolds.

Remark 3.5. Lemma 3.3 provides another proof of Corollary 12.13, i.e. that *non-degenerate critical points of a J-convex function have Morse indices $\leq n$.* Indeed, p be a critical point of a J-convex function ϕ. Suppose $\mathrm{ind}(p) > n$. Then there exists a subspace $W \subset T_pV$ of dimension $> n$ on which the Hessian of ϕ is negative definite. Since $W \cap JW \neq \{0\}$, W contains a complex line L. Let C be a J-holomorphic curve through p tangent to L. Then $\phi|_C$ attains a local maximum at p. But this contradicts the maximum principle because $\phi|_C$ is subharmonic by Lemma 3.3.

In view of Lemma 3.3 we can speak about *continuous J-convex functions* on almost complex manifolds as functions whose restrictions to all J-holomorphic curves are subharmonic. Such functions are also called *(strictly) plurisubharmonic*. For functions on $\mathbb{C}^n$, Lemma 3.1 and the proof of Lemma 3.3 show

Lemma 3.6. *A continuous function $\phi : \mathbb{C}^n \supset U \to \mathbb{R}$ is i-convex if and only if its restriction to every complex line is subharmonic. This means that there exists a positive continuous function $h : U \to \mathbb{R}$ such that*

$$\phi(z) + h(z)|w|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(z + we^{i\theta}) d\theta \tag{3.4}$$

for all $z \in U$ and $w \in \mathbb{C}^n$ for which the disk of radius $|w|$ around z in contained in U.

The following lemma follows from equation (3.1) via integration by parts.

Lemma 3.7. *If ϕ is a J-convex function on an almost complex manifold (V, J), then $\phi + \psi$ is J-convex for every sufficiently C^2-small C^2-function $\psi : V \to \mathbb{R}$.*

Our interest in continuous J-convex functions is motivated by the following

Lemma 3.8. *If ϕ and ψ are continuous J-convex functions on an almost complex manifold (V, J), then $\max(\phi, \psi)$ is J-convex. More generally, let $(\phi_\lambda)_{\lambda \in \Lambda}$ be a continuous family of continuous functions, parameterized by a compact metric space Λ, that are uniformly J-convex in the sense that on every J-holomorphic disk $U \subset V$ condition (3.3) holds for all ϕ_λ with functions h_λ depending continuously on λ. Then $\max_{\lambda \in \Lambda} \phi_\lambda$ is a continuous J-convex function.*

Proof. Continuity of $\max_{\lambda \in \Lambda} \phi_\lambda$ is an easy exercise. For J-convexity we use the criterion from Lemma 3.1. Let $U \subset V$ be a J-holomorphic disk and choose a local coordinate z on U. By hypothesis, condition (3.3) holds for all ϕ_λ with functions h_λ depending continuously on λ. Note that $h(z) := \min_{\lambda \in \Lambda} h_\lambda$ defines a positive continuous function on U. Set $\phi := \max_{\lambda \in \Lambda} \phi_\lambda$. At any point $z \in U$ we have $\phi = \phi_\lambda$ for some $\lambda \in \Lambda$ (depending on z). Now the lemma follows from

$$\begin{aligned}\phi(z) + h(z)r^2 \leq \phi_\lambda(z) + h_\lambda(z)r^2 &\leq \frac{1}{2\pi} \int \phi_\lambda(z + re^{i\theta}) d\theta \\ &\leq \frac{1}{2\pi} \int \phi(z + re^{i\theta}) d\theta.\end{aligned}$$

□

Remark 3.9. For example, the hypotheses of Lemma 3.8 are satisfied if all the J-convex functions ϕ_λ are C^2 and their first two derivatives depend continuously on λ. This follows immediately from the remark after Lemma 3.1.

3.2 Smoothing of J-convex functions

For integrable J, continuous J-convex functions can be approximated by smooth ones. The following proposition was proved by Richberg [54]. We give below a proof following [21].

Proposition 3.10. *Let ϕ be a continuous J-convex function on a (integrable) complex manifold (V, J). Then for every positive function $h : V \to \mathbb{R}_+$ there exists a smooth J-convex function $\psi : V \to \mathbb{R}$ such that $|\phi(x) - \psi(x)| < h(x)$ for all $x \in V$. If ϕ is already smooth on a neighborhood of a compact subset K, then we can achieve $\psi = \phi$ on K.*

Remark 3.11. A continuous weakly J-convex function cannot in general be approximated by smooth weakly J-convex functions, see [21] for a counterexample. We do not know whether the proposition remains true for almost complex manifolds.

The proof is based on an explicit smoothing procedure for functions on $\mathbb{R}^m$. Pick a smooth nonnegative function $\rho : \mathbb{C}^m \to \mathbb{R}$ with support in the unit ball and $\int_{\mathbb{R}^m} \rho = 1$. For $\delta > 0$ set $\rho_\delta(x) := \delta^{-m}\rho(x/\delta)$. Let $U \subset \mathbb{R}^m$ be an open subset and set

$$U_\delta := \{x \in U \mid \bar{B}_\delta(x) \subset U\}$$

For a continuous function $\phi : \mathbb{R}^m \supset U \to \mathbb{R}$ define the *mollified function* $\phi_\delta : U_\delta \to \mathbb{R}$,

$$\phi_\delta(x) := \int_{\mathbb{C}^n} \phi(x - y)\rho_\delta(y) d^{2n}y = \int_{\mathbb{C}^n} \phi(y)\rho_\delta(x - y) d^{2n}y. \tag{3.5}$$

The last expression shows that the functions ϕ_δ are smooth for every $\delta > 0$. The first expression shows that if ϕ is of class C^k for some $k \geq 0$, then $\phi_\delta \to \phi$ as $\delta \to 0$ uniformly on compact subsets of U.

Proposition 3.10 is an immediate consequence of the following lemma, via induction over a countable coordinate covering.

Lemma 3.12. *Let ϕ be a continuous J-convex function on a complex manifold (V, J). Let $A, B \subset V$ be compact subsets such that ϕ is smooth on a neighborhood of A and B is contained in a holomorphic coordinate neighborhood. Then for every $\varepsilon > 0$ and every neighborhood W of $A \cup B$ there exists a continuous J-convex function $\psi : V \to \mathbb{R}$ with the following properties.*

- *ψ is smooth on a neighborhood of $A \cup B$;*
- *$|\psi(x) - \phi(x)| < \varepsilon$ for all $x \in V$;*

- *$\psi = \phi$ on A and outside W.*

Proof. The proof follows [21]. First suppose that ϕ is i-convex on an open set $U \subset \mathbb{C}^n$. By Lemma 3.6, there exists a positive continuous function $h : U \to \mathbb{R}$ such that (3.4) holds for all $z \in U_{2\delta}$ and $w \in \mathbb{C}^n$ with $|w| \leq \delta$. Hence the mollified function ϕ_δ satisfies

$$\begin{aligned}\phi_\delta(x) + h_\delta(x)|w|^2 &= \int_{\mathbb{C}^n} \Big(\phi(x-y) + \delta(x-y)|w|^2\Big)\rho_\delta(y)d^{2n}y \\ &\leq \int_{\mathbb{C}^n}\int_0^{2\pi} \phi(x-y+we^{i\theta})d\theta\rho_\delta(y)d^{2n}y \\ &= \int_0^{2\pi} \phi_\delta(x+we^{i\theta})d\theta,\end{aligned}$$

so ϕ_δ is i-convex on $U_{2\delta}$.

Now let $\phi : V \to \mathbb{R}$ be as in the proposition. Pick a holomorphic coordinate neighborhood U and compact neighborhoods $A' \subset W$ of A and $B' \subset B'' \subset W \cap U$ of B with $A \subset \operatorname{int} A' \subset A' \subset W$, such that ϕ is smooth on A'. By the preceding discussion, there exists a smooth J-convex function $\phi_\delta : B'' \to \mathbb{R}$ with $|\phi_\delta(x) - \phi(x)| < \varepsilon/2$ for all $x \in B''$. Pick smooth cutoff functions $g, h : V \to [0,1]$ such that $g = 1$ on A, $g = 0$ outside A', $h = 1$ on B', and $h = 0$ outside B''. Define a continuous function $\tilde{\phi} : V \to \mathbb{R}$,

$$\tilde{\phi} := \phi + (1-g)h(\phi_\delta - \phi).$$

The function $\tilde{\phi}$ is smooth on $A' \cup B'$, $|\tilde{\phi}(x) - \phi(x)| < \varepsilon/2$ for all $x \in V$, $\tilde{\phi} = \phi_\delta$ on $B' \setminus A'$, and $\tilde{\phi} = \phi$ on A and outside B''. Since ϕ is C^2 on $A' \cap B''$, the function $(1-g)h(\phi_\delta - \phi)$ becomes arbitrarily C^2-small on this set for δ small. Hence by Lemma 3.7, $\tilde{\phi}$ is J-convex on $A' \cap B''$ for δ sufficiently small. So we can make $\tilde{\phi}$ J-convex on $A' \cup B'$. However, $\tilde{\phi}$ need not be J-convex on $B'' \setminus (A' \cup B')$.

Pick a compact neighborhood $W' \subset W$ of $A' \cup B''$. Without loss of generality we may assume that ε was arbitrarily small. Then by Lemma 3.7 there exists a continuous J-convex function $\tilde{\psi} : V \to \mathbb{R}$ (which differs from ϕ by a C^2-small function) satisfying $\tilde{\psi} = \phi - \varepsilon$ on $A \cup B$, $\tilde{\psi} = \phi + \varepsilon$ on $W' \setminus (A' \cup B')$, and $\tilde{\psi} = \phi$ outside W. Now the function $\psi := \max(\tilde{\phi}, \tilde{\psi})$ has the desired properties. □

Remark 3.13. The proof of Lemma 3.12 shows the following additional properties in Proposition 3.10:

(1) If ϕ_λ is a continuous family of J-convex functions depending on a parameter λ in a compact space Λ, then the ϕ_λ can be uniformly approximated by a continuous family of smooth J-convex functions ψ_λ.

(2) If $\phi_0 \leq \phi_1$ then the smoothed functions also satisfy $\psi_0 \leq \psi_1$. This holds because the proof only uses mollification $\phi \mapsto \phi_\delta$, interpolation and taking the maximum of two functions, all of which are monotone operations.

Lemma 3.8, the remark after it and Proposition 3.10 imply

Corollary 3.14. *The maximum of two smooth J-convex functions ϕ, ψ on a complex manifold (V,J) can be C^0-approximated by smooth J-convex functions. If* $\max(\phi,\psi)$ *is smooth on a neighborhood of a compact subset K, then we can choose the approximating sequence to be equal to* $\max(\phi,\psi)$ *on K.*

More generally, let $(\phi_\lambda)_{\lambda\in\Lambda}$ be a continuous family of J-convex C^2-functions whose first two derivatives depend continuously on λ in a compact metric space Λ. Then $\max_{\lambda\in\Lambda}\phi_\lambda$ *can be C^0-approximated by smooth J-convex functions. If* $\max_{\lambda\in\Lambda}\phi_\lambda$ *is smooth on a neighborhood of a compact subset K, then we can choose the approximating sequence to be equal to* $\max_{\lambda\in\Lambda}\phi_\lambda$ *on K.*

Finally, we show that we can arbitrarily prescribe a J-convex function near a totally real submanifold.

Proposition 3.15. *Let L be a totally real submanifold of a complex manifold (V,J) and $K\subset L$ a compact subset. Suppose that two smooth J-convex functions ϕ, ψ coincide along L together with their differentials, i.e. $\phi(x)=\psi(x)$ and $d\phi(x)=d\psi(x)$ for all $x\in L$. Then, given any neighborhood U of K in V, there exists a J-convex function ϑ which coincides with ϕ outside U and with ψ in a smaller neighborhood $U'\subset U$ of K. Moreover, ϑ can be chosen arbitrarily C^1-close to ϕ and such that ϑ agrees with ϕ together with its differential along L.*

The proof uses the following simple lemma. Consider an almost complex manifold (V,J) equipped with a Hermitian metric. To a smooth function $\phi:V\to\mathbb{R}$ we associate its *modulus of J-convexity* $m_\phi:V\to\mathbb{R}$,

$$m_\phi(x):=\min\{-dd^{\mathbb{C}}\phi(v,Jv)\mid v\in T_xV,\ |v|=1\}.$$

Thus ϕ is J-convex iff $m_\phi>0$.

Lemma 3.16. *Let $\phi,\psi,\beta:V\to\mathbb{R}$ be smooth functions on an almost complex manifold (V,J) such that*

$$|\phi(x)-\psi(x)|\,|dd^{\mathbb{C}}_x\beta|+2|d_x\beta|\,|d_x(\phi-\psi)|<\min\bigl(m_\phi(x),m_\psi(x)\bigr)$$

for all $x\in V$ (with respect to some Hermitian metric). Then $(1-\beta)\phi+\beta\psi$ is J-convex.

Proof. Adding up

$$dd^{\mathbb{C}}(\beta\psi)=\beta\,dd^{\mathbb{C}}\psi+d\beta\wedge d^{\mathbb{C}}\psi+d\psi\wedge d^{\mathbb{C}}\beta+\psi\,dd^{\mathbb{C}}\beta$$

and the corresponding equation for $(1-\beta)\phi$ at any point $x\in V$, we find

$$\begin{aligned}-dd^{\mathbb{C}}\bigl((1-\beta)\phi+\beta\psi\bigr)&=-(1-\beta)\,dd^{\mathbb{C}}\phi-\beta dd^{\mathbb{C}}\psi+d\beta\wedge d^{\mathbb{C}}(\phi-\psi)\\&\quad+d(\phi-\psi)\wedge d^{\mathbb{C}}\beta+(\phi-\psi)dd^{\mathbb{C}}\beta\\&\geq\min(m_\phi,m_\psi)-2|d\beta|\,|d(\phi-\psi)|-|\,\phi-\psi|\,|dd^{\mathbb{C}}\beta|\\&>0.\end{aligned}$$

□

Proof of Proposition 3.15. Fix a compact neighborhood $\tilde{K} \subset L \cap U$ of K in L. Pick a Hermitian metric on (V, J) and consider the function dist_L^2, square of the distance to L, defined on a tubular neighborhood of L. According to Proposition 2.13, this function is J-convex. Hence

$$\phi_\lambda := \phi + \lambda \mathrm{dist}_L^2$$

is J-convex for any $\lambda \geq 0$. Since ϕ and ψ agree up to first order along L, there exists a $\lambda > 0$ and a compact neighborhood $W \subset U$ of $\tilde{K}$ such that

$$\phi_\lambda > \psi \text{ on } W \setminus L.$$

For any $\varepsilon > 0$ we can find a $\delta < \varepsilon$ and a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\alpha(r) = r \text{ for } r \in [0, \delta], \quad \alpha(r) = 0 \text{ for } r \geq \varepsilon, \quad -\frac{3\delta}{\varepsilon} \leq \alpha' \leq 1, \quad |\alpha''| \leq \frac{3}{\varepsilon}.$$

Set

$$\tilde{\phi} := \phi + \lambda\alpha(\mathrm{dist}_L^2).$$

Then $\tilde{\phi}$ coincides with ϕ on $W \setminus U_\epsilon$ and with ϕ_λ on $W \cap U_\delta$, where $U_\varepsilon := \{\mathrm{dist}_L < \varepsilon\}$ denotes the ε-neighborhood of L. Let us show that $\tilde{\phi}$ is J-convex. Indeed,

$$dd^{\mathbb{C}}\tilde{\phi} = dd^{\mathbb{C}}\phi + \lambda\alpha'' d(\mathrm{dist}_L^2) \wedge d^{\mathbb{C}}(\mathrm{dist}_L^2) + \lambda\alpha' dd^{\mathbb{C}}(\mathrm{dist}_L^2).$$

On $W \cap U_\varepsilon$ we have $|d(\mathrm{dist}_L^2)| \leq C\varepsilon$, where the constant C depends only on the geometry of $L \cap W$. Since $d(\mathrm{dist}_L^2) \wedge d^{\mathbb{C}}(\mathrm{dist}_L^2)$ is a quadratic function of $d(\mathrm{dist}_L^2)$, the second term on the right hand side can be estimated by

$$|\lambda\alpha'' d(\mathrm{dist}_L^2) \wedge d^{\mathbb{C}}(\mathrm{dist}_L^2)| \leq C_1\lambda \cdot \frac{1}{\varepsilon} \cdot \varepsilon^2$$

for some constant C_1. The third term on the right hand side is estimated by

$$\lambda\alpha' dd^{\mathbb{C}}(\mathrm{dist}_L^2) \geq -\lambda\frac{3\delta}{\varepsilon}|dd^{\mathbb{C}}(\mathrm{dist}_L^2)| \geq -\frac{C_2\lambda\delta}{\varepsilon}$$

for some constant C_2. Thus the modulus of J-convexity of $\tilde{\phi}$ satisfies

$$m_{\tilde{\phi}} \geq m_\phi - C_1\lambda\varepsilon - C_2\lambda\delta/\varepsilon.$$

So if $a := \min_W m_\phi > 0$, then $m_{\tilde{\phi}} \geq a/2 > 0$ on W whenever ε and δ/ε are sufficiently small.

Note that $\tilde{\phi}$ is arbitrarily C^1-close to ϕ for ε small. Fix a cutoff function β with support in W and equal to 1 on a neighborhood $W' \subset W$ of $\tilde{K}$. The function

$$\bar{\phi} := (1 - \beta)\phi + \beta\tilde{\phi}.$$

satisfies $\bar{\phi} = \phi$ outside W and on L, and $\bar{\phi} > \psi$ on $W' \setminus L$. Moreover, since the estimates $m_\phi \geq a$ and $m_{\tilde{\phi}} \geq a/2$ are independent of ε and δ, Lemma 3.16 implies that $\bar{\phi}$ is J-convex if ε and δ/ε are sufficiently small.

Next pick a cutoff function γ with support in a smaller neighborhood $W'' \subset W'$ of $\tilde{K}$ and equal to 1 near $\tilde{K}$. The function

$$\hat{\phi} := \bar{\phi} - \mu\gamma$$

is J-convex for $\mu > 0$ sufficiently small. Moreover, it satisfies

$$\hat{\phi} < \psi \ \text{near } \tilde{K}, \qquad \hat{\phi} > \psi \ \text{on } W' \setminus (W'' \cup L), \qquad \hat{\phi} = \phi \text{ outside } W, \qquad \hat{\phi} \leq \psi \text{ on } L.$$

So the function

$$\hat{\vartheta} := \begin{cases} \max(\psi, \hat{\phi}) & \text{on } W', \\ \hat{\phi} & \text{outside } W' \end{cases}.$$

coincides with ψ near $\tilde{K}$ and on L and with ϕ outside W. Let $\tilde{\vartheta}$ be the J-convex function obtained by smoothing $\hat{\vartheta}$ as described in Corollary 3.14, leaving it unchanged near $\tilde{K}$ and outside W. Then $\tilde{\vartheta}$ coincides with ϕ outside W and with ψ near $\tilde{K}$. Moreover, since $\hat{\phi}$ is C^1-close to ϕ by construction, $\tilde{\vartheta}$ is C^1-close to ϕ by Corollary 3.25.

So $\tilde{\vartheta}$ has all the desired properties except that, due to the smoothing procedure, it may not agree with ϕ on $L \setminus \tilde{K}$. To remedy this, fix a cutoff function ρ with support in U which equals 1 near K and 0 on $L \setminus \tilde{K}$ and set

$$\vartheta := (1 - \rho)\phi + \rho\tilde{\vartheta}.$$

By Lemma 3.16, ϑ is J-convex if we choose $\tilde{\vartheta}$ sufficiently C^1-close to ϕ. Since ψ agrees with ϕ together with their differentials along L, the same holds for ϑ and ϕ. So ϑ is the desired function. □

Remark 3.17. Note that if the function ϕ (and hence, ψ) is regular at the points of L then the construction of Proposition 3.15 can be performed without creating any new critical points. Indeed, the constructed function ϑ is C^1-close to ϕ. See Lemma 3.27 below for a similar statement when ϕ has critical points along L.

The corresponding result for J-convex hypersurfaces is

Corollary 3.18. *Let Σ, Σ' be J-convex hypersurfaces in a complex manifold (V, J) that are tangent to each other along a totally real submanifold L. Then for any compact subset $K \subset L$ and neighborhood U of K, there exists a J-convex hypersurface Σ'' that agrees with Σ outside U and with Σ' near K. Moreover, Σ'' can be chosen C^1-close to Σ and tangent to Σ along L.*

Proof. Pick smooth functions ϕ, ψ with regular level sets $\Sigma = \phi^{-1}(0)$ and $\Sigma' = \psi^{-1}(0)$ such that $d\phi = d\psi$ along L. By Lemma 2.4, after composing ϕ and ψ with the same convex function, we may assume that ϕ, ψ are J-convex on a neighborhood $W \subset U$ of K. Let $\vartheta : W \to \mathbb{R}$ be the J-convex function from Proposition 2.13 which coincides with ψ near K and with ϕ outside a compact subset $W' \subset W$. Since ϑ is C^1-close to ϕ, it has 0 as a regular value and $\Sigma'' := \vartheta^{-1}(0)$ is the desired J-convex hypersurface. □

We will finish this section with the following

Lemma 3.19. *Let $\phi_0, \phi_1 : V \to \mathbb{R}_+$ be two exhausting J-convex functions. Then there exist smooth functions $h_0, h_1 : \mathbb{R}_+ \to \mathbb{R}_+$ with $h_0', h_1' \to \infty$ and $h_0'', h_1'' > 0$, a completely exhausting function $\psi : V \to \mathbb{R}_+$, and a sequence of compact domains V^k, $k = 1, \dots$, with smooth boundaries $\Sigma^k = \partial V^k$, such that*

- $V^k \subset \operatorname{Int} V^{k+1}$ *for all* $k \geq 1$*;*
- $\bigcup_k V^k = V$*;*
- Σ^{2j-1} *are level sets of the function* ϕ_1 *and* Σ^{2j} *are level sets of the function* ϕ_0 *for* $j = 1, \dots$*;*
- $\psi = h_1 \circ \phi_1$ *on* $\mathcal{O}p\left(\bigcup_{j=1}^{\infty} \Sigma^{2j-1}\right)$ *and* $\psi = h_0 \circ \phi_0$ *on* $\mathcal{O}p\left(\bigcup_{j=1}^{\infty} \Sigma^{2j}\right)$.

Proof of Lemma 3.19. We will call a diffeomorphism $h : \mathbb{R}^+ \to \mathbb{R}^+$ an *admissible function* if $h'' \geq 0$ and $h' \to \infty$. Take any $c_1 > 0$, and denote $V^1 := \{\phi_1 \leq c_1\}, \Sigma^1 := \partial V^1$. There exists an admissible function g_1 such that $\phi_0|_{\Sigma^1} < d_1 = g_1(c_1)$. Set $\psi_0 := \phi_0$, $\psi_1 := g_1 \circ \phi_1$. Take any $c_2 > d_1$ and denote $V^2 := \{\psi_0 \leq c_2\}, \Sigma^2 := \partial V^2$. Then $V^1 \subset \operatorname{Int} V^2$. There exists an admissible function g_2 such that $g_2(x) = x$ for $x \in [0, d_1]$ and $\psi_1|_{\Sigma^2} < d_2 = g_2(c_2)$. Set $\psi_2 := g_2 \circ \psi_0$. Continuing this process we will take $c_3 > d_2$ and denote $V^3 := \{\psi_1 \leq c_3\}, \Sigma^3 := \partial V^3$. There exists an admissible function g_3 such that $g_3(x) = x, x \in [0, d_2]$ and $\psi_2|_{\Sigma^3} < d_3 = g_3(c_3)$. Set $\psi_3 := g_3 \circ \psi_1$, and so on. Continuing this process, we construct two admissible functions h_0, h_1 and a sequence of compact domains V^k, $k = 1, \dots$, such that

- $V^k \subset \operatorname{Int} V^{k+1}$ for all $k \geq 1$ and $\bigcup_k V^k = V$;
- ϕ_1 is constant on Σ^j for odd j, and ϕ_0 is constant on Σ^j for even j;
- $\psi_{\text{even}} = h_0 \circ \phi_0 = \lim\limits_{j \to \infty} \psi_{2j}$ and $\psi_{\text{odd}} = h_1 \circ \phi_1 = \lim\limits_{j \to \infty} \psi_{2j-1}$;
- $\psi_1|_{\Sigma^{2j-1}} > \psi_0|_{\Sigma^{2j-1}}$, $\psi_0|_{\Sigma^{2j}} > \psi_1|_{\Sigma^{2j}}$ for all $j \geq 1$.

Then smoothing the continuous plurisubharmonic function $\max(\psi_0, \psi_1)$ we get the required smooth J-convex function ψ. □

3.3 Critical points of J-convex functions

We wish to control the creation of new critical points under the construction of taking the maximum of two J-convex functions and then smoothing. This is based on the following trivial observation: A smooth function $\phi : M \to \mathbb{R}$ on a manifold has no critical points iff there exist a vector field X and a positive function h with $X \cdot \phi \geq h$. Multiplying by a nonnegative volume form Ω on M with compact support, we obtain

$$\int_M (X \cdot \phi)\Omega \geq \int_M h\Omega.$$

Using $(X \cdot \phi)\Omega + \phi L_X \Omega = L_X(\phi\Omega) = d(\phi i_X \Omega)$ and Stokes' theorem (assuming M is orientable over $\mathrm{supp}\Omega$), we can rewrite the left hand side as

$$\int_M (X \cdot \phi)\Omega = -\int_M \phi L_X \Omega.$$

So we have shown: A smooth function $\phi : M \to \mathbb{R}$ on a manifold has no critical points iff there exist a vector field X and a positive function h such that

$$-\int_M \phi L_X \Omega \geq \int_M h\Omega$$

for all nonnegative volume forms Ω on M with sufficiently small compact support. This criterion obviously still makes sense if ϕ is merely continuous. However, for technical reasons we will slightly modify it as follows.

We say that a continuous function $\phi : M \to \mathbb{R}$ satisfies $X \cdot \phi \geq h$ (in the distributional sense) if around each $p \in M$ there exists a coordinate chart $U \subset \mathbb{R}^m$ on which X corresponds to a constant vector field such that

$$-\int_U \phi L_X \Omega \geq h(p) \int_U \Omega$$

for all nonnegative volume forms Ω with support in U. Writing $\Omega = g(x)d^m x$ for a nonnegative function g, this is equivalent to

$$-\int_U \phi(x)(X \cdot g)(x) d^m x \geq h(p) \int_U g(x) d^m x. \tag{3.6}$$

This condition ensures that smoothing does not create new critical points:

Lemma 3.20. *If a continuous function $\phi : \mathbb{R}^m \supset U \to \mathbb{R}$ satisfies (3.6) for a constant vector field X and a constant $h = h(p) > 0$, then each mollified function ϕ_δ defined by equation (3.5) also satisfies (3.6) with the same X, h.*

Proof. Let g be a nonnegative test function with support in U and $0 < \delta < \mathrm{dist}(\mathrm{supp} g, \partial U)$. Let $y \in \mathbb{R}^m$ with $|y| < \delta$. Applying (3.6) to the function $x \mapsto g(x+y)$ and using translation invariance of X, h and the Lebesgue measure $dx := d^m x$, we find

$$\begin{aligned} -\int_U \phi(x-y) X \cdot g(x) dx &= -\int_U \phi(x) X \cdot g(x+y) dx \\ &\geq h \int_U g(x+y) dx = h \int_U g(x) dx. \end{aligned}$$

Multiplying by the nonnegative function ρ_δ and integrating yields

$$\begin{aligned} -\int_U \phi_\delta(x) X\cdot g(x)dx &= -\int_U\int_{B_\delta} \phi(x-y)\rho_\delta(y)X\cdot g(x)dy\,dx \\ &\geq h\int_U\int_{B_\delta} g(x)\rho_\delta(y)dy\,dx = h\int_U g(x)dx. \end{aligned}$$

□

The next proposition shows that the condition $X\cdot\phi\geq h$ is preserved under taking the maximum of functions.

Proposition 3.21. *Suppose the continuous functions $\phi,\psi : M\to\mathbb{R}$ satisfy $X\cdot\phi\geq h$, $X\cdot\psi\geq h$ with the same X,h. Then $X\cdot\max(\phi,\psi)\geq h$.*

More generally, suppose $(\phi_\lambda)_{\lambda\in\Lambda}$ is a continuous family of functions $\phi_\lambda : M\to\mathbb{R}$, parametrized by a compact separable metric space Λ, such that all ϕ_λ satisfy $X\cdot\phi_\lambda\geq h$ with the same X,h. Then $X\cdot\max_{\lambda\in\Lambda}\phi_\lambda\geq h$.

Proof. Let $U\subset\mathbb{R}^m$ be a coordinate chart and $X, h := h(p)$ be as in (3.6). After a rotation and rescaling, we may assume that $X=\frac{\partial}{\partial x_1}$. Suppose first that ϕ,ψ are smooth and 0 is a regular value of $\phi-\psi$. Then $\theta:=\max(\phi,\psi)$ is a continuous function which is smooth outside the smooth hypersurface $\Sigma := \{x\in U \mid \phi(x)=\psi(x)\}$. Define the function $\frac{\partial\theta}{\partial x_1}$ as $\frac{\partial\phi(x)}{\partial x_1}$ if $\phi(x)\geq\psi(x)$ and $\frac{\partial\psi(x)}{\partial x_1}$ otherwise. We claim that $\frac{\partial\theta}{\partial x_1}$ is the weak x_1-derivative of θ. Indeed, for any test function g supported in U we have (orienting Σ as the boundary of $\{\phi\geq\psi\}$)

$$\begin{aligned} \int_U \frac{\partial\theta}{\partial x_1} g\, d^m x &= \int_{\{\phi\geq\psi\}} \frac{\partial\phi}{\partial x_1} g\, d^m x + \int_{\{\phi<\psi\}} \frac{\partial\psi}{\partial x_1} g\, d^m x \\ &= \int_\Sigma \phi g\, dx_2\dots dx_m - \int_{\{\phi\geq\psi\}} \phi\frac{\partial g}{\partial x_1} d^m x \\ &\quad - \int_\Sigma \psi g\, dx_2\dots dx_m - \int_{\{\phi<\psi\}} \frac{\partial\psi}{\partial x_1} g\, d^m x \\ &= -\int_U \theta\frac{\partial g}{\partial x_1} d^m x, \end{aligned}$$

since $\phi=\psi$ on Σ. This proves the claim. By hypothesis we have $\frac{\partial\theta}{\partial x}\geq h$, so the conclusion of the lemma follows via

$$-\int_U \theta\frac{\partial g}{\partial x_1} d^m x = \int_U \frac{\partial\theta}{\partial x_1} g\, d^m x \geq h\int_U g\, d^m x.$$

Next let $\phi,\psi : U\to\mathbb{R}$ be continuous functions satisfying (3.6). By Lemma 3.20, there exist sequences ϕ_k,ψ_k of smooth functions, converging locally uniformly to ϕ,ψ, such that $X\cdot\phi_k\geq h$ and $X\cdot\psi_k\geq h$ for all k. Perturb the ϕ_k to smooth functions $\tilde\phi_k$ such that 0 is a regular value of $\tilde\phi_k-\psi_k$, $\tilde\phi_k\to\phi$ locally uniformly, and $X\cdot\tilde{\phi}_k \geq h-1/k$ for all k. By the smooth case above, the function $\max(\tilde{\phi}_k,\psi_k)$ satisfies

$$-\int_U \max(\tilde{\phi}_k,\psi_k)X\cdot g\,d^m x \geq (h-1/k)\int_U g\,d^m x$$

for any nonnegative test function g supported in U. Since $\max(\tilde{\phi}_k,\psi_k) \to \max(\phi,\psi)$ locally uniformly, the limit $k\to\infty$ yields the conclusion of the lemma for two functions ϕ,ψ.

Finally, let $(\phi_\lambda)_{\lambda\in\Lambda}$ be a continuous family as in the lemma. Pick a dense sequence $\lambda_1,\lambda_2,\dots$ in Λ. Set $\psi_k := \max\{\phi_{\lambda_1},\dots,\phi_{\lambda_k}\}$ and $\psi := \max_{\lambda\in\Lambda}\phi_\lambda$. By the lemma for two functions and induction, the functions ψ_k satisfy (3.6) with the same X,h for all k. Thus the lemma follows in the limit $k\to\infty$ if we can show locally uniform convergence $\psi_k\to\psi$.

We first prove pointwise convergence $\psi_k\to\psi$. So let $x\in U$. Then $\psi(x)=\phi_\lambda(x)$ for some $\lambda\in\Lambda$. Pick a sequence k_ℓ such that $\lambda_{k_\ell}\to\lambda$ as $\ell\to\infty$. Then $\phi_{\lambda_{k_\ell}}(x)\to\phi_\lambda(x)=\psi(x)$ as $\ell\to\infty$. Since $\phi_{\lambda_{k_\ell}}(x)\leq\psi_{k_\ell}(x)\leq\psi(x)$, this implies $\psi_{k_\ell}(x)\to\psi(x)$ as $\ell\to\infty$. Now the convergence $\psi_k(x)\to\psi(x)$ follows from monotonicity of the sequence $\psi_k(x)$.

So we have an increasing sequence of continuous functions ψ_k that converges pointwise to a continuous limit function ψ. By a simple argument this implies locally uniform convergence $\psi_k\to\psi$: Let $\varepsilon>0$ and $x\in U$ be given. By pointwise convergence there exists a k such that $\psi_k(x)\geq\psi(x)-\varepsilon$. By continuity of ϕ_k and ψ, there exists a $\delta>0$ such that $|\psi_k(y)-\psi_k(x)|<\varepsilon$ and $|\psi(y)-\psi(x)|<\varepsilon$ for all y with $|y-x|<\delta$. This implies $\psi_k(y)\geq\psi(y)-3\varepsilon$ for all y with $|y-x|<\delta$. In view of monotonicity, this establishes locally uniform convergence $\psi_k\to\psi$ and hence concludes the proof of the proposition. □

Finally, we show that J-convex functions can be smoothed without creating critical points.

Proposition 3.22. *Let $\phi : V\to\mathbb{R}$ be a continuous J-convex function on a complex manifold satisfying $X\cdot\phi\geq h$ for a vector field X and a positive function $h : V\to\mathbb{R}$. Then the J-convex smoothing $\psi : V\to\mathbb{R}$ in Proposition 3.10 can be constructed so that it satisfies $X\cdot\psi\geq\tilde{h}$ for any given function $\tilde{h}<h$.*

Proof. The function ψ is constructed from ϕ in Lemma 3.12 by repeated application of the following 3 constructions:

(1) Mollification $\phi\mapsto\phi_\delta$. This operation preserves the condition $X\cdot\phi\geq h$ by Lemma 3.20.

(2) Taking the maximum of two functions. This operation preserves the condition $X\cdot\phi\geq h$ by Proposition 3.21.

(3) Adding a C^2-small function f to ϕ. Let $k : V\to\mathbb{R}$ be a small positive function such that $\sup_U(X\cdot f)(x)\geq -k(p)$ for each coordinate chart U around

p as in condition (3.6) (for this it suffices that f is sufficiently C^1-small). Then we find

$$-\int_U f(x)(X \cdot g)(x)dx = \int_U (X \cdot f)(x)g(x)dx \geq -k(p) \int_U g(x)dx,$$

so the function $\phi + f$ satisfies $X \cdot (\phi + f) \geq h - k$. In the proof of Lemma 3.12, this operation is applied finitely many times on each compact subset of V, so by choosing the function k sufficiently small we can achieve that $X \cdot \psi \geq \tilde{h}$. □

Propositions 3.21 and 3.22 together imply

Corollary 3.23. *If two smooth J-convex functions ϕ, ψ on a complex manifold V satisfy $X \cdot \phi > 0$ and $X \cdot \psi > 0$ for a vector field X, then the smoothing θ of* $\max(\phi, \psi)$ *also satisfies $X \cdot \theta > 0$.*

Remark 3.24. Inspection of the proofs shows that Propositions 3.21 and 3.22 remain valid if all inequalities are replaced by the reverse inequalities.

Corollary 3.25. *If two smooth J-convex functions ϕ, ψ on a complex manifold V are C^1-close, then the smoothing of* $\max(\phi, \psi)$ *is C^1-close to ϕ.*

Proof. Let X be a vector field and $h_\pm : V \to \mathbb{R}$ functions such that $h_- \leq X \cdot \phi, X \cdot \psi \leq h_+$. By the preceding remark, the smoothing ϑ of $\max(\phi, \psi)$ can be constructed such that $\tilde{h}_- \leq X \cdot \vartheta \leq \tilde{h}_+$ for any given functions $\tilde{h}_- < h_-$ and $\tilde{h}_+ > h_+$. Since X, h_-, h_+ were arbitrary, this proves C^1-closeness of ϑ to ϕ. □

Finally, we apply the preceding result to smoothing of J-convex hypersurfaces.

Corollary 3.26. *Let $(M \times \mathbb{R}, J)$ be a compact complex manifold and $\phi, \psi : M \to \mathbb{R}$ two functions whose graphs are J-convex cooriented by ∂_r, where r is the coordinate on $\mathbb{R}$. Then there exists a smooth function $\theta : M \to \mathbb{R}$ with J-convex graph which is C^0-close to* $\min(\phi, \psi)$ *and coincides with* $\min(\phi, \psi)$ *outside a neighborhood of the set $\{\phi = \psi\}$.*

Proof. For a convex increasing function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ consider the functions

$$\Phi(x, r) := f\big(r - \phi(x)\big), \qquad \Psi(x, r) := f\big(r - \psi(x)\big).$$

For f sufficiently convex, Φ and Ψ are J-convex and satisfy $\partial_r \Phi > 0, \partial_r \Psi > 0$ near their zero level sets. Thus by Propositions 3.21 and 3.22 the function $\max(\Phi, \Psi)$ can be smoothed, keeping it fixed outside a neighborhood U of the set $\{\max(\Phi, \Psi) = 0\}$, to a function Θ which is J-convex and satisfies $\partial_r \Theta > 0$ near its zero level set. The last condition implies that the smooth J-convex hypersurface $\Theta^{-1}(0)$ is the graph of a smooth function $\theta : M \to \mathbb{R}$. Now note that the zero level set $\{\max(\Phi, \Psi) = 0\}$ is the graph of the function $\min(\phi, \psi)$. This implies that θ is C^0-close to $\min(\phi, \psi)$ and coincides with $\min(\phi, \psi)$ outside U. □

We finish this section with the following analogue of Remark 3.17

Lemma 3.27. *Let L be a compact k-dimensional real submanifold of a complex manifold (V, J), $N \supset L$ be its tubular neighborhood in V with respect to some Hermitian metric and $\pi : N \to L$ the normal projection. Let $\phi : V \to \mathbb{R}$ be a J-convex function and $\varphi = \phi|_L$. Denote $\widetilde{\varphi} := \varphi \circ \pi : N \to \mathbb{R}$ and $\psi := \widetilde{\varphi} + C\mathrm{dist}_L^2$, where $C > 0$ is chosen sufficiently large so that the function ψ is J-convex in N. Suppose that $\varphi : L \to \mathbb{R}$ is a Morse function with a unique critical point $p \in \mathrm{Int}\, L$ of index $k' \leq k$. Suppose that ϕ and ψ have along L the same differentials, and p is also a critical point of ψ of the same index k'. Then, given any neighborhood U of L in V, there exists a J-convex function ϑ which coincides with ϕ outside U and with ψ in a smaller neighborhood $U' \subset U$ of L Moreover, ϑ can be chosen arbitrarily C^1-close to ϕ and such that ϑ agrees with ϕ together with its differential along L, and having the same critical points as ϕ.*

Proof. Let $y_1, \dots, y_l$ be local coordinates in a neighborhood Ω of the critical point p such that $L \cap \Omega = \{y_1 = \dots = y_l = 0\}$ and $\mathrm{dist}_L^2 = |y|^2 \sum\limits_1^l y_j^2$. A point $u \in \Omega$ can be assigned coordinates $(x, y_1, \dots, y_l)$, where $x = \pi(U) \in L$. Thus, we have $\psi(u) = \varphi(x) + C \sum\limits_1^l y_j^2$. On the other hand, the condition on the differentials of ϕ and ψ implies that the function ψ can be written as

$$\psi(u) = \varphi(x) + Q_x(y) + o(|y|^2),$$

where $Q_x(y)$ is a quadratic form of variables $(y_1, \dots, y_k)$ with the coefficients depending on $x \in L$. Moreover, the equality of indices of the critical point p for ϕ and ψ ensures that the form $Q_p(y)$ is positive definite. Consider the vector field

$$Y = \frac{1}{|y|} \sum_1^l y_j \frac{\partial}{\partial y_j}$$

in $\Omega \setminus L$. Then if the neighborhood Ω is chosen small enough then there exists an $\varepsilon > 0$ such that

$$d_{(x,y)}\phi(Y), d_{(x,y)}\psi(Y) > \varepsilon |y|$$

for all $u = (x, y) \in \Omega \setminus L$.

Next, we use the construction of the function ϑ in Proposition 3.15. Let us observe that this construction uses only the following operations:

a) Modifing functions ψ and ϕ to functions $\widetilde{\psi} = \psi + \alpha(\mathrm{dist}_L)$ and $\widetilde{\phi} = \phi + \beta(\mathrm{dist}_L)$;

b) smoothing the function $\max(\widetilde{\psi}, \widetilde{\phi})$.

Rewrite the proof of 3.15 accordingly

The functions α and β in a) can be chosen to have an arbitrarily small support and such that $\alpha'', \beta'' > \frac{\varepsilon}{2}$. This implies that $d_{(x,y)}\widetilde{\phi}(Y), d_{(x,y)}\widetilde{\psi}(Y) > \frac{\varepsilon}{2}|y|$ for all $u = (x, y) \in \Omega \setminus L$.

The operation in b) preserves the positivity of the derivative along the vector field Y. Hence, applying Corollary 3.23 we conclude that the function ϑ has no critical points in $\Omega \setminus L$. Taking into account Remark 3.17 we conclude tat ϑ has no critical points in the rest of the neighborhood N, provided that it was chosen small enough. □

3.4 From families of hypersurfaces to J-convex functions

The following result shows that a continuous family of J-convex hypersurfaces transverse to the same vector field gives rise to a smooth function with regular J-convex level sets. This will be extremely useful for the construction of J-convex functions with prescribed critical points.

Proposition 3.28. *Let $(M \times [0,1], J)$ be a compact complex manifold such that $M \times \{0\}$ and $M \times \{1\}$ are J-convex cooriented by ∂_r, where r is the coordinate on $[0,1]$. Suppose there exists a smooth family $(\Sigma_\lambda)_{\lambda \in [0,1]}$ of J-convex hypersurfaces transverse to ∂_r with $\Sigma_0 = M \times \{0\}$ and $\Sigma_1 = M \times \{1\}$. Then there exists a smooth foliation $(\tilde{\Sigma}_\lambda)_{\lambda \in [0,1]}$ of $M \times [0,1]$ by J-convex hypersurfaces transverse to ∂_r with $\tilde{\Sigma}_\lambda = M \times \{\lambda\}$ for λ near 0 or 1.*

Proof. Let $\varepsilon > 0$ be so small that the hypersurfaces $M \times \{\lambda\}$ are J-convex for $\lambda \le \varepsilon$ and $\lambda \ge 1 - \varepsilon$. Set $V := M \times [0,1]$ and $U := M \times (\varepsilon, 1 - \varepsilon)$. Reparametrize in λ such that $\Sigma_\lambda = M \times \{\lambda\}$ for $\lambda \le \varepsilon$ and $\lambda \ge 1 - \varepsilon$. After a C^2-small perturbation and decreasing ε, we may further assume that $\Sigma_\lambda \subset U$ for $\lambda \in (\varepsilon, 1 - \varepsilon)$. Pick a smooth family of J-convex functions ϕ_λ with regular level sets $\phi_\lambda^{-1}(0) = \Sigma_\lambda$. After composing each ϕ_λ with a suitable function $\mathbb{R} \to \mathbb{R}$, we may assume that $\phi_\lambda > \phi_\mu$ for all $\lambda < \mu$ with either $\lambda \le \varepsilon$ or $\mu \ge 1 - \varepsilon$.

The continuous functions

$$\psi_\lambda := \max_{\nu \ge \lambda} \phi_\nu$$

are J-convex by Lemma 3.8 and, by construction, satisfy

$$\psi_\lambda \ge \psi_\mu \text{ for } \lambda \le \mu. \tag{3.7}$$

Moreover, we have $\psi_\lambda = \phi_\lambda$ for $\lambda \le \varepsilon$ and $\lambda \ge 1 - \varepsilon$. By Proposition 3.21, the ψ_λ satisfy $\partial_r \cdot \psi_\lambda \ge h$ (in the distributional sense) for a positive function $h : M \times [0,1] \to \mathbb{R}$.

Next use Proposition 3.10 to approximate the ψ_λ by smooth J-convex functions $\hat{\psi}_\lambda$. By Remark 3.13, the resulting family $\hat{\psi}_\lambda$ is continuous in λ and still satisfies (3.7). By Proposition 3.22, the smoothed functions satisfy $\partial_r \cdot \hat{\psi}_\lambda \ge h/2 > 0$, hence the level sets $\hat{\Sigma}_\lambda := \hat{\psi}_\lambda^{-1}(0)$ are regular and transverse to ∂_r. We can modify the smoothing construction to achieve $\hat{\psi}_\lambda = \phi_\lambda$ near $\lambda = 0$ and 1, still satisfying J-convexity, transversality of the zero level to ∂_r, and (3.7). Note that as a result of the smoothing construction the functions $\hat{\psi}_\lambda$, and hence their level sets $\hat{\Sigma}_\lambda$, depend continuously on the parameter λ with respect to the C^2-topology.

Since $\hat{\Sigma}_\lambda$ is transverse to ∂_r, we can write it as the graph $\{r = f_\lambda(x)\}$ of a smooth function $f_\lambda : M \to [0,1]$. By construction, the functions f_λ depend continuously on λ with respect to the C^2-topology, $f_\lambda \le f_\mu$ for $\lambda \le \mu$, and $f_\lambda(x) = \lambda$ for $\lambda \le \varepsilon$ and $\lambda \ge 1-\varepsilon$, with some $\varepsilon > 0$ (possibly smaller than the one above). Note that $f_\mu(x) - f_\lambda(x) \ge \mu - \lambda$ for $\lambda \le \mu \le \varepsilon$ and $1-\varepsilon \le \lambda \le \mu$. Pick a function $g : [0,1] \to [0,1]$ satisfying $g(\lambda) = 0$ for $\lambda \le \varepsilon/2$ and $\lambda \ge 1-\varepsilon/2$, $g'(\lambda) \ge -1+\gamma$ for $\varepsilon/2 \le \lambda \le \varepsilon$ and $1-\varepsilon \le \lambda \le 1-\varepsilon/2$, and $g'(\lambda) \ge \gamma$ for $\varepsilon \le \lambda \le 1-\varepsilon$, with some $\gamma > 0$. For g sufficiently small, the graphs of the functions $\hat{f}_\lambda(x) := f_\lambda(x) + g(\lambda)$ are still J-convex, $\hat{f}_\lambda(x) = \lambda$ for $\lambda \le \varepsilon/2$ and $\lambda \ge 1-\varepsilon/2$, and

$$\hat{f}_\mu(x) - \hat{f}_\lambda(x) \ge \gamma(\mu - \lambda)$$

for all $\lambda \le \mu$. Now mollify the functions $\hat{f}_\lambda(x)$ in the parameter λ to

$$\tilde{f}_\lambda(x) := \int_{\mathbb{R}} \hat{f}_{\lambda-\mu}(x)\rho_\delta(\mu)d\mu,$$

with a cutoff function $\rho : \mathbb{R} \to \mathbb{R}$ as in equation (3.5). Since the functions $f_{\lambda-\mu}$ are C^2-close to f_λ for $\mu \in \mathrm{supp}(\rho_\delta)$ and δ small, the graph of $\tilde{f}_\lambda$ is C^2-close to the graph of f_λ and hence J-convex. Moreover, for $\lambda' \ge \lambda$ the $\tilde{f}_\lambda$ still satisfy

$$\tilde{f}_{\lambda'}(x) = \int_{\mathbb{R}} \hat{f}_{\lambda'-\mu}(x)\rho_\delta(\mu)d\mu \ge \int_{\mathbb{R}} \hat{f}_{\lambda-\mu}(x)\rho_\delta(\mu)d\mu + \gamma(\lambda'-\lambda) = \tilde{f}_\lambda(x) + \gamma(\lambda'-\lambda).$$

Modify the $\tilde{f}_\lambda$ such that $\tilde{f}_\lambda(x) = \lambda$ for $\lambda \le \varepsilon/2$ and $\lambda \ge 1-\varepsilon/2$, and so that their graphs are still J-convex and $\tilde{f}_\mu(x) - \tilde{f}_\lambda(x) \ge \gamma(\mu - \lambda)$ for all $\lambda \le \mu$. The last inequality implies that the map $(x, \lambda) \mapsto \big(x, f_\lambda(x)\big)$ is an embedding, thus the graphs of $\tilde{f}_\lambda$ form the desired foliation $\tilde{\Sigma}_\lambda$. □

Chapter 4

Shapes for i-convex hypersurfaces

In this chapter we introduce our main tool for the construction of J-convex functions and use it to construct specific i-convex functions on $\mathbb{C}^n$.

4.1 Shapes

Consider the map

$$\pi : \mathbb{C}^n \to \mathbb{R}^2, \qquad z \mapsto (r, R) := (|x|, |y|)$$

for $z = x + iy$, $x, y \in \mathbb{R}^n$. The image of the map π is the quadrant

$$Q := \{(r, R) \mid r, R \geq 0\} \subset \mathbb{R}^2.$$

A curve $C \subset Q$ defines a hypersurface $\Sigma_C := \pi^{-1}(C)$ in $\mathbb{C}^n$. We call C the *shape* of Σ_C. Our goal in this section is to determine conditions on C which guarantee i-convexity of Σ_C.

As a preliminary, let us compute the second fundamental form of a surface of revolution. Consider $\mathbb{R}^k \oplus \mathbb{R}^l$ with coordinates (x, y) and $\mathbb{R}^k \oplus \mathbb{R}$ with coordinates $(x, R = |y|)$. To a function $\Phi : \mathbb{R}^k \oplus \mathbb{R} \to \mathbb{R}$ we associate the *surface of revolution*

$$\Sigma_\Phi := \{(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^l \mid \Phi(x, |y|) = 0\}.$$

We coorient Σ_Φ by the gradient $\nabla\Phi$ of Φ (with respect to all variables). Denote by $\Phi_R = \frac{\partial \Phi}{\partial R}$ the partial derivative.

Lemma 4.1. *At every $z = (x, y) \in \Sigma_\Phi$ the splitting*

$$T_z\Sigma_\Phi = \Big(T_z\Sigma_\Phi \cap (\mathbb{R}^k \oplus \mathbb{R}y)\Big) \oplus \Big(T_z\Sigma_\Phi \cap (\mathbb{R}^k \oplus \mathbb{R}y)^\perp\Big)$$

is orthogonal with respect to the second fundamental form II. The second subspace is an eigenspace of II with eigenvalue $\Phi_R/|\nabla\Phi|R$.

Proof. The unit normal vector to Σ_Φ at $z=(x,y)$ is

$$\nu(z)=\frac{1}{|\nabla\Phi|}(\nabla_x\Phi,\frac{\Phi_R}{R}y),$$

where $\nabla_x\Phi$ denotes the gradient with respect to the x-variables. For $Y\perp y$ we get

$$D\nu(z)\cdot(0,Y)=\frac{1}{|\nabla\Phi|}(0,\frac{\Phi_R}{R}Y)+\mu\nu$$

for some $\mu\in\mathbb{R}$. From $\langle\nu(z),D\nu(z)\cdot(0,Y)\rangle=0$ we deduce $\mu=0$, so $T_z\Sigma_\Phi\cap(\mathbb{R}^k\oplus\mathbb{R}y)^\perp$ is an eigenspace of II with eigenvalue $\Phi_R/|\nabla\Phi|R$. From this it follows that

$$II\Big((0,Y),(X,\lambda y)\Big)=\langle D\nu\cdot(0,Y),(X,\lambda y)\rangle=0$$

for $(X,\lambda y)\in T_z\Sigma_\Phi\cap(\mathbb{R}^k\oplus\mathbb{R}y)$. □

Reduction to the case $n=2$. Now let $C\subset Q$ be a curve. At a point $z=x+iy\in\Sigma_C$ consider the subspace $\Lambda_{xy}\subset\mathbb{R}^n$ generated by the vectors $x,y\in\mathbb{R}^n$ and its complexification

$$\Lambda_{xy}^{\mathbb{C}}:=\Lambda_{xy}+i\Lambda_{xy}.$$

Let $\Lambda_\perp$ be the orthogonal complement of Λ_{xy} in $\mathbb{R}^n$ and $\Lambda_\perp^{\mathbb{C}}$ its complexification (which is the orthogonal complement of $\Lambda_{xy}^{\mathbb{C}}$ in $\mathbb{C}^n$). Note that $\Lambda_\perp^{\mathbb{C}}$ is contained in $T_z\Sigma_C$ and thus in the maximal complex subspace ξ_z. So the maximal complex subspace splits into the orthogonal sum (with respect to the metric)

$$\xi_z=\tilde\Lambda\oplus\Lambda_\perp^{\mathbb{C}}=\tilde\Lambda\oplus\Lambda_\perp\oplus i\Lambda_\perp,$$

where $\tilde\Lambda=\xi_z\cap\Lambda_{xy}^{\mathbb{C}}$. We claim that this splitting is orthogonal with respect to the second fundamental form II, and $\Lambda_\perp$ and $i\Lambda_\perp$ are eigenspaces with eigenvalues $\Phi_r/|\nabla\Phi|r$ and $\Phi_R/|\nabla\Phi|R$, respectively.

Indeed, Σ_C can be viewed as a surface of revolution in two ways, either rotating in the x- or the y-variables. So by Lemma 4.1, the splittings

$$\Big(\xi_z\cap(\mathbb{R}x\oplus i\mathbb{R}^n)\Big)\oplus\Big(\xi_z\cap(\mathbb{R}x\oplus i\mathbb{R}^n)^\perp\Big),$$
$$\Big(\xi_z\cap(\mathbb{R}^n\oplus i\mathbb{R}y)\Big)\oplus\Big(\xi_z\cap(\mathbb{R}^n\oplus i\mathbb{R}y)^\perp\Big)$$

are both orthogonal with respect to II and the right-hand spaces are eigenspaces. In particular, $\Lambda_\perp=\xi_z\cap(\mathbb{R}x\oplus i\mathbb{R}^n)^\perp$ and $i\Lambda_\perp=\xi_z\cap(\mathbb{R}^n\oplus i\mathbb{R}y)^\perp$ are eigenspaces orthogonal to each other with eigenvalues $\Phi_r/|\nabla\Phi|r$ and $\Phi_R/|\nabla\Phi|R$. Since $\Lambda_{xy}^{\mathbb{C}}$ is the orthogonal complement of $\Lambda_\perp\oplus i\Lambda_\perp$ in $\mathbb{C}^n$, the claim follows.

Now suppose that C is given near the point $\pi(z)$ by the equation $R = \phi(r)$, and the curve is cooriented by the gradient of the function $\Phi(r, R) = \phi(r) - R$. Since $|\nabla\Phi| = \sqrt{\Phi_r^2 + \Phi_R^2} = \sqrt{1 + \phi'(r)^2}$, the eigenvalues λ_r on $\Lambda_\perp$ and λ_R on $i\Lambda_\perp$ equal

$$\lambda_r = \frac{\Phi_r}{|\nabla\Phi| r} = \frac{\phi'(r)}{r\sqrt{1+\phi'(r)^2}},$$
$$\lambda_R = \frac{\Phi_R}{|\nabla\Phi| R} = -\frac{1}{\phi(r)\sqrt{1+\phi'(r)^2}}.$$

Hence by Proposition 2.9, the restriction of the normalized Levi form $\mathbb{L}_{\Sigma_C}$ to $\Lambda_\perp^{\mathbb{C}}$ is given by

$$\mathbb{L}_{\Sigma_C}(X) = \frac{1}{2\sqrt{1+\phi'(r)^2}}\Big(\frac{\phi'(r)}{r} - \frac{1}{\phi(r)}\Big)|X|^2.$$

Hence we have proved

Lemma 4.2. *Let Σ_C be the hypersurface given by the curve $C = \{\phi(r)-R = 0\}$, cooriented by the gradient of $\phi(r) - R$. Then the restriction of the Levi form L_{Σ_C} to $\Lambda_\perp^{\mathbb{C}}$ is positive definite if and only if*

$$\mathcal{L}^\perp(\phi) := \frac{\phi'(r)}{r} - \frac{1}{\phi(r)} > 0.$$

In particular, if $\phi'(r) \le 0$ the restriction is always negative definite.

Lemma 4.2 reduces the question about i-convexity of Σ_C to positivity of $\mathcal{L}^\perp(\phi)$ and the corresponding question about the intersection $\Sigma_C \cap \Lambda_{xy}^{\mathbb{C}}$. When $\dim_{\mathbb{C}} \Lambda_{xy}^{\mathbb{C}} = 1$, this intersection is a curve which is trivially i-convex, hence Σ_C is i-convex if and only if $\mathcal{L}^\perp(\phi) > 0$. The remaining case $\dim_{\mathbb{C}} \Lambda_{xy}^{\mathbb{C}} = 2$ just means that we have reduced the original question to the case $n = 2$, which we will now consider.

The case $n = 2$. We denote complex coordinates in $\mathbb{C}^2$ by $z = (\zeta, w)$ with $\zeta = s + it$, $w = u + iv$. The hypersurface $\Sigma_C \subset \mathbb{C}^2$ is given by the equation

$$\sqrt{t^2 + v^2} = R = \phi(r) = \phi(\sqrt{s^2 + u^2}).$$

We want to express the Levi form $\mathcal{L}$ at a point $z \in \Sigma_C$ in terms of ϕ. Suppose that $r, R > 0$ at the point z. After a unitary transformation

$$\zeta \mapsto \zeta\cos\alpha + w\sin\alpha, \qquad w \mapsto -\zeta\sin\alpha + w\cos\alpha$$

which leaves Σ_C invariant we may assume $t = 0$ and $v > 0$. Then near z we can solve the equation $R = \phi(r)$ for v,

$$v = \sqrt{\phi(\sqrt{s^2+u^2})^2 - t^2} =: \psi(s, t, u).$$

According to Lemma 2.21, the normalized Levi form of the hypersurface $\Sigma_C = \{v = \psi(s,t,u)\}$ is given by $\mathbb{L}(T) = \mathbb{L}_0|T|^2$, $T \in \xi$, where

$$\mathbb{L}_0 = \frac{1}{(1+\psi_s^2+\psi_t^2)^{\frac{3}{2}}}\big((\psi_{ss}+\psi_{tt})(1+\psi_u^2)+\psi_{uu}(\psi_s^2+\psi_t^2) + 2\psi_{su}(\psi_t - \psi_u\psi_s) - 2\psi_{tu}(\psi_s+\psi_u\psi_t)\big). \tag{4.1}$$

Note that at the point z we have $t = 0$ and $\psi(s,0,u) = \phi(r) = \phi(\sqrt{u^2+s^2})$. Using this, compute the derivatives at z,

$$\psi_s = \frac{\phi' s}{r}, \qquad \psi_{ss} = \frac{\phi'' s^2}{r^2} + \frac{\phi' u^2}{r^3}, \qquad \psi_u = \frac{\phi' u}{r}, \qquad \psi_{uu} = \frac{\phi'' u^2}{r^2} + \frac{\phi' s^2}{r^3},$$
$$\psi_{su} = \frac{\phi'' su}{r^2} - \frac{\phi' su}{r^3}, \qquad \psi_t = 0, \qquad \psi_{tt} = -\frac{1}{\phi}, \qquad \psi_{tu} = 0.$$

Inserting this in equation (4.1), we obtain

$$\begin{aligned}\frac{(r^2+s^2\phi'^2)^{\frac{3}{2}}}{r^3}\mathbb{L}_0 &= \Big(\frac{\phi'' s^2}{r^2} + \frac{\phi' u^2}{r^3} - \frac{1}{\phi}\Big)\Big(1 + \frac{\phi'^2 u^2}{r^2}\Big) \\ &\quad + \Big(\frac{\phi'' u^2}{r^2} + \frac{\phi' s^2}{r^3}\Big)\frac{\phi'^2 s^2}{r^2} - 2\Big(\frac{\phi'' su}{r^2} - \frac{\phi' su}{r^3}\Big)\frac{\phi'^2 su}{r^2} \\ &= \frac{\phi'' s^2}{r^2} + \frac{\phi' u^2}{r^3} + \frac{\phi'^3}{r} - \frac{1}{\phi}\Big(1 + \frac{\phi'^2 u^2}{r^2}\Big).\end{aligned}$$

We say that the curve C is *cooriented from above* if it is cooriented by the gradient of the function $\phi(r) - R$. Equivalently (since $t = 0$ at z), the hypersurface Σ_C is cooriented by the gradient of $\sqrt{\phi(\sqrt{s^2+u^2})^2 - t^2} - v$, which is the coorientation we have chosen above. The opposite coorientation will be called *coorientation from below*. Lemma 4.2 and the preceding discussion yield the following criteria for i-convexity of Σ_C.

Proposition 4.3. *The hypersurface $\Sigma_C = \{R = \phi(r)\}$ is i-convex cooriented from above at $r > 0$ if and only if ϕ satisfies the following two conditions:*

$$\mathcal{L}^\perp(\phi) := \frac{\phi'(r)}{r} - \frac{1}{\phi(r)} > 0, \tag{4.2}$$

$$\mathcal{L}^2(\phi) := \frac{\phi'' s^2}{r^2} + \frac{\phi' u^2}{r^3} + \frac{\phi'^3}{r} - \frac{1}{\phi}\Big(1 + \frac{\phi'^2 u^2}{r^2}\Big) > 0 \tag{4.3}$$

for all (s,u) with $s^2 + u^2 = r^2$. It is i-convex cooriented from below if and only if the reverse inequalities hold.

Remark 4.4. Let us note that for Σ_C cooriented from above and when $\phi' > 0$, the maximal absolute value of the negative normal curvature of Σ_C equals

$$M(\Sigma) = \max\{-II(T),\, T \in T\Sigma, |T| = 1\} = \max\left(\frac{-\phi''}{(1+\phi'^2)^{\frac{3}{2}}}, \frac{1}{\phi\sqrt{1+\phi'2}}\right).$$

Hence, in that case the inequality $\mu(\Sigma_C) > \varepsilon > 0$ for the modulus of J-convexity of Σ_c is equivalent to a system of inequalities, stronger than (4.2) and (4.3):

$$\mathcal{L}^{\perp}_{\varepsilon,1}(\phi) := \varepsilon\frac{\min(\phi'', 0)}{1+\phi'^2} + \frac{\phi'(r)}{r} - \frac{1}{\phi(r)} > 0, \tag{4.4}$$

$$\mathcal{L}^{\perp}_{\varepsilon,2}(\phi) := \frac{\phi'(r)}{r} - \frac{1+\varepsilon}{\phi(r)} > 0, \tag{4.5}$$

$$\mathcal{L}^{2}_{\varepsilon,1}(\phi) := (1-\varepsilon)\frac{\phi'' s^2}{r^2} + \frac{\phi' u^2}{r^3} + \frac{\phi'^3}{r} - \frac{1}{\phi}\Big(1 + \frac{\phi'^2 u^2}{r^2}\Big) > 0 \tag{4.6}$$

$$\mathcal{L}^{2}_{\varepsilon,2}(\phi) := \frac{\phi'' s^2}{r^2} + \frac{\phi' u^2}{r^3} + \frac{\phi'^3}{r} - \frac{1}{\phi}\Big(1 + \frac{\phi'^2 u^2}{r^2}\Big) - \frac{\varepsilon}{\phi}(1+\phi'^2\Big) > 0 \tag{4.7}$$

for all (s, u) with $s^2 + u^2 = r^2$.

The following corollary gives some useful sufficient conditions for i-convexity.

Corollary 4.5. *(a) If $\phi > 0$, $\phi' > 0$, $\phi'' \leq 0$ and*

$$\phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi}(1+\phi'^2) > 0, \tag{4.8}$$

then Σ_C is i-convex cooriented from above.
(b) If $\phi > 0$, $\phi' \leq 0$, $\phi'' \geq 0$ and

$$\phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi} < 0,$$

then Σ_C is i-convex cooriented from below.

Proof. (a) If $\phi' > 0$ and $\phi'' \leq 0$ we get

$$\mathcal{L}^2(\phi) \geq \phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi}(1+\phi'^2).$$

So positivity of the right hand side implies condition (4.3). Condition (4.2) is also a consequence of $\phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi}(1+\phi'^2) > 0$.
(b) If $\phi' \leq 0$ and $\phi'' \geq 0$ we get

$$\mathcal{L}^2(\phi) \leq \phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi}.$$

So negativity of the right hand side implies the reverse inequality (4.3). The reverse inequality (4.2) is automatically satisfied. □

Remark 4.6. Let us rewrite in the case a) the sufficient conditions (**??**)–(**??**). The following inequality guarantees the lower bound $\mu(\Sigma_C) > \varepsilon$

$$(1+\varepsilon)\phi'' + \frac{\phi'^3}{r} - \frac{1+\varepsilon}{\phi}(1+\phi'^2) > 0, \tag{4.9}$$

In the case $\phi' > 0, \phi'' \geq 0$ the bound $\mu(\Sigma_C) > \varepsilon$ follows from the inequalities

$$\begin{aligned} \mathcal{L}_\varepsilon^\perp &= \frac{\phi'}{r} - \frac{1+\varepsilon}{\phi} > 0; \\ \mathcal{L}_\varepsilon^2 &= \phi'' + \frac{\phi'^3}{r} - \frac{1+\varepsilon}{\phi}(1+\phi'^2) > 0. \end{aligned} \tag{4.10}$$

As a first application of Corollary 4.5 we have

Lemma 4.7. *For any $\varepsilon > 0$ and $\delta \in (\frac{2\sqrt{2}\varepsilon}{3}, \varepsilon)$ sufficiently small, the quarter circle*

$$\phi(r) := \varepsilon - \sqrt{\delta^2 - (\varepsilon - r)^2}, \qquad r \in [\varepsilon - \delta, \varepsilon]$$

defines an i-convex hypersurface $\{R = \phi(r)\}$ cooriented from below.

Proof. Denote $s := \sqrt{\delta^2 - (\varepsilon - r)^2}$. We have

$$\phi'(r) = -\frac{\varepsilon - r}{s}, \quad \phi''(r) = \frac{\delta^2}{s^3}.$$

Hence,

$$\phi'' + \frac{\phi'^3}{r} - \frac{1}{\phi} = \frac{1}{s^3}\left(\delta^2 - \frac{(\varepsilon - r)^3}{r} - \frac{s^3}{\varepsilon - s}\right).$$

Set $t = \varepsilon - r$. then we need to prove that

$$F(t) := \frac{t^3}{\varepsilon - t} + \frac{s^3}{\varepsilon - s} > \delta^2, \tag{4.11}$$

where $s = \sqrt{\delta^2 - t^2}$. We have

$$F'(t) = t\left(\frac{t(3\varepsilon - 2t)}{(\varepsilon - t)^2} - \frac{s(3\varepsilon - 2s)}{(\varepsilon - s)^2}\right).$$

The functiion $\frac{F'(t)}{t}$ has a unique zero when $t = s$, i.e. $t = \frac{\delta}{\sqrt{2}}$, negative on $[0, \frac{\delta}{\sqrt{2}})$ and positive on $(\frac{\delta}{\sqrt{2}}, \delta]$. Hence the function $F(t)$ has its minimum at the point $\frac{\delta}{\sqrt{2}}$. We compute $F(\frac{\delta}{\sqrt{2}}) = \frac{2\delta^3}{2\sqrt{2}\varepsilon - \delta}$, and taking into account that $\varepsilon < \frac{3\delta}{2\sqrt{2}}$ we conclude that $F(\frac{\delta}{\sqrt{2}}) > \delta^2$, which implies the inequality (4.11). □

For the remainder of this chapter we will only be interested in hypersurfaces $\{R = \phi(r)\}$ that are i-convex cooriented from *above*. We will call the corresponding function ϕ satisfying the conditions of Proposition 4.3 an *i-convex shape*. The following lemma lists some elementary properties of i-convex shapes.

Lemma 4.8 (Properties of i-convex shapes). *(a) If ϕ is an i-convex shape then so is $\phi + c$ for any constant $c \geq 0$ (i-convexity from above is preserved under upwards shifting).*

(b) If a ϕ is an i-convex shape at $r > 0$, then the function $\phi_\lambda(r) := \lambda\phi(r/\lambda)$ is an i-convex shape at λr for each $\lambda > 0$.

(c) If ϕ, ψ are i-convex shapes for $r \leq r_0$ resp. $r \geq r_0$ such that $\phi(r_0) = \psi(r_0)$ and $\phi'(r_0) = \psi'(r_0)$, then the function

$$\theta(r) := \begin{cases} \phi(r) & \text{for } r \leq r_0, \\ \psi(r) & \text{for } r \geq r_0 \end{cases}$$

can be C^1-perturbed to a smooth i-convex shape which agrees with θ outside a neighborhood or r_0.

(d) If ϕ, ψ are i-convex shapes, then the function

$$\theta := \max(\phi, \psi)$$

can be C^0-perturbed to a smooth i-convex shape which agrees with θ outside a neighborhood or the set $\{\phi = \psi\}$.

Proof. (a) If ϕ satisfies one of the inequalities (4.2), (4.3) and (4.8), then $\phi + c$ satisfies the same inequality for any constant $c \geq 0$.

(b) can be seen by applying the biholomorphism $z \mapsto \lambda z$ on $\mathbb{C}^n$, or from Proposition 4.3 as follows: The function ϕ_λ has derivatives $\phi_\lambda(\lambda r) = \lambda\phi(r)$, $\phi'_\lambda(\lambda r) = \phi'(r)$, $\phi''_\lambda(\lambda r) = \phi(r)/\lambda$, and the replacement $r \mapsto \lambda r$, $\phi \mapsto \lambda\phi$, $\phi' \mapsto \phi'$, $\phi'' \mapsto \phi''/\lambda$ leaves both conditions in Proposition 4.3 unchanged.

(c) follows from the fact that for given r, ϕ, ϕ', the set of ϕ'' such that condition (4.3) holds is convex.

(d) After C^2-perturbing ϕ we may assume that the graphs of ϕ and ψ intersect transversally. Consider an intersection point r_0 such that $\phi(r_0) = \psi(r_0)$ and $\phi'(r_0) < \psi'(r_0)$, so near r_0 we have

$$\theta(r) = \begin{cases} \phi(r) & \text{for } r \leq r_0, \\ \psi(r) & \text{for } r \geq r_0 \end{cases}.$$

Pick $r_- < r_0 < r_+$ with $|r_+ - r_-| < \delta$ small. Let $\chi'' : [r_-, r_+] \to \mathbb{R}$ be a continuous function which near r_- increases steeply from $\chi''(r_-) = \phi''(r_-)$ to a constant $m >> 0$, near r_+ decreases steeply from m to $\chi''(r_+) = \phi''(r_+)$, and such that $\int_{r_-}^{r_+} \chi''(r)dr = \psi'(r_+) - \phi'(r_-)$. So the function $\chi'(r) := \phi'(r_-) + \int_{r_-}^{r} \chi''(s)ds$ satisfies $\chi'(r_-) = \phi'(r_-)$ and $\chi'(r_+) = \psi(r_+)$. The function $\chi(r) := \phi(r_-) + \int_{r_-}^{r} \chi'(s)ds$ satisfies $\chi(r_-) = \phi(r_-)$ and $|\chi(r_+) - \psi(r_+)| \leq C\delta$ for a constant C independent of δ. Moreover, its first and second derivatives agree with those of ϕ resp. ψ at r_0 resp. r_+.

It remains to show i-convexity of χ; the desired function is then obtained by interpolating from χ to ψ to the right of r_+ for small δ. Condition (4.2) holds for χ because it holds for ϕ, ψ and up to error of order δ for $r \in [r_0, r_+]$ we have $r \cong r_0$, $\chi(r) \cong \phi(r_0) = \psi(r_0)$ and $\chi'(r) \in [\phi'(r_0), \psi'(r_0)]$. Next note that condition (4.3) for $s = 0$ becomes

$$\left(\frac{\chi'}{r} - \frac{1}{\chi}\right)(1 + \chi'^2) > 0,$$

which is satisfied in view of condition (4.2). Since $\chi''(r)$ is uniformly bounded from below independently of δ, there exists a constant $\sigma > 0$ independent of δ such that χ satisfies condition (4.3) for all $|s| \leq \sigma$. Moreover, near r_- resp. r_+ condition (4.3) holds for χ because it holds for ϕ, ψ and χ'' is larger than ϕ'' resp. ψ''. So it remains to consider the region where $\chi'' \equiv m$ in the case $|s| \geq \sigma$. In this region r, χ, χ' are bounded independently of δ. On the other hand, since the constant m is of order $1/\delta$, the term $\chi'' s^2/r^2$ becomes arbitrarily large as $\delta \to 0$, so condition (4.3) holds for δ sufficiently small. □

The following lemma extends i-convex shapes to the subcritical case.

Lemma 4.9. *For $k < n$ set $r := \sqrt{x_1^2 + \cdots + x_n^2 + y_{k+1}^2 + \cdots + y_n^2}$ and $R := \sqrt{y_1^2 + \cdots + y_k^2}$. Let $\phi(r)$ be an i-convex shape. Then $\Sigma := \{R = \phi(r)\}$ is an i-convex hypersurface cooriented from above. Moreover, Σ intersects the subspace $i\mathbb{R}^n$ i-orthogonally.*

Proof. Set $\bar{r} := \sqrt{x_1^2 + \cdots + x_n^2}$ and $\bar{R} := \sqrt{y_1^2 + \cdots + y_n^2}$. By assumption, the hypersurface $\bar{\Sigma} := \{\bar{R} = \phi(\bar{r})\}$ is i-convex cooriented from above. Let $\bar{\psi}(\bar{r}, \bar{R})$ be an increasing function of $\phi(\bar{r}) - \bar{R}$ which is i-convex on a neighborhood of $\bar{\Sigma}$. The unitary group $U(n-k)$ acts on the second factor of $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^{n-k}$ and the functions $z \mapsto \bar{\psi}(gz)$, $g \in U(n-k)$, form a smooth family of i-convex functions. Therefore, by Lemma 3.8, the continuous function

$$\psi(z) := \max_{g \in U(n-k)} \bar{\psi}(gz)$$

is i-convex. Set $z' := (z_1, \ldots, z_k)$ and $z'' := (z_{k+1}, \ldots, z_n)$. Since $\bar{\phi}$ is increasing, the function

$$g \mapsto \bar{\phi}\big(\sqrt{\mathrm{Re}\,(z')^2 + \mathrm{Re}\,(gz'')^2}\big) - \sqrt{\mathrm{Im}\,(z')^2 + \mathrm{Im}\,(gz'')^2}$$

for fixed (z', z'') is maximized iff $\mathrm{Im}\,(gz'') = 0$, so we have $\psi(z) = \bar{\psi}(r, R)$. This implies that $\psi(z) = \bar{\psi}(r, R)$ is smooth and i-convex, hence its level set Σ is also i-convex.

The i-orthogonality of Σ to $i\mathbb{R}^n$ is clear from the definition. □

4.2 Construction of special shapes

We will now construct special i-convex shapes satisfying the differential inequality in Corollary 4.5 (a). In fact, we will solve a slightly stronger differential inequality (4.9) which ensures the lower bound for the modulus of J-convexity for the constructed J-convex hypersurface.

One such solution with the desired properties has been constructed in [14]. The following simplified construction was pointed out to us by M. Struwe. We will find the function ϕ as a solution of *Struwe's differential equation*

$$\phi'' + \frac{\phi'^3}{2r} = 0 \tag{4.12}$$

with $\phi' > 0$ and hence $\phi'' < 0$. Then the inequality (4.9) with $\varepsilon = \frac{1}{2}$ reduces to

$$\frac{\phi'^3}{2r} - \frac{3}{\phi}(1 + \phi'^2) > 0. \tag{4.13}$$

Lemma 4.10. *For any $d, K, \delta, \lambda > 0$ satisfying $K \geq e^{4/d^2}$ and $12K\delta \leq (\ln K)^{-3/2}$ there exists a solution $\phi : [\lambda\delta, K\lambda\delta] \to \mathbb{R}$ of (4.12) with the following properties:*

(a) $\phi'(\lambda\delta) = +\infty$ and $\phi(\lambda\delta) \geq \lambda + d\lambda\delta$;

(b) $\phi(K\lambda\delta) = \lambda + dK\lambda\delta$ and $\phi'(K\lambda\delta) \leq d$;

(c) ϕ satisfies (4.13) and hence is the shape of an i-convex hypersurface cooriented from above.

Proof. First note that if ϕ satisfies equation (4.12) and inequality (4.13), then so does the rescaled function $\lambda\phi(r/\lambda)$. Thus it suffices to consider the case $\lambda = 1$. The differential equation (4.12) is equivalent to

$$\left(\frac{1}{\phi'^2}\right)' = -\frac{2\phi''}{\phi'^3} = \frac{1}{r},$$

thus $1/\phi'^2 = \ln(r/\delta)$ for some constant $\delta > 0$, or equivalently, $\phi'(r) = 1/\sqrt{\ln(r/\delta)}$. By integration, this yields a solution ϕ for $r \geq \delta$ which is strictly increasing and concave and satisfies $\phi'(\delta) = +\infty$. Note that $\int_\delta^{K\delta} \phi'(r)dr = \delta K_1$ with

$$K_1 := \int_1^K \frac{du}{\sqrt{\ln u}} < \infty.$$

Fix the remaining free constant in ϕ by setting $\phi(K\delta) := 1 + dK\delta$, thus

$$\phi(\delta) = 1 + dK\delta - K_1\delta.$$

Estimating the logarithm on $[1, K]$ from below by the linear function with the same values at the endpoints,

$$\ln u \geq \frac{\ln K}{K-1}(u-1),$$

we obtain an upper estimate for K_1:

$$K_1 \leq \int_1^K \frac{du}{\sqrt{\frac{\ln K}{K-1}(u-1)}} = \sqrt{\frac{K-1}{\ln K}} \int_0^{K-1} \frac{du}{\sqrt{u}} = \frac{2(K-1)}{\sqrt{\ln K}}. \tag{4.14}$$

By hypothesis we have $\sqrt{\ln K} \geq 2/d$, hence $K_1 \leq d(K-1)$. This implies

$$\phi(\delta) \geq 1 + dK\delta - d(K-1)\delta = 1 + d\delta.$$

Concavity of ϕ implies $\phi(r) \geq 1 + dr$ for all $r \in [\delta, K\delta]$, and in particular $\phi'(K\delta) \leq d$. So it only remains to check inequality (4.13). Denoting by $\sim$ equality up to a positive factor, we compute

$$\begin{aligned} \frac{\phi'^3}{2r} - \frac{3}{\phi}(1+\phi'^2) &\geq \frac{\phi'^3}{2r} - \frac{3}{1+dr}(1+\phi'^2) \\ &\sim \frac{\phi'^3}{r}(1+dr) - 6 - 6\phi'^2 \\ &\sim \frac{1}{r} + d(1-\varepsilon) - 6\ln(r/\delta)^{3/2} - 6\ln(r/\delta)^{1/2}. \end{aligned}$$

The function on the right hand side is decreasing in r. So its minimum is achieved for $r = K\delta$ and has the value

$$d + \frac{1}{K\delta} - 6(\ln K)^{3/2} - 6(\ln K)^{1/2} > \frac{1}{K\delta} - 12(\ln K)^{3/2} \geq 0$$

by hypothesis. □

For numbers $\lambda, a, b, c, d \geq 0$ consider the following functions:

$S_\lambda(r) = \sqrt{\lambda^2 + ar^2}$ (standard function),

$Q_\lambda(r) = \lambda + br + cr^2/2\lambda$ (quadratic function),

$L_\lambda(r) = \lambda + dr$ (linear function).

Let us first determine in which ranges they satisfy the inequalities (4.2) and (4.3).

Lemma 4.11. *(a) The function $S_\lambda(r)$ is the shape of an i-convex hypersurface for $\lambda \geq 0$, $a > 1$ and $r > 0$.*

(b) The function $Q_\lambda(r)$ is the shape of an i-convex hypersurface for $\lambda > 0$, $b \geq 0$, $c > 1$ and $r > 0$.

(c) The function $Q_\lambda(r)$ is the shape of an i-convex hypersurface for $\lambda > 0$, $b = 4 - c$, $0 \leq c \leq 4$ and $0 < r \leq 2\lambda$.

(d) The function $L_\lambda(r)$ is the shape of an i-convex hypersurface for $\lambda \geq 0$, $d > 1$ and $r > 0$.

(e) The function $L_\lambda(r)$ is the shape of an i-convex hypersurface for $\lambda > 0$, $d > 0$, $r > 0$ and $r(1 - d^4) < \lambda d^3$.

Proof. First note that by Lemma 4.8 (b) we only need to prove the statements for $\lambda = 1$. Set $S := S_1$, $Q := Q_1$, $L := L_1$. We denote by $\sim$ equality up to multiplication by a positive factor.

(a) This holds because $R = S(r)$ describes a level set of the i-convex function $\phi(r, R) = ar^2 - R^2$ for $a > 1$.

(b) Condition (4.2) follows from

$$Q'(r)Q(r) - r = (b + cr)(1 + br + \frac{cr^2}{2}) - r \geq b + cr - r = b + (c-1)r > 0,$$

and condition (4.3) from

$$\begin{aligned}
\mathcal{L}^2(Q) &\geq \frac{c(r^2 - u^2)}{r^2} + \frac{(b+cr)u^2}{r^3} + \frac{(b+cr)^3}{r} - 1 - \frac{(b+cr)^2u^2}{r^2} \\
&\sim cr(r^2 - u^2) + (b+cr)u^2 + r^2(b+cr)^3 - r^3 - r(b+cr)^2u^2 \\
&= (c-1)r^3 + bu^2 + r^2(b+cr)^3 - ru^2(b+cr) \\
&\geq (c-1)r^3 + +r^2(b+cr)^3 - r^3(b+cr) \\
&= (c-1)r^3 + r^2(b+cr)^2\big(b + (c-1)r\big) > 0.
\end{aligned}$$

(c) Condition (4.2) follows as in (b) from

$$Q'(r)Q(r) - r \geq b + cr - r = 4 - c(1-r) - r \geq 4 - 4(1-r) = 4r - r > 0.$$

For condition (4.3) it suffices, by (b), to show that

$$\begin{aligned}
A &:= (c-1)r + (b+cr)^2\big(b + (c-1)r\big) \\
&= (c-1)r + \big(4 - c(1-r)\big)^2\big(4 - c(1-r) - r\big) > 0.
\end{aligned}$$

For $c > 1$ this follows from (b). For $c \leq 1$ we have $4 - c(1-r) \geq 3$ and $4 - c(1-r) - r \geq 3 - r$, hence

$$A \geq -r + 9(3 - r) = 27 - 10r > 0$$

for $r \leq 2$.

(d) Condition (4.2) follows from

$$L'(r)L(r) - r = d(1 + dr) - r = d + (d^2 - 1)r > 0,$$

and condition (4.3) from

$$\begin{aligned}
\mathcal{L}^2(L) &= \frac{du^2}{r^3} + \frac{d^3}{r} - \frac{1}{1+dr}\Big(1 + \frac{d^2u^2}{r^2}\Big) \\
&\sim (1+dr)du^2 + d^3r^2(1+dr) - r^3 - d^2ru^2 \\
&= du^2 + d^3r^2 + (d^4 - 1)r^3 > 0.
\end{aligned}$$

(e) Condition (4.2) follows from $r(1-d^2) < d^3/(1+d^2)$ via

$$L'(r)L(r) - r = d + (d^2-1)r \geq d - \frac{d^3}{1+d^2} = \frac{d}{1+d^2} > 0,$$

and condition (4.3) as in (d) from

$$\mathcal{L}^2(L) = du^2 + d^3r^2 + (d^4-1)r^3 \geq r^2\big(d^3 + (d^4-1)r\big) > 0.$$

□

Remark 4.12. It is useful note the bounds for the modulus of convexity μ_Σ for all J-convex shapes reviewed in Lemma 4.11

a) The normalized Levi form of the hyperboloid $ar^2 - R^2 = -\lambda^2$ is equal to $\mathbb{L}(T) = \frac{2a-2}{(a^2+a)r^2+R^2}|T|^2$ for $T \in \xi$ and the minimum $-M(\Sigma)$ of the negative normal curvature is equal to $-\frac{2}{(a^2+a)r^2+R^2}$. Hence $\mu(\Sigma) = a-1$.

b) Assuming $c = 1$ it can be deduced from the above proof that $\mu(\Sigma) \geq c-1$.

c) TO BE CONTINUED

Lemma 4.13. *(a) For $\lambda, c > 0$ and $d > b > 0$ the functions $Q_\lambda(r)$ and $L_\lambda(r)$ intersect at a unique point $\lambda r_{QL} > 0$, where $r_{QL} = 2(d-b)\lambda/c$.*

(b) For $\lambda > 0$ and $a > d^2 > 0$ the functions $L_\lambda(r)$ and $S_\lambda(r)$ intersect at a unique point $\lambda r_{SL} > 0$, where $r_{SL} = 2d\lambda/(a-d^2)$.

(c) For $\lambda, b > 0$, $a > c \geq 0$ and $2b^2(a+c)^2 < (a-c)^3$ the functions $S_\lambda(r)$ and $Q_\lambda(r)$ intersect at precisely two points $\lambda r_{SQ}, \lambda r'_{SQ}$ satisfying $0 < r_{SQ} < 4b/(a-c) < r'_{SQ}$. Moreover, the points r_{SQ} and r'_{SQ} depend smoothly on a, b, c.

Proof. (a) and (b) are simple computations, so we only prove (c). Again, by rescaling it suffices to consider the case $\lambda = 1$. First observe that for $x > 0$ and $\mu < 1$ we have $\sqrt{1+x} > 1 + \mu x/2$ provided that $1 + x > 1 + \mu x + \mu^2 x^2/4$, or equivalently, $x < 4(1-\mu)/\mu^2$. Applying this to $x = ar^2$, we find that $S(r) > 1 + \mu ar^2/2$ provided that

$$r^2 < \frac{4(1-\mu)}{a\mu^2}. \tag{4.15}$$

Hence if

$$1 + \frac{\mu ar^2}{2} = Q(r) = 1 + br + \frac{cr^2}{2}$$

for some $r > 0$ and $\mu < 1$ satisfying (4.15), then $S(r) > Q(r)$. Assuming $\mu a > c$, we solve the last equation for $r = 2b/(\mu a - c)$. Inequality (4.15) becomes

$$r^2 = \frac{4b^2}{(\mu a - c)^2} < \frac{4(1-\mu)}{a\mu^2},$$

or equivalently,

$$ab^2\mu^2 < (1-\mu)(\mu a - c)^2. \tag{4.16}$$

Now pick $\mu := (a+c)/2a$. The hypothesis $a > c$ implies $\mu < 1$ and $\mu a = (a+c)/2 > c$. With $\mu a - c = (a-c)/2$ and $1-\mu = (a-c)/2a$, inequality (4.16) becomes

$$ab^2\left(\frac{a+c}{2a}\right)^2 < \frac{a-c}{2a}\left(\frac{a-c}{2}\right)^2,$$

or equivalently,

$$2b^2(a+c)^2 < (a-c)^3.$$

Assume this inequality holds, so $S(r_+) > Q(r_+)$ at the point

$$r_+ = \frac{2b}{\mu a - c} = \frac{4b}{a-c}.$$

Now $f(r) := Q(r)^2 - S(r)^2$ is a polynomial of degree 4 satisfying $f(0) = 0$ and $f(r) \to +\infty$ as $r \to \pm\infty$. Since $b > 0$, we have $f(r) > 0$ for $r > 0$ close to zero and $f(r) < 0$ for $r < 0$ close to zero, so $f(r_-) = 0$ for some $r_- < 0$. By the preceding discussion we have $f(r_+) > 0$, so f has two more zeroes r_{SQ}, r'_{SQ} with $0 < r_{SQ} < r_+ < r'_{SQ}$. Since the 4 zeroes of f are distinct they are all nondegenerate, which implies smooth dependence on the parameters a, b, c. □

Now we can show

Lemma 4.14. *For every $a > 1$ and $\gamma > 0$ there exists a $0 < d < \gamma$ and an i-convex shape $\phi(r)$ which agrees with $S(r) = \sqrt{1+ar^2}$ for $r \geq \gamma$ and with $L(r) = 1 + dr$ for r close to 0.*

Proof. Pick $1 < c < a$. Pick $0 < b < 1$ such that $2b^2(a+c)^2 < (a-c)^3$ and $4b < \gamma(a-c)$. By Lemma 4.13, the i-convex shapes $S(r)$ and $Q(r) = 1+br+cr^2/2$ intersect at a point $0 < r_2 < 4b/(a-c) < \gamma$. Now pick $b < d < 1$ such that $r_1 := 2(d-b)/c$ satisfies $r_1 < r_2$ and $r_1 < d^3/(1-d^4)$. By Lemma 4.13, the functions $Q(r)$ and $L(r)$ intersect at the point r_1, and by Lemma 4.11 the function $L(r)$ is i-convex for $r \leq r_1$. Now the desired function is a smoothing of the function which equals $L(r)$ for $r \leq r_1$, $Q(r)$ for $r_1 \leq r \leq r_2$ and $S(r)$ for $r \geq r_2$. □

Combining the preceding lemma with Lemma 4.10 (for $\lambda = 1$), we obtain

Corollary 4.15. *For every $a > 1$ and $\gamma > 0$ there exists a $0 < \delta < \gamma$ and an i-convex shape $\phi(r)$ which agrees with $S(r) = \sqrt{1+ar^2}$ for $r \geq \gamma$ and satisfies $\phi'(\delta) = +\infty$ and $\phi(\delta) > 1$.*

4.3 Families of special shapes

In this section we construct a family of i-convex shapes interpolating between the function in Corollary 4.15 and the standard functions S_λ.

We begin by constructing another family of solutions to Struwe's differential equation (4.12).

Lemma 4.16. *For any $\delta > 0$ and $d \geq 4$ there exists a solution $\phi : [\delta, 2\delta] \to \mathbb{R}$ of (4.12) with the following properties:*

(a) $\phi'(\delta) = +\infty$ and $\phi(\delta) \geq d\delta$;

(b) $\phi(2\delta) = 2d\delta$ and $\phi'(2\delta) \leq d$;

(c) ϕ satisfies (4.13) and hence is an i-convex shape.

Proof. The proof is similar to the proof of Lemma 4.10. By rescaling, it suffices to consider the case $\delta = 1$. Define the solution ϕ by $\phi'(r) := 1/\sqrt{\ln r}$ and $\phi(2) := 2d$, thus

$$\phi(1) = 2d - \int_1^2 \frac{du}{\sqrt{\ln u}}.$$

Estimating the integral as in (4.14) and using $d \geq 4$, we find

$$\phi(1) \geq 2d - \frac{2}{\sqrt{\ln 2}} \geq d + 4 - \frac{2}{\sqrt{\ln 2}} \geq d,$$

since $\sqrt{\ln 2} \geq 1/2$. Concavity of ϕ implies $\phi(r) \geq dr$ for all $r \in [1, 2]$, and in particular $\phi'(2) \leq d$. So it only remains to check inequality (4.13). Denoting by $\sim$ equality up to a positive factor, we compute

$$\begin{aligned}\frac{\phi'^3}{2r} - \frac{1}{\phi}(1 + \phi'^2) &\geq \frac{\phi'^3}{2r} - \frac{1}{dr}(1 + \phi'^2) \\ &\sim d\phi'^3 - 2 - 2\phi'^2 \\ &\sim d - 2(\ln r)^{3/2} - 2(\ln r)^{1/2}.\end{aligned}$$

The function on the right hand side is decreasing in r. So its minimum is achieved for $r = 2$ and has the value

$$d - 2(\ln 2)^{3/2} - 2(\ln 2)^{1/2} > 4 - 2 - 2 = 0,$$

since $d \geq 4$ and $\sqrt{\ln 2} < 1$. □

Remark 4.17. For ϕ as in Lemma 4.16 and any constant $c \in \mathbb{R}$, the part of the function $\phi + c$ that lies above the linear function dr is i-convex. Indeed, the last part of the proof applied to $\phi + c$ estimates the quantity in inequality 4.13 by $d - 2(\ln r_1)^{3/2} - 2(\ln r_1)^{1/2}$, where r_1 is the larger intersection point of $\phi + c$ and dr. Since $r_1 \leq 2$, this is positive.

Extend the standard function to $\lambda < 0$ and $a > 1$ by

$$S_\lambda(r) := \sqrt{ar^2 - \lambda^2}, \qquad r \geq |\lambda|/\sqrt{a}.$$

Note that S_λ is the shape of an i-convex hypersurface because its graph is a level set of the i-convex function $\phi(r,R) = ar^2 - R^2$.

We say that a family of i-convex shapes $\psi_\lambda : [\delta, \gamma] \to \mathbb{R}_+$ with $\psi_\lambda'(\delta) = \infty$ is *smooth* if their graphs $\{R = \psi_\lambda(r)\}$, extended by the vertical line below $\psi_\lambda(\delta)$, form a smooth family of lines in the positive quadrant $Q \subset \mathbb{R}^2$.

Lemma 4.18. *Let $L_\lambda(r) = \lambda + d_\lambda r$, $0 < r \leq \gamma$, $0 \leq \lambda \leq 1$, be an increasing smooth family of i-convex shapes, where $\lambda \mapsto d_\lambda$ is decreasing with $d_0 = 8$ and $0 < d_1 \leq 1$. Then for any sufficiently small $\delta \in (0, \gamma/4)$ there exists a smooth family of increasing i-convex shapes $\psi_\lambda : [\delta, \gamma] \to \mathbb{R}$, $-8\delta \leq \lambda \leq 1$, with the following properties:*

(a) $\psi_{-8\delta}(r) = \sqrt{64r^2 - 64\delta^2}$ for all $r \geq \delta$;

(b) $\psi_\lambda(r) = \sqrt{64r^2 - \lambda^2}$ for $-8\delta \leq \lambda \leq 0$ and $r \geq \gamma/2$;

(c) $\psi_\lambda(r) = L_\lambda(r)$ for $0 \leq \lambda \leq 1$ and $r \geq \gamma/2$.

(d) $\psi_\lambda'(\delta) = \infty$ for all λ;

(e) $\psi_1(\delta) > 1$.

Proof. (1) For each $\lambda \in (0,1]$, set $K_\lambda := e^{4/d_\lambda^2}$. Pick a smooth family of $\delta_\lambda > 0$ such that $\lambda\delta_\lambda$ increases with λ and

$$4K_\lambda\delta_\lambda \leq (\ln K_\lambda)^{-3/2}, \qquad K_\lambda\lambda\delta_\lambda < \gamma/2.$$

By Lemma 4.10, there exist i-convex solutions $\phi_\lambda : [\lambda\delta_\lambda, K_\lambda\lambda\delta_\lambda] \to \mathbb{R}$ of (4.12) satisfying

- $\phi_\lambda'(\lambda\delta_\lambda) = +\infty$ and $\phi_\lambda(\lambda\delta_\lambda) \geq \lambda + d_\lambda\lambda\delta_\lambda$;
- $\phi_\lambda(K_\lambda\lambda\delta_\lambda) = \lambda + d_\lambda K_\lambda\lambda\delta_\lambda$ and $\phi_\lambda'(K_\lambda\lambda\delta_\lambda) \leq d_\lambda$.

(2) From $d_0 = 8$ and $d_1 < 1$ we conclude $K_0 = e^{1/16} < 2$ and $K_1 \geq e^4 > 2$. Hence there exists a $0 < \bar\lambda < 1$ with $K_{\bar\lambda} = 2$. Set $\bar\delta := \bar\lambda\delta_{\bar\lambda} < \gamma/4$. By Lemma 4.16 (with $d = 8$), there exists an i-convex solution $\bar\phi : [\bar\delta, 2\bar\delta] \to \mathbb{R}$ of (4.12) satisfying

- $\bar\phi'(\bar\delta) = +\infty$ and $\bar\phi(\bar\delta) \geq 8\bar\delta$;
- $\bar\phi(2\bar\delta) = 16\bar\delta$ and $\bar\phi'(2\bar\delta) \leq 8$.

By Lemma 4.8 (a), the functions

$$\bar\phi_\lambda := \bar\phi(r) + L_\lambda(2\bar\delta) - L_0(2\bar\delta) \geq \bar\phi(r)$$

are i-convex for $0 \le \lambda \le \bar\lambda$ and $\bar\delta \le r \le 2\bar\delta$. Note that the functions $\phi_{\bar\lambda}$ and $\bar\phi_{\bar\lambda}$ have the same value at $r = 2\bar\delta$ and derivative ∞ at $r = \bar\delta$. Since they both solve the second order differential equation (4.12), they coincide on $[\bar\delta, 2\bar\delta]$. Thus the families constructed in (1) and (2) fit together to a continuous family $(\hat\phi_\lambda)_{\lambda\in[0,1]}$ with $\hat\phi_\lambda = \phi_\lambda : [\lambda\delta_\lambda, K_\lambda\lambda\delta_\lambda] \to \mathbb{R}_+$ for $\lambda \ge \bar\lambda$, and $\hat\phi_\lambda = \bar\phi_\lambda : [\bar\delta, 2\bar\delta] \to \mathbb{R}_+$ for $\lambda \le \bar\lambda$. Set $\bar\delta_\lambda := \lambda\delta_\lambda$ for $\lambda \ge \bar\lambda$ and $\bar\delta_\lambda := \bar\delta$ for $\lambda \le \bar\lambda$ and define $\tilde\phi_\lambda : [\bar\delta_\lambda, \gamma] \to \mathbb{R}_+$ by

$$\tilde\phi_\lambda(r) := \begin{cases} \hat\phi_\lambda(r) & \text{for } r \le K_\lambda\delta_\lambda, \\ L_\lambda(r) & \text{for } r \ge K_\lambda\delta_\lambda. \end{cases}$$

After smoothing, the family $\tilde\phi_\lambda$ is i-convex and agrees with L_λ for $r \ge \gamma/2$.

(3) For $-8\bar\delta \le \tau \le 0$ consider the functions $\bar\phi_\tau := \bar\phi + \tau : [\bar\delta, 2\bar\delta] \to \mathbb{R}_+$. By Remark 4.17, the portion of $\bar\phi_\tau$ above the linear function L_0 is i-convex. Thus for $0 < \delta < \bar\delta/2$ sufficiently small, the portion of $\bar\phi_\tau$ above the function $S_{-8\delta}$ is i-convex. Here $S_\lambda(r) = \sqrt{64r^2 - \lambda^2}$ is the standard function defined above with $a = 64$ and $\lambda \in [-8\delta, 0]$. For $-8\delta \le \lambda \le 0$ define $\tilde\phi_\lambda : [\bar\delta, \gamma] \to \mathbb{R}_+$ by

$$\tilde\phi_\lambda(r) := \begin{cases} \bar\phi(r) + S_\lambda(2\bar\delta) - S_0(2\bar\delta) & \text{for } r \le 2\bar\delta, \\ S_\lambda(r) & \text{for } r \ge 2\bar\delta. \end{cases}$$

Since $S_\lambda(r) - S_0(r)$ is increasing in r for $\lambda > 0$, the condition $\bar\phi(\bar\delta) \ge 8\bar\delta$ ensures that $\tilde\phi_\lambda$ lies above S_λ. Thus after smoothing, the family $\tilde\phi_\lambda$ is i-convex for $-8\delta \le \lambda \le 1$ and agrees with L_λ (if $\lambda \ge 0$) resp. S_λ (if $\lambda \le 0$) for $r \ge \gamma/2$. Now define $\tilde\psi_\lambda : [\delta, \gamma] \to \mathbb{R}_+$ by

$$\tilde\phi_\lambda(r) := \begin{cases} S_{-8\delta(r)} & \text{for } r \le \bar\delta_\lambda, \\ \tilde\phi(r) & \text{for } r \ge \bar\delta_\lambda. \end{cases}$$

After smoothing, the family $\tilde\psi_\lambda$ is i-convex for $-8\delta \le \lambda \le 1$ and satisfies conditions (b-d).

(4) To arrange condition (a), note that $\tilde\psi_{-8\delta} = \max(S_{-8\delta}, \bar\phi_{\bar\tau})$ for some $\bar\tau < 0$. By the discussion above, the functions $\max(S_{-8\delta}, \bar\phi_\tau)$ are i-convex for $-8\bar\delta \le \tau \le 0$. For δ sufficiently small, we have $\max(S_{-8\delta}, \bar\phi_{-8\bar\delta}) = S_{-8\delta}$. After rescaling in the parameter λ, this yields a family $\tilde\psi_\lambda$ satisfying condition (a-d).

(5) To arrange condition (e), set $\delta_t := (2-t)\delta_1 + (t-1)\delta$ for $t \in [1,2]$ and let $\phi_t : [\delta_t, K_1\delta_t] \to \mathbb{R}$ be the i-convex shape from Lemma 4.10 with $\lambda = 1$ and δ replaced by δ_t. For $\lambda \in [1,2]$ define $\tilde\psi_\lambda : [\delta, \gamma] \to \mathbb{R}_+$ by

$$\tilde\phi_\lambda(r) := \begin{cases} S_{-8\delta(r)} & \text{for } r \le \delta_\lambda, \\ \phi_\lambda(r) & \text{for } \delta_\lambda \le r \le \delta_1, \\ L_1(r) & \text{for } r \ge \delta_1. \end{cases}$$

For $\lambda = 1$ this matches the previous family $\tilde\psi_\lambda$, so rescaling in λ yields the desired family ψ_λ. □

The following result is a family version of Lemma 4.14.

Lemma 4.19. *For any $\rho > 0$ there exists a constant $0 < \gamma < \rho$ and a smooth family of increasing i-convex shapes $\phi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, $\lambda \in [0,1]$, with the following properties:*

(a) $\phi_0(r) = 8r$ for all r;

(b) $\phi_\lambda(r) = \lambda + d_\lambda r$ for $r \le \gamma$ and all λ, where $\lambda \mapsto d_\lambda$ is decreasing with $d_0 = 8$ and $0 < d_1 \le 1$;

(c) $\phi_\lambda(r) = \sqrt{64r^2 + \lambda^2}$ for $r \ge \rho$ and all λ.

Proof. Set $a := 64$ and $c := 2$. With this choice and $\lambda \in (0,1]$ we consider the functions

$$S_\lambda(r) = \sqrt{\lambda^2 + ar^2}, \qquad Q_{b,\lambda}(r) = \lambda + br + cr^2/2\lambda, \qquad L_{d,\lambda}(r) = \lambda + dr$$

as above. Here the constants b, d will vary in the course of the proof but always satisfy the condition

$$0 < b < d \le b + b^3 < 8. \tag{4.17}$$

Then the numerical condition in Lemma 4.13 (c), $2b^2(64+2)^2 < (64-2)^3$, holds because $b < 4$. Hence all the numerical conditions in Lemma 4.13 are satisfied, so the functions $S_\lambda, Q_{b,\lambda}, L_{d,\lambda}$ intersect at points $\lambda r_{QL}(b,d), \lambda r_{SL}(d), \lambda r_{SQ}(b)$ satisfying

$$r_{QL}(b,d) = \frac{2(d-b)}{c}, \qquad r_{SL}(d) = \frac{2d}{a-d^2}, \qquad 0 < r_{SQ}(b) < \frac{4b}{a-c}.$$

By condition (4.17) we have

$$r_{QL}(b,d)(1-d^4) \le r_{QL}(b,d) \le b^3 < d^3,$$

so the numerical condition in Lemma 4.11 (e) is satisfied for $r \le \lambda r_{QL}(b,d)$. It follows that the shape functions $S_\lambda(r)$ and $Q_{b,\lambda}(r)$ are i-convex for all r, and $L_{d,\lambda}(r)$ is i-convex for $r \le \lambda r_{QL}(b,d)$. For each triple (b,d,λ) we consider the function

$$\psi_{b,d,\lambda} := \max(S_\lambda, Q_{b,\lambda}, L_{d,\lambda}) = \lambda\psi_{b,d,1}(\cdot/\lambda).$$

This function will be i-convex provided that the region where it coincides with $L_{d,\lambda}(r)$ is contained in the interval $[0, \lambda r_{QL}(b,d)]$. We say that $\psi_{b,d,\lambda}$ is of type

(a) if $r_{QL}(b,d) \le r_{SL}(d) <\le r_{SQ}(b)$;

(b) if $r_{SQ}(b) <\le r_{SL}(d) \le r_{QL}(b,d)$;

(c) if $r_{SQ}(b) <\le r_{QL}(b,d) \le r_{SL}(d)$;

see Figure [fig:???] (where we have droped the parameters b, d, λ). Thus the function $\psi_{b,d,\lambda}$ is i-convex for types (a) and (b), but not necessarily for type (c).

After these preparations, we now construct the family ϕ_λ in 4 steps.

Step 1. Consider $\lambda = 1$. Pick a pair (b_1, d_1) satisfying (4.17) and such that

$$r_{QL}(b_1, d_1) = \frac{2(d_1 - b_1)}{c} < r_{SQ}(b_1) < \frac{4b_1}{a - c} < \rho.$$

Then the shape function $\psi_{b_1,d_1,\lambda}$ is of type (a) and therefore i-convex for all $\lambda > 0$, and it agrees with S_λ for $r \geq \gamma$. Note that in particular we have $r_{SL}(d_1) < \rho$.

Step 2. Fix a parameter $0 < \lambda^* < \rho/8$. This condition ensures that for any pair (b, d) satisfying (4.17) we have $\lambda^* r_{QL}(b, d), \lambda^* r_{SQ}(b) < \rho$. We may assume that b_1 is Step 1 is chosen so small that $b_1^2 < c/(a - b^2)$ for all $b \in [0, b_1]$. Then for any $b \in [0, b_1]$ such that (b, d_1) satisfies (4.17) we have

$$r_{QL}(b, d_1) = \frac{2(d_1 - b)}{c} \leq \frac{2b^3}{c} < \frac{2b}{a - b^2} < \frac{2d_1}{a - d_1^2} = r_{SL}(d_1) < \rho.$$

Let $b_1^* \in (0, b_1]$ be the solution of $b_1^* + (b_1^*)^3 = d_1$. We claim that for all $b \in [b_1^*, b_1]$ the function ψ_{b,d_1,λ^*} is of type (a) and therefore i-convex. Indeed, by Step 1 this holds for $b = b_1$. Since $r_{SQ}(b)$ depends smoothly on b, if ψ_{b,d_1,λ^*} changes its type there must exist a $b \in [b_1^*, b_1]$ for which $r_{SQ}(b) = r_{SL}(d_1)$. But this implies also $r_{QL}(b, d_1) = r_{SL}(d_1)$, contradicting the preceding inequality.

Step 3. For $b > 0$ consider the function

$$f(b) := \frac{r_{QL}(b,d)}{r_{SL}(d)}\Big|_{d=b+b^3} = \frac{(d-b)(a-d^2)}{cd}\Big|_{d=b+b^3} = \frac{b^2\big(a - (b+b^3)^2\big)}{c(1+b^2)}.$$

A short computation shows that $f(0) = 0$, $f(1) > 1$ and $f'(b) > 0$ for all $b \in (0, 1)$. Thus there exists a unique $b_2^* \in (0, 1)$ with $f(b_2^*) = 1$, i.e. $r_{QL}(b, b + b^3) = r_{SL}(b + b^3)$ precisely for $b = b_2^*$. Since $b_1^* < b_2^*$, the function $\psi_{b,b+b^3,\lambda^*}$ is of type (a) and therefore i-convex for all $b \in [b_1^*, b_2^*]$. For $b \in [b_2^*, 1]$ we have $r_{QL}(b, b + b^3) \geq r_{SL}(b + b^3)$, so the function $\psi_{b,b+b^3,\lambda^*}$ is of type (b) and therefore also i-convex. Combining this, we see that the function $\psi_{b,b+b^3,\lambda^*}$ is i-convex for all $b \in [b_1^*, 1]$. Moreover, $\lambda^* r_{SL}(b + b^3) < \rho$ for all $b \in [b_1^*, 1]$, so $\psi_{b,b+b^3,\lambda^*}(r) = S^*_\lambda(r)$ for $r \geq \rho$.

Step 4. The previous step leads for $b = 1$ and $d = b + b^3 = 2$ to the function $\psi_{1,2,\lambda^*}$. For $d \in [2, 8]$ define b_d, λ_d by the conditions

$$b_d + b_d^3 = d, \qquad \lambda_d r_{SL}(d) = \rho,$$

so

$$\lambda_d = \frac{\rho(a - d^2)}{2d}.$$

Note that $b_2 = 1$, $\lambda_2 > \lambda^*$, and $\psi_{1,\lambda,2}$ is i-convex for all $\lambda \in [\lambda^*, \lambda_2]$ and agrees with S_λ for $r \geq \rho$. The same holds for the functions ψ_{b_d,d,λ_d} for all $d \in [2, 8]$. In the limit $d \to 8$ we find $\lambda_8 = 0$ and thus the linear function

$$\psi_{b_8,8,0}(r) = 8r.$$

Now we combine the homotopies of i-convex functions $\psi_{b,d,\lambda}$ in Steps 1-4: Starting from $(b_1, d_1, 1)$ we first decrease λ to (b_1, d_1, λ^*) (Step 1), then decrease b to (b_1^*, d_1, λ^*) (Step 2), next increase (b, d) simultaneously do $(1, 2, \lambda^*)$ (Step 3), and finally increase (b, d) and decrease λ simultaneously to $(b_8, 8, 0)$. By construction, each function $\psi_{b,d,\lambda}$ during this homotopy coincides with the corresponding standard function S_λ for $r \geq \rho$ and with the linear function L_λ for $r \leq \gamma$ for some small $\gamma > 0$. Moreover, during the homotopy λ is non-increasing and d is non-decreasing. Smoothen the functions $\psi_{b,d,\lambda}$ and perturb the homotopy such that λ is strictly decreasing from 1 to 0 and d is strictly increasing from $d_1 \leq 1$ to 8. The resulting homotopy, parametrized by $\lambda \in [0, 1]$, is the desired family ϕ_λ. □

Now we can prove the main result of this chapter.

Proposition 4.20. *For all $0 < \rho < \varepsilon$ and any sufficiently small $\delta \in (0, \rho)$ there exists a smooth family of increasing i-convex shapes $\psi_\lambda : [\delta, \varepsilon] \to \mathbb{R}$, $-8\delta \leq \lambda \leq 1$, with the following properties:*

(a) $\psi_{-8\delta}(r) = \sqrt{64r^2 - 64\delta^2}$ for all $r \geq \delta$;

(b) $\psi_\lambda(r) = \sqrt{64r^2 - \lambda^2}$ for $r \geq \rho$ and all λ;

(c) $\psi'_\lambda(\delta) = \infty$ for all λ;

(e) $\psi_1(\delta) > 1$.

Proof. Let $(\phi_\lambda)_{\lambda \in [0,1]}$ be the family of i-convex shapes from Lemma 4.19. They agree with the standard functions $\sqrt{64r^2 - \lambda^2}$ for $r \geq \rho$ and with the linear functions $\lambda + d_\lambda r$ for $r \leq \gamma$. Since $\phi_0(r) = 8r$, we can extend the family by $\phi_\lambda(r) := \sqrt{64r^2 - \lambda^2}$ for $\lambda < 0$ and $r \geq \lambda/8$. Check!

Let $(\psi_\lambda)_{\lambda \in [-8\delta, 1]}$ be the family from Lemma 4.18 which agrees with $\lambda + d_\lambda r$ for $r \geq \gamma/2$ and $\lambda \in [0, 1]$, and with $\sqrt{64r^2 - \lambda^2}$ for $r \geq \gamma/2$ and $\lambda \in [-8\delta, 0]$. So the families ψ_λ and ϕ_λ fit together to a family of i-convex shapes $[\delta, \varepsilon]$ with the desired properties. □

Change below here!

Define the standard i-convex function,

$$\psi_{\rm st}(r, R) := 64r^2 - R^2 + 1.$$

on the handle H_ε.

Proposition 4.21. *For all $0 < \rho < \varepsilon$ and $\beta > 0$ there exists a smooth family of i-convex functions $\psi_t : H_\varepsilon \to \mathbb{R}$, $t \in [0, 1]$, with the following properties:*

(a) $\psi_0 = \psi_{\rm st}$;

(b) $\psi_t = \psi_{\rm st}$ on the set $\{\psi_{\rm st} \leq -\beta\}$ and near $r = 0$ for all $t \in [0, 1]$;

(c) $\psi_t = f_t \circ \psi_{\rm st}$ for $r \geq \rho$, where $f_t : [-\beta, 1 + 64\varepsilon^2] \to \mathbb{R}$ are strictly increasing with $f_0 = \mathbb{1}$, $f_t(x) = x$ near $x = -\beta$, and $f_1(0) > 1$;

(d) all ψ_t have 0 *as the only critical point.*

Proof. Let $(\phi_\lambda)_{\lambda\in[0,1]}$ be the family of i-convex shape functions from Lemma 4.19. They agree with the linear functions $\lambda + d_\lambda r$ for $r \leq \gamma$. Let $(\psi_\lambda)_{\lambda\in[-8\delta,1]}$ be the family from Lemma 4.18 which agrees with $\lambda + d_\lambda r$ for $r \geq \gamma/2$ and $\lambda \in [0,1]$. Since $\phi_0(r) = 8r$ and $\psi_{-8\delta}(r) = \sqrt{64r^2 - 64\delta^2}$ we can extend the families ϕ_λ and ψ_λ by $S_\lambda(r) = \sqrt{64r^2 - \lambda^2}$ for $\lambda \in [-8\varepsilon, 0]$ resp. $\lambda \in [-8\varepsilon, -8\delta]$. The extended families ψ_λ and ϕ_λ fit together to a family of i-convex shape functions $\chi_\lambda : [0,\varepsilon] \to \mathbb{R}_+$, $\lambda \in [-8\varepsilon, 1]$, with the following properties:

(a) $\chi_\lambda(r) = S_\lambda(r)$ for $r \geq \rho$ and all λ, as well as for $\lambda \leq -8\delta$ and all r;

(b) $\chi_\lambda(r) = S_{-8\delta}(r)$ for $\delta \leq r \leq 2\delta$ and $-8\delta \leq \lambda \leq 1$.

It will be convenient to perform a bijective continuous change of parameters from λ to $\mu = \mu(\lambda) \in [0, 1 + 64\varepsilon^2]$ defined by

$$\mu(\lambda) := \begin{cases} 1 + \lambda^2 : & \lambda < 0, \\ 1 - \lambda^2 : & \lambda \geq 0, \end{cases}$$

so $S_\lambda(r) = \sqrt{64r^2 + 1 - \mu}$. Note that the family of hypersurfaces $\{R = \chi_\mu(r)\}$, $\mu \in [0, 1 + 64\varepsilon^2]$, is transverse to the vector field

$$X = \sum_{i=1}^{k}\left(x_i\frac{\partial}{\partial x_i} - y_i\frac{\partial}{\partial y_i}\right) + \sum_{j=k+1}^{n}\left(x_j\frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial y_j}\right).$$

Hence by Proposition 3.28 we can perturb it to a foliation by i-convex hypersurfaces Σ_μ transverse to X with the following properties:

(a) $\Sigma_\mu = \{R = \sqrt{64r^2 + 1 - \mu}\}$ for $r \geq \rho$ and all μ;

(b) $\Sigma_0 = \{R = \sqrt{64r^2 - 64\delta^2}\}$ for $\delta \leq r \leq 2\delta$.

Extend the foliation Σ_μ to $\mu \in [-\alpha, 0]$, for $\alpha \in (0,\beta)$ sufficiently small, such that $\Sigma_\mu(r) = \sqrt{64r^2 + 1 - \mu}$ for $r \geq \rho$ and all $\mu \in [-\alpha, 1 + 64\varepsilon^2]$. Define a function $\psi : H_\varepsilon \to \mathbb{R}$ by $\psi := \mu$ on Σ_μ, and $\psi := -\alpha$ otherwise. Pick an increasing convex smooth function $f : [-\alpha, 1+64\varepsilon^2] \to \mathbb{R}$ such that $f(\mu) = \mu$ near $\mu = -\alpha$, $f(0) > 1$, and $f \circ \psi$ is i-convex on $\cup_\mu \Sigma_\mu$. Note that $f \circ \psi = f \circ \psi_{\rm st}$ on the region $\{\psi_{\rm st} \geq -\alpha, r \geq \rho\}$ and $f \circ \psi \equiv -\alpha \leq \psi_{\rm st}$ on the region $\{\psi_{\rm st} \geq -\alpha\} \setminus \cup_\mu \Sigma_\mu$.

Pick a constant $c > \max_{H_\varepsilon}(f\circ\psi) - \min_{H_\varepsilon}\psi_{\rm st}$ and for $t \in [0,1]$ define $\psi_t : H_\varepsilon \to \mathbb{R}$ by

$$\psi_t := \max\{\psi_{\rm st}, f \circ \psi + c(t-1)\}$$

on $\{\psi_{\rm st} \geq -\alpha\}$ and $\psi_t := \psi_{\rm st}$ on $\{\psi_{\rm st} \leq -\alpha\}$. After smoothing, the functions ψ_t will be i-convex and we claim that they have the desired properties. Indeed, properties (a) and (b) are immediate from the construction. Property (c) holds with (a smoothing of) the function $f_t := \max\{\mathbb{1}, f + c(t-1)\}$. Property (d) holds because $\psi_t = \psi_{\rm st}$ near 0, and away from 0 the level sets of the functions $\psi_{\rm st}$ and ψ are transverse to the vector field X above, so by Corollary 3.23 taking the maximum and smoothing does not create any new critical points. □

Part II

Further Techniques

Chapter 5

Symplectic and Contact Preliminaries

In this chapter we collect some relevant facts from symplectic and contact geometry. For more details see [47].

5.1 Symplectic vector spaces

A *symplectic vector space* (V,ω) is a (finite dimensional) vector space V with a nondegenerate skew-symmetric bilinear form ω. Here nondegenerate means that $v \mapsto \omega(v,\cdot)$ defines an isomorphism $V \mapsto V^*$. A linear map $\Psi : (V_1,\omega_1) \to (V_2,\omega_2)$ between symplectic vector spaces is called *symplectic* if $\Psi^*\omega_2 \equiv \omega_2(\Psi\cdot,\Psi\cdot) = \omega_1$.

For any vector space U the space $U \oplus U^*$ carries the *standard symplectic structure*

$$\omega_{\rm st}\Big((u,u^*),(v,v^*)\Big) := v^*(u) - u^*(v).$$

In coordinates q_i on U and dual coordinates p_i on U^*, the standard symplectic form is given by

$$\omega_{\rm st} = \sum dq_i \wedge dp_i.$$

Define the ω*-orthogonal complement* of a linear subspace $W \subset V$ by

$$W^\omega := \{v \in V \mid \omega(v,w) = 0 \text{ for all } w \in W\}.$$

Note that $\dim W + \dim W^\omega = 2n$, but $W \cap W^\omega$ need not be $\{0\}$. W is called

- *symplectic* if $W \cap W^\omega = \{0\}$;
- *isotropic* if $W \subset W^\omega$;

- *coisotropic* if $W^\omega \subset W$;
- *Lagrangian* if $W^\omega = W$.

Note that $\dim W$ is even for W symplectic, $\dim W \leq n$ for W isotropic, $\dim W \geq n$ for W coisotropic, and $\dim W = n$ for W Lagrangian. Note also that $(W^\omega)^\omega = W$, and $\big(W/(W \cap W^\omega), \omega\big)$ is a symplectic vector space.

Consider a subspace W of a symplectic vector space (V, ω) and set $N := W \cap W^\omega$. Choose subspaces $V_1 \subset W$, $V_2 \subset W^\omega$ and $V_3 \subset (V_1 \oplus V_2)^\omega$ such that

$$W = V_1 \oplus N, \quad W^\omega = N \oplus V_2, \quad (V_1 \oplus V_2)^\omega = N \oplus V_3.$$

Then the decomposition

$$V = V_1 \oplus N \oplus V_2 \oplus V_3$$

induces a symplectic isomorphism

$$\begin{aligned} (V, \omega) &\to (W/N, \omega) \oplus (W^\omega/N, \omega) \oplus (N \oplus N^*, \omega_{\text{st}}), \\ v_1 + n + v_2 + v_3 &\mapsto \Big(v_1, v_2, (n, -i_{v_3}\omega)\Big). \end{aligned} \tag{5.1}$$

Every symplectic vector space (V, ω) of dimension $2n$ possesses a *symplectic basis* $e_1, f_1, \dots, e_n, f_n$, i.e. a basis satisfying

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0, \qquad \omega(e_i, e_j) = \delta_{ij}.$$

Moreover, given a subspace $W \subset V$, the basis can be chosen such that

- $W = \text{span}\{e_1, \dots, e_{k+l}, f_1, \dots, f_k\}$;
- $W^\omega = \text{span}\{e_{k+1}, \dots, e_n, f_{k+l+1}, \dots, f_n\}$;
- $W \cap W^\omega = \text{span}\{e_{k+1}, \dots, e_{k+l}\}$.

In particular, we get the following normal forms:

- $W = \text{span}\{e_1, f_1, \dots, e_k, f_k\}$ if W is symplectic;
- $W = \text{span}\{e_1, \dots, e_k\}$ if W is isotropic;
- $W = \text{span}\{e_1, \dots, e_n, f_1, \dots, f_k\}$ if W is coisotropic;
- $W = \text{span}\{e_1, \dots, e_n\}$ if W is Lagrangian.

This reduces the study of symplectic vector spaces to the *standard symplectic space* $(\mathbb{R}^{2n}, \omega_{\text{st}} = \sum dq_i \wedge dp_i)$.

A pair (ω, J) of a symplectic form ω and a complex structure J on a vector space V is called *compatible* if

$$g_J := \omega(\cdot, J\cdot)$$

is an inner product (i.e. symmetric and positive definite). This is equivalent to saying that

$$H(v, w) := \omega(Jv, w) - i\omega(v, w)$$

defines a Hermitian metric. Therefore, we will also call a compatible pair (ω, J) a *Hermitian structure.*

Lemma 5.1. *(a) The space of symplectic forms compatible with a given complex structure is nonempty and contractible.*

(b) The space of complex structures compatible with a given symplectic form is nonempty and contractible.

Proof. (a) immediately follows from the fact that the Hermitian metrics for a given complex structure form a convex space.

(b) is a direct consequence of the following fact (see [47]): For a symplectic vector space (V, ω) there exists a continuous map from the space of inner products to the space of compatible complex structures which maps each induced inner product g_J to J.

To see this fact, note that an inner product g defines an isomorphism $A : V \to V$ via $\omega(\cdot, \cdot) = g(A\cdot, \cdot)$. Skew-symmetry of ω implies $A^T = -A$. Recall that each positive definite operator P possesses a unique positive definite square root $\sqrt{P}$, and $\sqrt{P}$ commutes with every operator with which P commutes. So we can define

$$J_g := (AA^T)^{-\frac{1}{2}} A.$$

It follows that $J_g^2 = -\mathbb{1}$ and $\omega(\cdot, J\cdot) = g(\sqrt{AA^T}\cdot, \cdot)$ is an inner product. Continuity of the mapping $g \mapsto J_g$ follows from continuity of the square root. Finally, if $g = g_J$ for some J then $A = J = J_g$. □

Let us call a subspace $W \subset V$ of a complex vector space (V, J)

- *totally real* if $W \cap JW = \{0\}$,
- *totally coreal* if $W + JW = V$,
- *maximally real* if $W \cap JW = \{0\}$ and $W + JW = V$,
- *complex* if $JW = W$.

Note that $\dim W \leq n$ if W is totally real, $\dim W \geq n$ if W is totally coreal, and $\dim W = n$ if W is maximally real.

Recall that a Hermitian vector space (V, J, ω) is a complex vector space with a J-invariant symplectic form ω. Denote by $W^\perp$ the orthogonal complement with respect to the metric $\langle v, w \rangle := \omega(v, Jw)$. The following lemma relates the symplectic and complex notions on a Hermitian vector space. It follows easily from the relation $W^\omega = (JW)^\perp = J(W^\perp)$.

Lemma 5.2. *Let (V, J, ω) be a Hermitian vector space and $W \subset V$ a subspace. Then*

(a) W isotropic $\iff JW \subset W^\perp \implies W$ totally real;

(b) W coisotropic $\iff W^\perp \subset JW \implies W$ totally coreal;

(a) W Lagrangian $\iff JW = W^\perp \implies W$ maximally real;

(c) W complex $\implies W$ symplectic.

5.2 Symplectic vector bundles

The discussion of the previous section immediately carries over to vector bundles. For this, let $E \to M$ be a real vector bundle of rank $2n$ over a manifold. A *symplectic structure* on E is a smooth section ω in the bundle $\Lambda^2 E^* \to M$ such that each $\omega_x \in \Lambda^2 E_x^*$ is a linear symplectic form. A pair (ω, J) of a symplectic and a complex structure on E is called *compatible*, or a *Hermitian structure*, if $\omega(\cdot, J\cdot)$ defines an inner product on E. Lemma 5.1 immediately yields the following facts, where the spaces of sections are equipped with any reasonable topology, e.g. the $C^\infty_{\rm loc}$ topology:

(a) The space of compatible complex structures on a symplectic vector bundle (E, ω) is nonempty and contractible.

(b) The space of compatible symplectic structures on a complex vector bundle (E, J) is nonempty and contractible.

This shows that the homotopy theories of symplectic, complex and Hermitian vector bundles are the same. In particular, obstructions to trivialization of a symplectic vector bundle (E, ω) are measured by the *Chern classes* $c_k(E, \omega) = c_k(E, J)$ for any compatible complex structure J.

Remark 5.3. The homotopy equivalence between symplectic, complex and Hermitian vector bundles can also be seen in terms of their structure groups: The symplectic group [1]

$$Sp(2n) := \{\Psi \in GL(2n, \mathbb{R}) \mid \Psi^*\omega = \omega\} = \{\Psi \in GL(2n, \mathbb{R}) \mid \Psi^T J \Psi = J\}$$

and the general complex linear group $GL(n, \mathbb{C})$ both deformation retract onto the unitary group

$$U(n) = Sp(2n) \cap O(2n) = O(2n) \cap GL(n, \mathbb{C}) = GL(n, \mathbb{C}) \cap Sp(2n).$$

We end this section with a normal form for subbundles of symplectic vector bundles.

Proposition 5.4. *Let (E, ω) be a rank $2n$ symplectic vector bundle and $W \subset E$ a rank $2k + l$ subbundle such that $N := W \cap W^\omega$ has constant rank l. Then*

$$(E, \omega) \cong (W/N, \omega) \oplus (W^\omega/N, \omega) \oplus (N \oplus N^*, \omega_{\rm st}).$$

[1] $Sp(2n)$ is *not* the "symplectic group" $Sp(n)$ considered in Lie group theory. E.g., the latter is compact, while our symplectic group is not.

Proof. Pick a compatible almost complex structure J on (E, ω). Then

$$V_1 := W \cap JW, \quad V_2 := W^\omega \cap JW^\omega, \quad V_3 := JN$$

are smooth subbundles of E. Now the isomorphism 5.1 of the previous section yields the desired decomposition. □

5.3 Symplectic manifolds

A *symplectic manifold* (V, ω) is a manifold V with a closed nondegenerate 2-form ω. A map $f : (V_1, \omega_1) \to (V_2, \omega_2)$ between symplectic manifolds is called *symplectic* if $f^*\omega_2 = \omega_1$, and a symplectic diffeomorphism is called *symplectomorphism*. The following basic results states that every symplectic manifold of dimension $2n$ is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_{\rm st})$. In other words, every symplectic manifold possesses a *symplectic atlas*, i.e. an atlas all of whose transition maps are symplectic.

Proposition 5.5 (symplectic Darboux Theorem)**.** *Let (V, ω) be a symplectic manifold of dimension $2n$. Then every $x \in V$ possesses a coordinate neighborhood U and a coordinate map $\phi : U \to U' \subset \mathbb{R}^{2n}$ such that $\phi^*\omega_{st} = \omega$.*

The symplectic Darboux Theorem is a special case of the Symplectic neighborhood Theorem which will be proved in the next section. Now let us discuss some examples of symplectic manifolds.

Cotangent bundles. Let $T^*Q \xrightarrow{\pi} Q$ be the cotangent bundle of a manifold Q. The 1-form $\sum p_i dq_i$ is independent of coordinates q_i on Q and dual coordinates p_i on T^*_qQ and thus defines the *Liouville 1-form* $\lambda_{\rm st}$ on T^*Q. Intrinsically,

$$(\lambda_{\rm st})_{(q,p)} \cdot v = \langle p, T_{(q,p)}\pi \cdot v \rangle \quad \text{for } v \in T_{(q,p)}T^*Q,$$

where $\langle \;, \; \rangle$ is the pairing between T^*_qQ and T_qQ. The 2-form $\omega_{\rm st} := -d\lambda_{\rm st}$ is Sign?
clearly closed, and the coordinate expression $\omega_{\rm st} = \sum dq_i \wedge dp_i$ shows that it is also nondegenerate. So $\omega_{\rm st}$ defines the *standard symplectic form* on T^*Q. The standard form on $\mathbb{R}^{2n}$ is a particular case of this construction.

Almost complex submanifolds. A pair (ω, J) of a symplectic form and an almost complex structure on V is called *compatible* if $\omega(\cdot, J\cdot)$ defines a Riemannian metric. It follows that ω induces a symplectic form on every almost complex submanifold $W \subset V$ (which is compatible with $J|_W$).

J-convex functions. If (V, J) is an almost complex structure and $\phi : V \to \mathbb{R}$ a J-convex function, then the 2-form $\omega_\phi = -dd^{\mathbb{C}}\phi$ is symplectic. Moreover, ω_ϕ is compatible with J if J is integrable (see Section 2.2). In particular, every J-convex function on a Stein manifold induces a symplectic form compatible with J.

Kähler manifolds. A Kähler manifold is a complex manifold (V, J) with a *Kähler metric*, i.e. a Hermitian metric $H = g - i\omega$ on TV such that the 2-form

ω is closed. Thus the *Kähler form* ω is a symplectic form compatible with J. Note that every complex submanifold of a Kähler manifold is again Kähler.

The two basic examples of Kähler manifolds are $\mathbb{C}^n$ with the standard complex structure and Hermitian metric, and the complex projective space $\mathbb{C}P^n = (\mathbb{C}^{n+1}\setminus 0)/(\mathbb{C}\setminus 0)$ with the induced complex structure and Hermitian metric (the latter is defined by restricting the Hermitian metric of $\mathbb{C}^{n+1}$ to the unit sphere and dividing out the standard circle action). Passing to complex submanifolds of $\mathbb{C}^n$, we see again that Stein manifolds are Kähler. Passing to complex submanifolds of $\mathbb{C}P^n$, we see that smooth projective varieties are Kähler. This gives us a rich source of examples of closed symplectic manifolds.

Remark 5.6. While cotangent bundles and Kähler manifolds provide obvious examples of symplectic manifolds, it is not obvious how to go beyond them. The first example of a closed symplectic manifold that is not Kähler was presented by Thurston in 1976. In 1995 Gompf [24] proved that every finitely presented group is the fundamental group of a closed symplectic 4-manifold, in stark contrast to the many restrictions on the fundamental groups of closed Kähler surfaces.

Problem 5.1. Show that a Riemannian metric g on a manifold Q induces a natural almost complex structure J_g on T^*Q, compatible with $\omega_{\rm st}$, which interchanges the horizontal and vertical subspaces defined by the Levi-Civita connection. Prove that J_g is integrable if and only if the metric g is flat.

5.4 Moser's trick and symplectic normal forms

An (embedded or immersed) submanifold W of a symplectic manifold (V,ω) is called *symplectic (isotropic, coisotropic, Lagrangian)* if $T_xW \subset T_xV$ is symplectic (isotropic, coisotropic, Lagrangian) for every $x \in W$ in the sense of Section 5.1. In this section we derive normal forms for neighborhoods of such submanifolds.

All the normal forms can be proved by the same technique which we will refer to as *Moser's trick*. It is based on Cartan's formula $L_X\alpha = i_Xd\alpha + d\,i_X\alpha$ for a vector field X and a k-form α. Suppose we are given k-forms α_0, α_1 on a manifold M, and we are looking for a diffeomorphism $\phi : M \to M$ such that $\phi^*\alpha_1 = \alpha_0$. Moser's trick is to construct ϕ as the time-1 map of a time-dependent vector field X_t. For this, let α_t be a smooth family of k-forms connecting α_0 and α_1, and look for a vector field X_t whose flow ϕ_t satisfies

$$\phi_t^*\alpha_t \equiv \alpha_0. \tag{5.2}$$

Then the time-1 map $\phi = \phi_1$ the solves our problem. Now equation (5.2) follows by integration (provided the flow of X_t exists, e.g. if X_t has compact support) once its linearized version

$$0 = \frac{d}{dt}\phi_t^*\alpha_t = \phi_t^*(\dot\alpha_t + L_{X_t}\alpha_t)$$

holds for every t. Inserting Cartan's formula, this reduces the problem to the algebraic problem of finding a vector field X_t that satisfies

$$\dot{\alpha}_t + d\, i_{X_t}\alpha_t + i_{X_t} d\alpha_t = 0. \tag{5.3}$$

Here is a first application of this method. Here, as well as thoughout the book, by *diffeotopy* we denote a smooth family of diffeomorphisms ϕ_t, $t \in [0.1]$, with $\phi_0 = \mathbb{1}$.

Theorem 5.7 (Moser's Stability Theorem). *Let W be a compact manifold with (possibly empty) boundary ∂W. Let ω_t, $t \in [0,1]$, be a smooth family of symplectic forms on W which coincide along ∂W and such that the relative cohomology class $[\omega_t - \omega_0] \in H^2(W, \partial W; \mathbb{R})$ is independent of t. Then there exists a diffeotopy ϕ_t with $\phi_t|_{\partial W} = \mathbb{1}$ such that $\phi_t^*\omega_t = \omega_0$.*

Proof. For every t the closed 2-form $\dot{\omega}_t$ vanishes along ∂W and is trivial in relative cohomology $H^2(W, \partial W; \mathbb{R})$, so there exists a 1-form β_t vanishing along ∂W such that $d\beta_t = \dot{\omega}_t$. The forms β_t are not unique, but they can be chosen to depend smoothly on t. This can be achieved either by local arguments in coordinate charts (cf. [47], Theorem 3.17), or by Hodge theory as follows: Pick a Riemannian metric on the manifold V and let $d^* : \Omega^2(V) \to \Omega^1(V)$ be the L^2-adjoint of d. By Hodge theory, $\mathrm{im}(d^*) = \ker(d)^\perp$, so d is an isomorphism from $\mathrm{im}(d^*)$ to the exact 2-forms. The inverse of this isomorphism provides the particular choice for β_t.

Adapt: Hodge theory with boundary

Now we can solve equation (5.3),

$$0 = \dot{\omega}_t + d\, i_{X_t}\omega_t + i_{X_t} d\omega_t = d(\beta_t + i_{X_t}\omega_t)$$

by solving $\beta_t + i_{X_t}\omega_t = 0$, which has a unique solution X_t due to the nondegeneracy of ω_t. Since X_t vanishes on ∂W, its flow ϕ_t exists and gives the desired family of diffeomorphisms. □

Corollary 5.8. *Let V be a manifold (without boundary but not necessarily compact). Let ω_t, $t \in [0,1]$, be a smooth family of symplectic forms on V which coincide outside a compact set and such that the cohomology class with compact support $[\omega_t - \omega_0] \in H^2_c(V; \mathbb{R})$ is independent of t. Then there exists a diffeotopy ϕ_t with $\phi_t = \mathbb{1}$ outside a compact set such that $\phi_t^*\omega_t = \omega_0$.*

In particular, this applies if $\omega_t = d\lambda_t$ for a smooth family of 1-forms λ_t which coincide outside a compact set, and in this case there exists a smooth family of functions $f_t : V \to \mathbb{R}$ with compact support such that

$$\phi_t^*\lambda_t - \lambda_0 = df_t$$

Proof. Pick a compact subset $W \subset V$ with smooth boundary such that the ω_t coincide outside a compact subset $W' \subset \mathrm{Int}\, W$ and $[\dot{\omega}_t] = 0 \in H^2(W, \partial W; \mathbb{R})$. Construct a smooth family of 1-forms β_t on W as in the proof of Theorem 5.7. Then β_t vanishes along ∂W and is closed on a neighborhood of ∂W, so $\beta_t = df_t$

near ∂W for a (unique) function vanishing on ∂W. After cutting off f_t outside a neighborhood of ∂W and replacing β_t by $\beta_t - df_t$, we may assume $\beta_t = 0$ near ∂W. Then the diffeomorphisms $\phi_t : W \to W$ constructed in the proof of Theorem 5.7 extend by the identity to the desired diffeomorphisms of V.

In the case $\omega_t = d\lambda_t$ we pick $\beta_t := \dot{\lambda}_t$. Then the defining equation for X_t becomes $\dot{\lambda}_t + i_{X_t} d\lambda_t = 0$ and we find

$$\frac{d}{dt}\phi_t^*\lambda_t = \phi_t^* d(i_{X_t}\lambda_t),$$

which integrates to

$$\phi_t^*\lambda_t - \lambda_0 = d\left(\int_0^t i_{X_s}\lambda_s ds\right).$$

□

Our second application of Moser's trick is the following lemma, which is the basis of all the normal form theorems below.

Lemma 5.9. *Let W be a compact submanifold of a manifold V, and let ω_0, ω_1 be symplectic forms on V which agree at all points of W. Then there exist tubular neighborhoods U_0, U_1 of W and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi|_W = \mathbb{1}$ and $\phi^*\omega_1 = \omega_0$.*

Proof. Set $\omega_t := (1-t)\omega_0 + \omega_1$. Since $\omega_t \equiv \omega_0$ along W, ω_t are symplectic forms on some tubular neighborhood U of W. By the relative de Rham Theorem, since $\dot{\omega}_t = \omega_1 - \omega_0$ is closed and vanishes along W, there exists a of 1-form β on U such that $\beta = 0$ along W and $d\beta = \dot{\omega}_t$ on U. As in the proof of Theorem 5.7, we solve equation (5.3) by setting $\beta + i_{X_t}\omega_t = 0$.

To apply Moser's trick, a little care is needed because U is noncompact, so the flow of X_t may not exist until time 1. However, since $\beta = 0$ along W, X_t vanishes along W. Thus there exists a tubular neighborhood U_0 of W such that the flow $\phi_t(x)$ of X_t exists for all $x \in U_0$ and $t \in [0,1]$, and $\phi_t(U_0) \subset U$ for all $t \in [0,1]$. Now $\phi_1 : U_0 \to U_1 := \phi_1(U_0)$ is the desired diffeomorphism with $\phi_1^*\omega_1 = \omega_0$. □

Now we are ready for the main result of this section.

Proposition 5.10 (symplectic normal forms)**.** *Let ω_0, ω_1 be symplectic forms on a manifold V and $W \subset V$ a compact submanifold such that $\omega_0|_W = \omega_1|_W$. Suppose that $N := \ker(\omega_0|_W) = \ker(\omega_1|_W)$ has constant rank, and the bundles $(TW^{\omega_0}/N, \omega_0$, $(TW^{\omega_1}/N, \omega_1$ over W are isomorphic as symplectic vector bundles. Then there exist tubular neighborhoods U_0, U_1 of W and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi|_W = \mathbb{1}$ and $\phi^*\omega_1 = \omega_0$.*

Proof. By Proposition 5.4,

$$(TV|_W, \omega_0) \cong (TW/N, \omega_0) \oplus (TW^{\omega_0}/N, \omega_0) \oplus (N \oplus N^*, \omega_{\rm st}),$$

and similarly for ω_1. By the hypotheses, the right-hand sides are isomorphic for ω_0 and ω_1. More precisely, there exists an isomorphism

$$\Psi : (TV|_W, \omega_0) \to (TV|_W, \omega_1)$$

with $\Psi|_{TW} = 1\!\!1$. Extend Ψ to a diffeomorphism $\psi : U_0 \to U_1$ of tubular neighborhoods such that $\psi|_W = 1\!\!1$ and $\psi^*\omega_1 = \omega_0$ along W, and apply Lemma 5.9. □

All the normal forms are easy corollaries of this result.

Corollary 5.11 (Symplectic neighborhood Theorem)**.** *Let ω_0, ω_1 be symplectic forms on a manifold V and $W \subset V$ a compact submanifold such that $\omega_0|_W = \omega_1|_W$ is symplectic, and the symplectic normal bundles $(TW^{\omega_0}, \omega_0)$, $(TW^{\omega_1}, \omega_1)$ over W are isomorphic (as symplectic vector bundles). Then there exist tubular neighborhoods U_0, U_1 of W and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi|_W = 1\!\!1$ and $\phi^*\omega_1 = \omega_0$.*

Corollary 5.12 (Isotropic neighborhood Theorem)**.** *Let ω_0, ω_1 be symplectic forms on a manifold V and $W \subset V$ a compact submanifold such that $\omega_0|_W = \omega_1|_W = 0$, and the symplectic normal bundles $(TW^{\omega_0}/TW, \omega_0)$, $(TW^{\omega_1}/TW, \omega_1)$ are isomorphic (as symplectic vector bundles). Then there exist tubular neighborhoods U_0, U_1 of W and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi|_W = 1\!\!1$ and $\phi^*\omega_1 = \omega_0$.*

Corollary 5.13 (Coisotropic neighborhood Theorem)**.** *Let ω_0, ω_1 be symplectic forms on a manifold V and $W \subset V$ a compact submanifold such that $\omega_0|_W = \omega_1|_W$ and W is coisotropic for ω_0 and ω_1. Then there exist tubular neighborhoods U_0, U_1 of W and a diffeomorphism $\phi : U_0 \to U_1$ such that $\phi|_W = 1\!\!1$ and $\phi^*\omega_1 = \omega_0$.*

Corollary 5.14 (Weinstein's Lagrangian neighborhood Theorem)**.** *Let $W \subset (V, \omega)$ be a compact Lagrangian submanifold of a symplectic manifold. Then there exist tubular neighborhoods U of the zero section in T^*W and U' of W in V and a diffeomorphism $\phi : U \to U'$ such that $\phi|_W$ is the inclusion and $\phi^*\omega = \omega_{\rm st}$.*

Proof. Since W is Lagrangian, the map $v \mapsto i_v\omega$ defines an isomorphism from the normal bundle $TV/TW|_W$ to T^*W. Extend the inclusion $W \subset V$ to a diffeomorphism $\psi : U \to U'$ of tubular neighborhoods of the zero section in T^*W and of W in V. Now apply the Coisotropic neighborhood Theorem to the zero section in T^*W and the symplectic forms $\omega_{\rm st}$ and $\psi^*\omega$. □

5.5 Contact manifolds and their Legendrian submanifolds

A *contact structure* ξ on a manifold M is a completely non-integrable tangent hyperplane field. According to the Frobenius condition, this means that for every nonzero local vector field $X \in \xi$ there exists a local vector field $Y \in \xi$ such that their Lie bracket satisfies $[X, Y] \notin \xi$. If α is any 1-form locally defining ξ, i.e. $\xi = \ker \alpha$, this means

$$d\alpha(X, Y) = -\frac{1}{2}\alpha([X, Y]) \neq 0.$$

So the restriction of the 2-form $d\alpha$ to ξ is nondegenerate, i.e. $(\xi, d\alpha|_\xi)$ is a symplectic vector bundle. This implies in particular that $\dim \xi$ is even and $\dim M = 2n + 1$ is odd. In terms of a local defining 1-form α, the contact condition can also be expressed as $\alpha \wedge (d\alpha)^n \neq 0$.

Remark 5.15. If $\dim M = 4k + 3$ the sign of the volume form $\alpha \wedge (d\alpha)^{2k+1}$ is independent of the sign of the defining local 1-form α, so a contact structure defines an orientation of the manifold. In particular, in these dimensions contact structures can exist only on orientable manifolds. On the other hand, a contact structure ξ on a manifold of dimension $4k + 1$ is itself orientable.

Contact structures ξ in this book will always be *cooriented*, i.e., they are globally defined by a 1-form α. In this case the symplectic structure on each of the hyperplanes ξ is defined uniquely up to a positive conformal factor.

Given a J-convex hypersurface M (which is by definition cooriented) in an almost complex manifold (V, J), the field ξ of complex tangencies defines a contact structure on M which is cooriented by $J\nu$, where ν is a vector field transverse to M defining the coorientation. Conversely, any cooriented contact structure ξ arises as a field of complex tangencies on a J-convex hypersurface in an almost complex manifold: Just chose a complex multiplication J on ξ compatible with the symplectic form $d\alpha$ in the sense that $d\alpha(\cdot, J\cdot)$ is a (positive definite) inner product on ξ and extend J arbitrarily to an almost complex structure on $V := M \times (-\epsilon, \epsilon)$.

Remark 5.16. If $\dim M = 3$ then J can always be chosen integrable. However, in dimensions ≥ 5 this is not always the case, see Example ??? below.

Let $(M, \xi = \ker \alpha)$ be a contact manifold of dimension $2n + 1$. An immersion $\phi : \Lambda \to M$ is called *isotropic* if it is tangent to ξ. Then at each point $x \in \Lambda$ we have $d\phi(T_x\Lambda) \subset \xi_{\phi(x)}$ and $d\alpha|_{d\phi(T_x\Lambda)} = d\big(\alpha|_{\phi(\Lambda)}\big)(x) = 0$. Hence $d\phi(T_xL)$ is an isotropic subspace in the symplectic vector space $(\xi_x, d\alpha)$. In particular,

$$\dim \Lambda \leq \frac{1}{2}\dim \xi = n.$$

Isotropic immersions of the maximal dimension n are called *Legendrian.*

1-jet spaces. Let L be a manifold of dimension n. The space J^1L of 1-jets of functions on L can be canonically identified with $T^*L \times \mathbb{R}$, where T^*L is the cotangent bundle of L. A point in J^1L is a triple (q, p, z) where q is a point in L, p is a linear form on T_qL, and $z \in \mathbb{R}$ is a real number. Pick local coordinates $(q_1, \dots, q_n)$ are local coordinates on L and write covectors in T^*L as $\sum p_i dq_i$.

It is easy to check that the 1-form

$$p\,dq := \sum_{i=1}^{n} p_i dq_i$$

is independent of the choice of such coordinates. It is called the *canonical 1-form on* T^*L. The 2-form $dp \wedge dq := d(p\,dq)$ is called the *canonical sympletic form on* T^*L. The 1-form $dz - p\,dq$ defines the *canonical contact structure*

$$\xi_{\mathrm{can}} := \ker(dz - p\,dq)$$

on J^1L. A function $f : L \to \mathbb{R}$ defines a section

$$q \mapsto j^1 f(q) := \Big(q, df(q), f(q)\Big)$$

of the bundle $J^1L \to L$. Since $f^*(dz - p\,dq) = df - df = 0$, this section is a Legendrian embedding in the contact manifold (J^1L, ξ). Consider the following diagram, where all arrows represent the obvious projections:

[to be added]

We call P_{Lag} the *Lagrangian projection* and P_{front} the *front projection*. Given a Legendrian submanifold $\Lambda \subset J^1L$, consider its images

$$P_{\mathrm{Lag}}(\Lambda) \subset T^*L, \qquad P_{\mathrm{front}}(\Lambda) \subset L \times \mathbb{R}.$$

The map $P_{\mathrm{Lag}} : \Lambda \to T^*L$ is a Lagrangian immersion with respect to the standard symplectic structure $dp \wedge dq = d(p\,dq)$ on T^*L. Indeed, the contact hyperplanes of ξ_{can} are transverse to the z-direction which is the kernel of the projection P_{Lag}. Hence Λ is transverse to the z-direction as well and $P_{\mathrm{Lag}}|_\Lambda$ is an immersion. It is Lagrangian because

$$P_{\mathrm{Lag}}^* dp \wedge dq = d(p\,dq|_\Lambda) = d(dz|_\Lambda) = 0.$$

Conversely, any *exact Lagrangian immersion* $\phi : \Lambda \to T^*L$, i.e. an immersion for which the form ϕ^*, dq is exact, lifts to a Legendrian immersion $\hat{\phi} : \Lambda \to J^1L$. It is given by the formula $\hat{\phi} := (\phi, H)$, where H is a primitive of the exact 1-form $\phi^* p\,dq$ so that $\hat{\phi}^*(dz - p\,dq) = dH - \phi^* p\,dq = 0$. The lift $\hat{\phi}$ is unique up to a translation along the z-axis.

Remark 5.17. More generally, a *Liouville structure* on an even-dimensional manifold is a 1-form α such that $d\alpha$ is symplectic. For example, the form $p\,dq$ is the canonical Liouville form on the cotangent bundle T^*L. An immersion $\phi : L \to V$ into a Liouville manifold (V, α) is called *exact Lagrangian* if $\phi^*\alpha$ is exact.

Let us now turn to the front projection. The image $P_{\mathrm{front}}(\Lambda)$ is called the *(wave) front* of the Legendrian submanifold $\Lambda \subset J^1L$. If the projection $\pi|_\Lambda : \Lambda \to L$ is nonsingular and injective, then Λ is a graph $\{\big((q, \alpha(q), f(q)\big) \mid q \in \pi(\Lambda)\}$ over

$\pi(\Lambda) \subset L$. The Legendre condition implies that the 1-form α is given by $\alpha = df$. So

$$\Lambda = \{\big((q, df(q), f(q)\big) \mid q \in \pi(\Lambda)\}$$

is the graph of the 1-jet $j^1 f$ of a function $f : \pi(\Lambda) \to \mathbb{R}$. In this case the front $P_{\text{front}}(\Lambda)$ is just the graph of the function f.

In general, the front of a Legendrian submanifold $\Lambda \subset J^1 L$ can be viewed as the graph of a multivalued function. Note that since the contact hyperplanes are transverse to the z-direction, the singular points of the projection $\pi|_\Lambda$ coincide with the singular points of the projection $P_{\text{front}}|_\Lambda$. Hence near each of its nonsingular points the front is indeed the graph of a function.

In general, the front can have quite complicated singularities. But when the projection $\pi|_\Lambda : \Lambda \to L$ has only "fold type" singularities, then the front itself has only "cuspidal" singularities along its singular locus as shown in Figure [fig:???].

Let us discuss this picture in more detail. Consider first the 1-dimensional case when $L = \mathbb{R}$. Then $J^1 L = \mathbb{R}^3$ with coordinates (q, p, z) and contact structure $\ker(dz - p\, dq)$. Consider the curve in $\mathbb{R}^3$ given by the equations

$$q = 3p^2, \qquad z = 2p^3. \tag{5.4}$$

This curve is Legendrian because $dz = 6p^2 dp = p\, dq$. Its front is given by (5.4) viewed as parametric equations for a curve in the (q, z)-plane. This is a semicubic parabola as shown in Figure [fig:???].

Generically, any singular point of a Legendrian curve in $\mathbb{R}^3$ looks like this. This means that, after a C^∞-small perturbation of the given curve to another Legendrian curve, there exists a contactomorphism (i.e. a diffeomorphism which preserves the contact structure) of a neighborhood of the singularity which transforms the curve to the curve described by (5.4) (see [4], Chapter 1 §4). If we want to construct just C^1 Legendrian curves (and any C^1 Legendrian curve can be further C^1-approximated by C^∞ or even real analytic Legendrian curves, see Corollary 7.25), then the following characterization of the front near its cusp points will be convenient. Suppose that the two branches of the front which form the cusp are given locally by the equations $z = f(q)$ and $z = g(q)$, where the functions $f, g : [0, \epsilon) \to \mathbb{R}$ satisfy $f \le g$ (see Figure [fig:???]). Then the front lifts to a C^1 Legendrian curve if and only if

$$f(0) = g(0), \qquad f'(0) = g'(0),$$
$$f''(q) \to -\infty \text{ as } q \to 0, \qquad g''(q) \to +\infty \text{ as } q \to 0.$$

In higher dimensions, suppose that a Legendrian submanifold $\Lambda \subset J^1 L$ projects to L with only "fold type" singularities. Then along its singular locus the front consists of the graphs of two functions $f \le g$ defined on an immersed strip $S \times [0, \epsilon)$. Denoting coordinates on $S \times [0, \epsilon)$ by (s, t), the front lifts to a C^1

Legendrian submanifold if and only if

$$f(s,0)=g(s,0), \qquad \frac{\partial f}{\partial t}(s,0)=\frac{\partial g}{\partial t}(s,0),$$
$$\frac{\partial^2 f}{\partial t^2}(s,t)\to -\infty \text{ as } t\to 0, \qquad \frac{\partial^2 g}{\partial t^2}(s,t)\to +\infty \text{ as } t\to 0.$$

However, in higher dimensions not all singularities are generically of fold type.

Example 5.18. Given a contact manifold $(M,\xi=\ker\alpha)$ and a Liouville manifold (V,β), their product $M\times V$ is a contact manifold with the contact form $\alpha\oplus\beta$. For example, if $M=J^1N$ and $V=T^*W$ with the canonical contact and Liouville forms, then $M\times V=J^1(N\times W)$ with the canonical contact form. A product $\Lambda\times L$ of a Legendrian submanifold $\Lambda\subset M$ and an exact Lagrangian submanifold $L\subset V$ is a Legendrian submanifold of $M\times V$. In particular, the product of a Legendrian submanifold $\Lambda\subset J^1N$ and an exact Lagrangian submanifold $L\subset T^*W$ is a Legendrian submanifold in $J^1(N\times W)$.

5.6 Contact normal forms

Let $(M^{2n+1},\xi=\ker\alpha)$ be a contact manifold and $\Lambda^k\subset M$, $0\le k\le n$, be an isotropic submanifold. The following result is due to Darboux in the case that Λ is a point (see e.g. Appendix 4 of [3]); the extension to general Λ is straightforward and left to the reader.

Proposition 5.19 (contact Darboux Theorem)**.** *Near each point on Λ there exist coordinates $(q_1,\dots,q_n,p_1,\dots,p_n,z)\in\mathbb{R}^{2n+1}$ in which $\alpha=dz-\sum p_i dq_i$ and $\Lambda=\mathbb{R}^k\times\{0\}$.*

To formulate a more global result, recall that the form $\omega=d\alpha$ defines a natural (i.e., independent of α) conformal symplectic structure on ξ. Denote the ω-orthogonal on ξ by a superscript ω. Since Λ is isotropic, $T\Lambda\subset T\Lambda^\omega$. So the normal bundle of Λ in M is given by

$$TM/T\Lambda = TM/\xi\oplus\xi/(T\Lambda)^\omega\oplus(T\Lambda)^\omega/T\Lambda\cong\mathbb{R}\oplus T^*\Lambda\oplus CSN(\Lambda).$$

Here TM/ξ is trivialized by the Reeb vector field R_α, the bundle $\xi/(T\Lambda)^\omega$ is canonically isomorphic to T^Λ via $v\mapsto i_v\omega$, and $CSN(\Lambda):=(T\Lambda)^\omega/T\Lambda$ denotes the *conformal symplectic normal bundle* which carries a natural conformal symplectic structure induced by ω. Thus $CSN(\Lambda)$ has structure group $Sp(n-k)$, which can be reduced to $U(n-k)$ by choosing a compatible complex structure.

Let (M,ξ_M) and (N,ξ_N) be two contact manifolds. A map $f:M\to N$ is called *isocontact* if $f^*\xi_N=\xi_M$, where $f^*\xi_N:=\{v\in TM\mid df\cdot v\in\xi_N\}$. Equivalently, f maps any defining 1-form α_N for ξ_N to a defining 1-form $f^*\alpha_M$ for ξ_M. In particular, f must be an immersion and thus $\dim M\le\dim N$. Moreover, $df:\xi_M\to\xi_N$ is *conformally symplectic*, i.e., symplectic up to a scaling factor. We call a monomorphism $F:TM\to TN$ *isocontact* if $F^*\xi_N=\xi_M$ and $F:\xi_M\to\xi_N$ is conformally symplectic.

Proposition 5.20 (Isotropic neighborhood Theorem, Contact Version [63]). *Let (M, ξ_M), (N, ξ_N) be contact manifolds with $\dim M \leq \dim N$ and $\Lambda \subset M$ an isotropic submanifold. Let $f : \Lambda \to N$ be an isotropic immersion covered by an isocontact monomorphism $F : TM \to TN$. Then there exists an isocontact immersion $g : U \to N$ of a neighborhood $U \subset M$ of Λ with $g|_\Lambda = f$ and $dg = F$ along Λ.*

Remark 5.21. (a) If ϕ is an embedding then ψ is also an embedding on a sufficiently small neighborhood. It follows that a neighborhood of a Legendrian submanifold Λ is contactomorphic to a neighborhood of the zero section in the 1-jet space $J^1\Lambda$ (with its canonical contact structure).

(b) A Legendrian immersion $f : \Lambda \to (M, \xi)$ extends to a an isocontact immersion of a neighborhood of the zero section in $J^1\Lambda$.

(c) Suppose that the conformal symplectic normal bundle of an isotropic submanifold Λ is the complexification of a real bundle $W \to \Lambda$ (i.e., the structure group of $CSN(\Lambda)$ reduces from $U(n-k)$ to $O(n-k)$). Then a neighborhood of Λ is contactomorphic to a neighborhood of the zero section in $J^1\Lambda \oplus (W \oplus W^*)$ (with its canonical contact structure, see Example 5.18). In this case (and only in this case) the isotropic submanifold Λ extends to a Legendrian submanifold (the total space of the bundle W).

We will also need the following refinement of the Isotropic neighborhood Theorem. Following Weinstein [63], let us denote by *isotropic setup* a quintuple $(V, \omega, X, \Sigma, \Lambda)$, where (V, ω) is a symplectic manifold with Liouville vector field X, $\Sigma \subset V$ is a codimension one hypersurface transverse to X, and $\Lambda \subset \Sigma$ is a closed isotropic submanifold for the contact structure $\ker(i_X\omega)|_\Sigma$. Let $(T\Lambda)^\omega/T\Lambda \subset \xi$ be the *symplectic normal bundle* over Λ.

Proposition 5.22 (Weinstein [63]). *Let $(V_i, \omega_i, X_i, \Sigma_i, \Lambda_i)$, $i = 0, 1$ be isotropic setups. Given a diffeomorphism $f : \Lambda_0 \to \Lambda_1$ covered by an isomorphism Φ of symplectic normal bundles, there exists an isomorphism of isotropic setups*

$$F : (U_0, \omega_0, X_0, \Sigma_0 \cap U_0, \Lambda_0) \to (U_1, \omega_1, X_1, \Sigma_1 \cap U_1, \Lambda_1)$$

between neighborhoods U_i of Λ_i in V_i inducing f and Φ.

This is redundant. Adapt to prove Proposition 5.22?

We will need a stronger form of Weinstein theorem 5.20. Not only the contact structure, but even the contact form can be standardized near an isotropic submanifold.

Proposition 5.23. *Let λ_0, λ_1 be two contact forms for the same contact structure ξ defined on a neighborhood of an isotropic submanifold $\Lambda \subset V$. Then there exists a fixed along Λ contact isotopy $h_t : \mathcal{O}p\,(\Lambda) \to \mathcal{O}p\,(\Lambda)$ such that $\lambda_1 = h_1^*\lambda_0$.*

Proof. We are following here the standard Moser homotopic method. Set $\lambda_t = (1-t)\lambda_0 + t\lambda_1$, $t \in [0,1]$. Then λ_t is a contact form for ξ for all $t \in [0,1]$.

Differentiating the equation $h_t * \lambda_0 - \lambda_t$, we get, using Carna's formula for the Lie derivative:

$$i(X_t)d\lambda_t + d(\lambda_t(X_t)) = \mu, \tag{5.5}$$

where

$$X_t(h_t(x)) = \frac{dh_t(x)}{dt} \quad \text{and} \quad \mu = \lambda_1 = \lambda_0.$$

Let R_t denotes the Reeb vector field of the form λ_t, i.e. $\lambda_t(R_t) = 1$ and $i(R_t)d\lambda_t = 0$. Let us write $X_t = a_tR_t + Y_t$, where $Y_t \in \xi$ and denote $b_t := \mu(R_t)$ and $\alpha := \mu|_\xi$. Then (5.5) is equivalent to the system

$$\begin{aligned} da_t(R_t) &= b_t, \\ i(Y_t)d\lambda_t &= \alpha - da_t|_\xi. \end{aligned} \tag{5.6}$$

Let us consider a germ Σ along Λ of a hypersurface tangent to ξ along Λ. There exists a smooth function f on Σ such that $f|_\Lambda = 0$ and $df|_{\xi_\Lambda} = \alpha|_{\xi|_\Lambda}$. Note that for each t the vector field R_t is transverse to Σ on $\mathcal{O}p\,\Lambda$. Hence the first of equations (5.6) has a solution a_t on $\mathcal{O}p\,\Lambda$ which satisfies an initial condition $a_t|_\Sigma = f$. The second equation is a non-differential non-degenerate linear system of equation with respect to Y_t and hence it has a unique solution Y_t after a_t is found. Note that by our choice of f the right-hand side of the second equation vanishes along Λ, and hence $X_t|_\Lambda = (a_tR_t + Y_t)|_\Lambda = 0$. Hence the vector field X_t can be integrated to the required isotopy $h_t : \mathcal{O}p\,\Lambda \to \mathcal{O}p\,\Lambda$, fixed along Λ.

□

All the properties discussed in this section also hold for families of isotropic submanifolds. Moreover, any isotropic submanifold with boundary can be extended beyond the boundary to a slightly bigger isotropic submanifold of the same dimension.

Finally, we mention that a similar homotopy argument proves Gray's Stability Theorem, which states that on a closed manifold all deformations of a contact structure are diffeomorphic to the original one.

Theorem 5.24 (Gray's Stability Theorem [27]). *Let $(\xi_t)_{t\in[0,1]}$ be a smooth homotopy of contact structures on a closed manifold M. Then there exists a diffeotopy $\phi_t : M \to M$ with $\phi_t^*\xi_t = \xi_0$ for all $t \in [0,1]$.*

5.7 Stabilization of Legendrian submanifolds

The goal of this section is the proof of the following

Proposition 5.25. *Let $\Lambda_0 \subset (M^{2n+1}, \xi = \ker\alpha)$ be a closed orientable Legendrian submanifold and k an integer. Suppose that $n > 1$. Then there exists a Legendrian submanifold $\Lambda_1 \subset M$ and a Legendrian regular homotopy Λ_t, $t \in [0,1]$, such that the self-intersection index of the immersion $L := \cup_{t\in[0,1]}\Lambda_t \times \{t\} \subset M \times [0,1]$ equals k (mod 2 if n is even).*

A local construction. The proof of Proposition 5.25 is base on a *stabilization* procedure which we will now describe. Consider the front projection of a (not necessarily closed) orientable Legendrian submanifold $\Lambda_0 \subset \mathbb{R}^{2n+1}$. Suppose that $P_{\mathrm{front}}(\Lambda_0)$ intersects $B^n \times [-1,2]$ in the two oppositely oriented branches $\{z=0\}$ and $\{z=1\}$. Let $f : B^n \to (-1,2)$ be a function which equals zero near ∂B^n and has no critical points on level 1. Replacing the branch $\{z=0\}$ over B^n by $\{z = tf(q)\}$ we obtain a family of Legendrian immersions $\Lambda_t \subset \mathbb{R}^{2n+1}$, $t \in [0,1]$. Note that the set $\{q \in B^n \mid f(q) \geq 1\}$ is a smooth n-manifold with boundary. Denote by $\chi(\{f \geq 1\})$ its Euler characteristic.

Lemma 5.26. *The self-intersection index of the immersion* $L := \cup_{t\in[0,1]}\Lambda_t \times \{t\} \subset M \times [0,1]$ *equals*

$$I_L = (-1)^{n(n-1)/2}\chi(\{f \geq 1\})$$

(mod 2 if n *is even).*

Proof. Perturb f such that all critical points above level 1 are nondegenerate and lie on distinct levels. Self-intersections of L occur precisely when $t_0 f$ has a critical point q_0 on level 1 for some $t_0 \in (0,1)$. By the Morse Lemma, we find coordinates near q_0 in which $q_0 = 0$ and f has the form

$$f(q) = a_0 - \frac{1}{2}\sum_{i=1}^{k} q_i^2 + \frac{1}{2}\sum_{i=k+1}^{n} q_i^2,$$

where $a_0 = f(q_0) = 1/t_0$ and k is the Morse index of q_0. The p-coordinates on the branch $\{z = tf(q)\}$ of Λ_t near q_0 are given by

$$p_i = \frac{\partial(tf)}{\partial q_i} = \begin{cases} -tq_i & i \leq k, \\ +tq_i & i \geq k+1. \end{cases}$$

Thus the tangent spaces in $T(\mathbb{R}^{2n+1} \times [0,1]) = \mathbb{R}^{2n+2}$ of the two intersecting branches of L corresponding to $\{z=1\}$ and $\{z = t_0 f(q)\}$ are given by

$$T_1 = \{p_1 = \cdots = p_n = 0, z = 0\},$$
$$T_2 = \{p_i = -t_0 q_i \text{ for } i \leq k, p_i = +t_0 q_i \text{ for } i \geq k+1, z = a_0 t\}.$$

Without loss of generality (because the self-intersection index does not depend on the orientation of L) suppose that the basis $(\partial_{q_1}, \ldots, \partial_{q_n}, \partial_t)$ represents the orientation of T_1. Since the two branches of Λ_0 are oppositely oriented, the orientation of T_2 is then represented by the basis

$$\big(\partial_{q_1} - t_0\partial_{p_1}, \ldots, \partial_{q_n} + t_0\partial_{p_n}, -(\partial_t + a_0\partial_z)\big).$$

Hence the orientation of (T_1, T_2) is represented by

$$(\partial_{q_1}, \ldots, \partial_{q_n}, \partial_t, -\partial_{p_1}, \ldots, -\partial_{p_n}, -\partial_z),$$

which equals $(-1)^{k+n+n(n-1)/2}$ times the complex orientation

$$(\partial_{q_1}, \partial_{p_1}, \dots, \partial_{q_n}, \partial_{p_n}, \partial_z, \partial_t)$$

of $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$. So the local intersection index of L at a critical point q equals

$$I_L(q) = (-1)^{\mathrm{ind}_f(q)+n+n(n-1)/2}$$

(mod 2 if n is even), where $\mathrm{ind}_f(q)$ is the Morse index of q.

On the other hand, for a vector field v on a compact manifold N with boundary which is outward pointing along the boundary and has only nondegenerate zeroes we have *Poincaré-Hopf Index Theorem* holds: The sum of the indices of v at all its zeroes equals the Euler characteristic of M (see [33]). Note that if v is the gradient vector field of a Morse function f, then the index of v at a critical point q of f equals $(-1)^{\mathrm{ind}_f(q)}$. Applying the Poincaré-Hopf Index Theorem to the gradient of the Morse function $-f$ on the manifold $\{f \geq 1\} = \{-f \leq -1\}$ (which is outward pointing along the boundary because f has no critical point on level 1), we obtain

$$\begin{aligned}\chi(\{f \geq 1\} &= \sum_q \mathrm{ind}_{\nabla(-f)}(q) = \sum_q (-1)^{\mathrm{ind}_{-f}(q)} = \sum_q (-1)^{n-\mathrm{ind}_f(q)} \\ &= (-1)^{n(n-1)/2} \sum_q I_L(q) = (-1)^{n(n-1)/2} I_L.\end{aligned}$$

□

Proof of Proposition 5.25. Since all Legendrian submanifolds are locally isomorphic, a neighborhood in M of a point on Λ_0 is contactomorphic to a neighborhood in $\mathbb{R}^{2n+1}$ of a point on a standard cusp $3z^2 = 2q_1^2$. Thus the front consists of two branches $\{z = \pm\sqrt{\frac{2}{3}q_1^3}\}$ joined along the singular locus $\{z = q_1 = 0\}$. Now deform the branches to $\{z = \pm\varepsilon\}$ over a small ball disjoint from the singular locus, thus (after rescaling) creating two parallel branches over a ball as in Lemma B.9. Now deform Λ_0 to Λ_1 as in Lemma B.9, for some function $f : B^n \to (-1, 2)$. Then Proposition B.7 follows from Lemma B.9, provided that we arrange $\chi(\{f \geq 1\}) = k$ for a given integer k if $n > 1$.

Thus it only remains to find for $n > 1$ an n-dimensional submanifold-with-boundary $N \subset \mathbb{R}^n$ of prescribed Euler characteristic $\chi(N) = k$ (then write $N = \{f \geq 1\}$ for a function $f : N \to [1, 2)$ without critical points on the boundary). Let N_+ be a ball in $\mathbb{R}^n$, thus $\chi(N_+) = +1$. Let N_- be a smooth tubular neighborhood in $\mathbb{R}^n$ of a figure eight in $\mathbb{R}^2$, thus $\chi(N_-) = -1$ (here we use $n \geq 2$!). So we can arrange $\chi(N)$ to be any integer by taking disjoint unions of copies of $N_\pm$. □

Remark 5.27. The preceding proof fails for $n = 1$ because a 1-dimensional manifold with boundary always has Euler characteristic $\xi \geq 0$. Therefore for $n = 1$ the local construction in Lemma 5.26 allows us only to realize *positive* values of the self-intersection index I_L. As explained in Appendix B, this failure to create negative I_L is unavoidable in view of Bennequin's inequality. However, no analog of Bennequin's inequality exists in *overtwisted* contact 3-manifolds, and we will show in Section 6.6 how to realize any value of the self-intersection index in that case.

Chapter 6

The h-principles

6.1 Immersions and embeddings

We begin by reviewing some facts about smooth immersions and embeddings. For a closed subset $A \subset X$ of a topological space, we denote by $\mathcal{O}p\, A$ a sufficiently small (*but not specified*) open neighborhood of A.

The h-principle for immersions. Let M, N be manifolds. A *monomorphism* $F : TM \to TN$ is a fibrewise injective bundle homomorphism covering a continuous map $M \to N$. Any immersion $f : M \to N$ gives rise to a monomorphism $df : TM \to TN$. We denote by $\mathrm{Mon}(TM, TN)$ the space of monomorphisms, and by $\mathrm{Imm}(M, N)$ the space of immersions. Given a (possibly empty) closed subset $A \subset M$ and an immersion $h : \mathcal{O}p\, A \to N$, we denote by $\mathrm{Imm}(M, N; A, h)$ the subspace of $\mathrm{Imm}(M, N)$ which consists of immersions equal to h on $\mathcal{O}p\, A$. Similarly, the notation $\mathrm{Mon}(TM, TN; A, dh)$ stands for the subspace of $\mathrm{Mon}(TM, TN)$ of monomorphisms which coincide with dh on $\mathcal{O}p\, A$. Extending S. Smale's theory of immersions of spheres (see [58, 59]) M .Hirsch proved the following h-principle (see also [32],[18]):

Theorem 6.1 (Hirsch [37]). *For* $\dim M < \dim N$ *and any immersion* $h : \mathcal{O}p\, A \to N$*, the map* $f \mapsto df$ *defines a homotopy equivalence between the spaces* $\mathrm{Imm}(M, N; A, h)$ *and* $\mathrm{Mon}(TM, TN; A, dh)$*. In particular, any monomorphism* $F \in \mathrm{Mon}(TM, TN; A, dh)$ *is homotopic to the differential* df *of an immersion* $f : M \to N$ *which coincides with* h *on* $\mathcal{O}p\, A$*. Given a homotopy* $F_t \in \mathrm{Mon}(TM, TN; A, dh)$*,* $t \in [0, 1]$*, between the differentials* $F_0 = df_0$ *and* $F_1 = df_1$ *of two immersions* $f_0, f_1 \in \mathrm{Imm}(M, N; A, h)$*, one finds a regular homotopy* $f_t \in \mathrm{Imm}(M, N; A, h)$*,* $t \in [0, 1]$*, such that the paths* F_t *and* df_t*,* $t \in [0, 1]$*, are homotopic with fixed ends.*

For example, if M is parallelizable, i.e. $TM \cong M \times \mathbb{R}^k$, the inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$ gives rise to a monomorphism $TM = M \times \mathbb{R}^k \to T(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n$, $(x, v) \mapsto (0, v)$. Thus Hirsch's theorem implies that every parallelizable closed manifold

M^k can be immersed into $\mathbb{R}^{k+1}$.

Immersions of half dimension. Next we describe results of Whitney [65] on immersions of half dimension. Fix a closed connected manifold M^n of dimension $n \geq 2$ and an oriented manifold N^{2n} of double dimension. Let $f : M \to N$ be an immersion whose only self-intersections are transverse double points. Then if M is orientable and n is even we assign to every double point $z = f(p) = f(q)$ an integer $I_f(z)$ as follows. Choose an orientation of M. Set $I_f(z) := \pm 1$ according to whether the orientations of $df(T_pM)$ and $df(T_qM)$ together determine the orientation of N or not. Note that this definition depends neither on the order of p and q (because n if even), nor on the orientation of M. Define the *self-intersection index*

$$I_f := \sum_z I_f(z) \in \mathbb{Z}$$

as the sum over all self-intersection points z. If n is odd or M non-orientable define $I_f \in \mathbb{Z}_2$ as the number of self-intersection points modulo 2.

Theorem 6.2 (Whitney [65])**.** *For a closed connected manifold M^n and an oriented manifold N^{2n}, $n \geq 2$, the following holds.*

(a) The self-intersection index is invariant under regular homotopies.

(b) The self-intersection index of a totally regular immersion $f : M \to N$ can be changed to any given value by a local modification (which is of course not a regular homotopy).

(c) If $n \geq 3$, any immersion $f : M \to N$ is regularly homotopic to an immersion with precisely $|I_f|$ transverse double points (where $|I_f|$ means 0 *resp.* 1 *for $I_f \in \mathbb{Z}_2$).*

Since every immersion of half dimension is regularly homotopic to an immersion with transverse self-intersections ([64], see also [38]), Part (a) allows to define the self-intersection index for every immersion $f : M \to N$. Since every n-manifold immerses into $\mathbb{R}^{2n}$, Parts (b) and (c) imply (the cases $n = 1, 2$ are treated by hand)

Corollary 6.3 (Whitney Embedding Theorem [65])**.** *Every closed n-manifold M^n, $n \geq 1$, can be embedded in $\mathbb{R}^{2n}$.*

Remark 6.4. The preceding results continue to hold if M has boundary, provided that for immersions and during regular homotopies no self-intersections occur on the boundary.

Remark 6.5. For $n = 1$ Whitney [65] defines a self-intersection index $I_f \in \mathbb{Z}$. With this definition, all the preceding results continue to hold for $n = 1$ (note e.g. that $\pi_1 V_{2,1} = \mathbb{Z}$).

Isotopies. Finally, we discuss *isotopies*, i.e. homotopies through embeddings. Consider a closed connected orientable k-manifold M^k and an oriented $(2k+1)$-manifold N^{2k+1}. Let $f_t : M \to N$ be a regular homotopy between embeddings

$f_0, f_1 : M \hookrightarrow N$. Define the immersion of half dimension $F : M \times [0,1] \to N \times [0,1]$, $F(x,t) := \big(f_t(x), t\big)$. Its self-intersection index $I_{\{f_t\}} := I_F$ is an invariant of f_t in the class of regular homotopies with fixed endpoints f_0, f_1. Recall that $I_{\{f_t\}}$ takes values in $\mathbb{Z}$ if k is odd and $\mathbb{Z}_2$ if k is even.

Theorem 6.6 (Whitney)**.** *If $k > 1$ and N is simply connected, then f_t can be deformed through regular homotopies with fixed endpoints to an isotopy if and only if $I_{\{f_t\}} = 0$.*

The proof uses the following

Lemma 6.7. *Let M, N, Λ be manifolds and $F : \Lambda \times M \to N$ a smooth map. If $2\dim M + \dim \Lambda < \dim N$, then F can be C^∞-approximated by a map $\tilde F$ such that $\tilde F(\lambda, \cdot)$ is an embedding for all $\lambda \in \Lambda$. Moreover, if F is already an embedding on a compact subset $K \subset \Lambda \times M$ we can choose $\tilde F = F$ on K.*

The case $\Lambda = [0,1]$ is due to Whitney [64].

Proof of Theorem 6.6. The argument is an adjustment of the Whitney trick [65]. Take two self-intersection points $Y_0 = (y_0, t_0), Y_1 = (y_1, t_1) \in N \times (0,1)$ of the immersion $F : M^k \times [0,1] \to N^{2k+1} \times [0,1]$ defined above. If $k+1$ is even we assume that the intersection indices of these points have opposite signs. Each of the double points y_0, y_1 is the image of two distinct points $x_0^\pm, x_1^\pm \in M$, i.e. we have $f_{t_0}(x_0^\pm) = y_0$ and $f_{t_1}(x_1^\pm) = y_1$. As $k > 1$, we find two embedded paths $\gamma^\pm : [t_0, t_1] \to M$ such that $\gamma^\pm(t_0) = x_0^\pm$, $\gamma^\pm(t_1) = x_1^\pm$, and $\gamma^+(t) \neq \gamma^-(t)$ for all $t \in [t_0, t_1]$. We claim that there exists a smooth family of paths $\delta_t : [-1,1] \to M$, $t \in [t_0, t_1]$, such that

- $\delta_t(\pm 1) = \gamma^\pm(t)$ for all $t \in [t_0, t_1]$;
- $\delta_{t_0}(s) = y_0$, $\delta_{t_1}(s) = y_1$ for all $s \in [-1,1]$;
- δ_t is an embedding for all $t \in (t_0, t_1)$.

Indeed, a family with the first two properties exists because N is simply connected. Moreover, we can arrange that δ_t is an embedding for $t \neq t_0, t_1$ close to t_0, t_1. Now we can achieve the third property by Lemma 6.7 because $2 \cdot 1 + 1 < 2k + 1$. Define

$$\Delta : [\, t_0, t_1] \times [-1,1] \to N \times [0,1], \qquad (t,s) \mapsto \big(\delta_t(s), t\big).$$

Then Δ is an embedding on $(t_0, t_1) \times [-1,1]$ and $\Delta(t_0 \times [-1,1]) = Y_0$, $\Delta(t_1 \times [-1,1]) = Y_1$. Thus Δ serves as a Whitney disk for elimination of the double points Y_0, Y_1 of the immersion F. Due to the special form of Δ, Whitney's elimination construction ([65], see also [50]) can be performed in such a way that the modified immersion $\tilde F$ has the form $\tilde F(x,t) := \big(\tilde f_t(x), t\big)$ for a regular homotopy $\tilde f_t : M \to N$ such that the paths $f_t, \tilde f_t \in \mathrm{Imm}(M,N)$, $t \in [0,1]$, are homotopic. Hence the repeated elimination of pairs of opposite index intersection points of the immersion F results in the required isotopy between f_0 and f_1. □

6.2 The h-principle for isotropic immersions

The following h-principle was proved by Gromov in 1986 ([32], see also [18]).

Let (M, ξ) be a contact manifold of dimension $2n+1$ and J a compatible almost complex structure on ξ. Let Λ be a manifold of dimension $k \leq n$ and $A \subset \Lambda$ a closed subset. Let $h : \mathcal{O}p\,A \to M$ be an isotropic immersion. We denote by $\mathrm{Iso}(\Lambda, M; A, h)$ the space of isotropic immersions $\Lambda \to M$ which coincide with h on $\mathcal{O}p\,A$, and by $\mathrm{Real}(T\Lambda, \xi; A, dh)$ the space of injective totally real homomorphisms $T\Lambda \to \xi$ which coincide with dh on $\mathcal{O}p\,A$. The map $f \mapsto df$ provides an inclusion $d : \mathrm{Iso}(\Lambda, M; A, h) \hookrightarrow \mathrm{Real}(T\Lambda, \xi; A, dh)$.

A need not be a submanifold, right?

Theorem 6.8 (Gromov's h-principle for isotropic immersions; contact case, see [32] and also [18])**.** *The map $d : \mathrm{Iso}(\Lambda, M; A, h) \hookrightarrow \mathrm{Real}(T\Lambda, \xi; A, dh)$ is a homotopy equivalence. In particular, given $F \in \mathrm{Real}(T\Lambda, \xi; A, dh)$ one finds $f \in \mathrm{Iso}(\Lambda, M; A, h)$ such that df and F are homotopic in $\mathrm{Real}(T\Lambda, \xi; A, dh)$. Moreover, f can be chosen C^0-close to the map $\Lambda \to M$ covered by the homomorphism F. Given two isotropic immersions $f_0, f_1 \in \mathrm{Iso}(\Lambda, M; A, h)$ and a homotopy $F_t \in \mathrm{Real}(T\Lambda, \xi; A, dh)$, $t \in [0,1]$, connecting df_0 and df_1 one finds a regular homotopy $f_t \in \mathrm{Iso}(\Lambda, M; A, h)$ connecting df_0 and df_1 such that the paths F_t and df_t, $t \in [0,1]$, are homotopic in $\mathrm{Real}(T\Lambda, \xi; A, dh)$ with fixed ends. Moreover, the f_t can be chosen C^0-close to the family of maps $\Lambda \to M$ covered by the homotopy F_t.*

Combining the preceding theorem with Hirsch's Immersion Theorem 6.1 yields

Corollary 6.9. *Let Λ, M, A, h be as in Theorem 6.8. Suppose that $f_0 : \Lambda \to M$ is an immersion which coincides with the isotropic immersion h on $\mathcal{O}p\,A$ and F_t is a family of monomorphisms $T\Lambda \to TN$ such that $F_0 = df_0$, $F_t = dh$ on $\mathcal{O}p\,A$ for all $t \in [0,1]$, and $F_1 \in \mathrm{Real}(T\Lambda, TM; A, dh)$. Then there exists a regular homotopy $f_t : \Lambda \to M$ such that*

(i) $f_1 \in \mathrm{Iso}(\Lambda, M; A, h)$;

(ii) $f_t = h$ *on* $\mathcal{O}p\,A$ *for all* $t \in [0,1]$;

(iii) *there exists a homotopy F_t^s, $s \in [0,1]$, of paths in $\mathrm{Mon}(T\Lambda, TM; A, dh)$ such that $F_t^0 = df_t$ and $F_t^1 = F_t$ for all $t \in [0,1]$, $F_0^s = df_0$ and $F_1^s \in \mathrm{Real}(T\Lambda, \xi; A, dh)$ for all $s \in [0,1]$.*

Proof. We first use Theorem 6.8 to construct an isotropic immersion $g_2 \in \mathrm{Iso}(\Lambda, M; A, h)$ and a homotopy of totally real monomorphisms $F_t \in \mathrm{Real}(T\Lambda, TM; A, dh)$, $t \in [1,2]$, such that $F_2 = dg_2$. Next we apply Hirsch's Theorem 6.1 to get a regular homotopy $g_t \in \mathrm{Imm}(\Lambda, M; A, h)$, $t \in [0,2]$, such that $g_0 = f_0$ and the paths dg_t, F_t, $t \in [0,2]$, are homotopic with fixed ends. Let

$$G : [0,2] \times [0,1] \to \mathrm{Mon}(T\Lambda, TM; A, dh), \qquad (t,s) \mapsto G_t^s$$

be this homotopy, i.e. $G_t^0 = dg_t$, $G_t^1 = F_t$ for all $t \in [0,2]$ and $G_0^s = df_0$, $G_2^s = dg_2$ for all $s \in [0,1]$. The required paths are now defined by $f_t := g_{2t}$, $t \in [0,1]$, and

$$F := G \circ \phi : [0,1] \times [0,1] \to \mathrm{Mon}(T\Lambda, TM; A, dh), \qquad (t,s) \mapsto F_t^s,$$

where $\phi : [0,1] \times [0,1] \to [0,2] \times [0,1]$ is any homeomorphism mapping the boundary as follows (see Figure [fig:h-isotropic]):

$$[0,1] \times 0 \to [0,2] \times 0, \qquad [0,1] \times 1 \to [0,1] \times 1,$$
$$0 \times [0,1] \to 0 \times [0,1], \qquad 1 \times [0,1] \to (2 \times [0,1]) \cup ([1,2] \times 1).$$

□

For later use, let us reformulate the homotopy conditions in Theorem 6.8. Fix compatible complex structures J_M, J_N on ξ_M, ξ_N and positive transversal vector fields v_M, v_N. Since $Sp(2n)$ and $Gl(n,\mathbb{C})$ both deformation retract onto $U(n)$, the space of totally real monomorphisms $TM \to TN$ is homotopy equivalent to the space of monomorphisms $F : TM \to TN$ for which $F(v_M) = v_N$ and $F : (\xi_M, J_M) \to (\xi_N, J_N)$ is complex linear. Since the spaces of compatible complex structures and positive transverse vector fields are contractible, this homotopy equivalence does not depend on the choice of J_M, J_N, v_M, v_N. Adapt, maybe move to other place.

Here is yet another reformulation. Extend J_M to an almost complex structure on $\mathbb{R} \times M$ such that $\eta_M := -J_M v_M$ has positive $\mathbb{R}$-component, and similarly for J_N. Then any monomorphism $F : TM \to TN$ with $F(v_M) = v_N$ and $F|_\xi : \xi_M \to \xi_N$ complex linear extends canonically to a complex linear monomorphism $F^{\mathrm{st}} : T(\mathbb{R} \times M) \to T(\mathbb{R} \times N)$ via $F^{\mathrm{st}}(\eta_M) := \eta_N$. Conversely, if $\dim M < \dim N$ or the manifold M is open, then any complex monomorphism $G : T(\mathbb{R} \times M) \to T(\mathbb{R} \times N)$ is homotopic in the space of complex isomorphisms to a stabilization F^{st} of a monomorphism $F : TM \to TN$. Indeed, this amounts to finding a non-vanishing homotopy between the two sections $G(\eta_M)$ and η_N of the $(\dim N + 1)$-dimensional bundle $g^*T(\mathbb{R} \times N) \to M$, where $g : M \to N$ is the map underlying G. This is always possible if $\dim M < \dim N$ or M is open because the only obstruction, the relative Euler class, lives in $H^{\dim N + 1}(M \times [0,1], M \times \{0,1\}) = 0$.

6.3 The h-principle for isotropic embeddings

We will use the following general position observation.

Proof???

Lemma 6.10. *Let* $\dim \Lambda = k = n - q$, $q \geq 0$. *Then any q-dimensional family of isotropic immersions $\Lambda \to (M, \xi)$ can be C^∞-approximated by a family of isotropic embeddings.*

In particular, if $k < n$ then the word "immersion" in Corollary 6.9 can be replaced by "embedding".

It turns out that if $n > 1$ then, using the stabilization trick from Section 5.7 and Whitney's Theorem 6.6, this can be done even for $k = n$, i.e. one can prove the following h-principle for isotropic *embeddings* rather than immersions. For $n = 1$ the analogous claim is false, see Section 6.6 below.

Do we need this formulation, or should we adapt it to the notation in this section?

Proposition 6.11. *Let (M^{2n+1}, ξ), $n > 1$, be a contact manifold with compatible almost complex structure J on $\mathbb{R} \times M$. Let Λ^k, $k \leq n$, be a closed manifold. Let $f_0 : \Lambda \hookrightarrow M$ be an embedding and $F_t : T(\mathbb{R} \times J^1\Lambda)|_\Lambda \to T(\mathbb{R} \times M)$ be a homotopy of real monomorphisms such that $F_0 = \mathbb{1} \times df_0|_\Lambda$ and F_1 is complex linear. Then there exists an* isotopy *of embeddings $f_t : \mathcal{O}p\,\Lambda \to M$ on an open neighborhood $\mathcal{O}p\,\Lambda \subset J^1\Lambda$ of the zero section such that f_1 is an isocontact embedding, and there exists a homotopy F_t^s, $s \in [0,1]$ of paths in $\mathrm{Mon}\big(T(\mathbb{R} \times \mathcal{O}p\,\Lambda), T(\mathbb{R} \times M)\big)$ such that $F_t^0 = \mathbb{1} \times df_t|_\Lambda$ and $F_t^1 = F_t$ for all $t \in [0,1]$, $F_0^s = \mathbb{1} \times df_0|_\Lambda$ and F_1^s is complex linear for all $s \in [0,1]$. Moreover, we can arrange that $f_t(\Lambda)$ is C^0-close to $f_0(\Lambda)$ for all $t \in [0,1]$.*

Proof. By applying Corollary 6.9 we can satisfy all the conditions of the theorem, except that f_1 will be an immersion rather than an embedding and f_t will be a regular homotopy rather than an isotopy. Of course, it is enough to arrange for the restriction $f_t|_\Lambda$ to be an isotopy. We will keep the notation f_t for this restriction.

By Lemma 6.10, after a C^∞-small isotropic regular homotopy, we may assume that f_1 is an isotropic embedding.

In the subcritical case $k < n$, a generic perturbation of f_t, fixing f_0 and f_1, will turn f_t into a smooth isotopy (Lemma 6.7).

Consider now the Legendrian case $k = n$. We will deform the regular homotopy f_t to an isotopy, keeping the end f_0 fixed and changing f_1 via a Legendrian isotopy. According to Whitney's Theorem 6.6, in order to deform the path f_t to an isotopy keeping *both* ends fixed we need the equality $I_{\{f_t\}} = 0$. On the other hand, according to Proposition 5.25, if $n > 1$ then for any Legendrian embedding g_0 there exists a Legendrian regular homotopy g_t with any prescribed value of the Whitney invariant $I_{\{g_t\}}$. Hence combining f_t, $t \in [0,1]$, with an appropriate Legendrian regular homotopy f_t, $t \in [1,2]$, we obtain a regular homotopy f_t, $t \in [0,2]$, with

$$I_{\{f_t\}_{t \in [0,2]}} = 0.$$

By Whitney's Theorem 6.6, $\{f_t\}$ can be further deformed, keeping the ends f_0 and f_2 fixed, to the required isotopy. $\square$

6.4 The h-principle for totally real embeddings

Proposition 6.12. *[see [32], [18]] Let (V, J) be an almost complex manifold of dimension $2n$, and $f : L \to V$ a smooth real embedding of a k-dimensional manifold L. Suppose that there exists a homotopy F_t, $t \in [0,1]$, of monomorphisms such that $F_0 = df$ and $F_1 : TL \to TV$ is totally real. Then there exists* *a C^0-small isotopy of f to a totally real embedding $g : L \to V$. If the embedding f is totally real on a neighborhood $\mathcal{O}p\, A$ of a closed subset $A \subset L$, and the homotopy F_t is fixed on $\mathcal{O}p\, A$, then the isotopy f_t can also be chosen fixed on $\mathcal{O}p\, A$.*

6.5 Disks attached to J-convex boundary

Theorem 6.14 below, which is a combination of h-principles discussed in this chapter, will play an important role in proving the main results of this book.

Let (V, J) be an almost complex manifold and $W \subset V$ a domain with smooth boundary ∂W. Given a k-disk $D \subset V \setminus \mathrm{Int}\, W$ with $D \cap \partial W = \partial D$ and which transversely intersects ∂W, we say that D is *transversely attached to W in V*. We say that D is *J-orthogonally* attached to W if $J(TD|_{\partial D}) \subset T(\partial W)$. Note that this implies that ∂D is tangent to the distribution $\xi = T(\partial W) \cap JT(\partial W)$. In particular, if ∂W is J-convex then ∂D is an isotropic submanifold for the contact structure ξ.

Remark 6.13. Note that any totally real manifold transversely attached to ∂W along an isotropic submanifold is isotopic relative its boundary to a J-orthogonal one through a totally real isotopy.

Theorem 6.14. *Suppose that (V, J) is an almost complex manifold of dimension $2n$, $n > 2$. Let $W \subset V$ be a domain with smooth J-convex boundary and D a k-disk, $k \le n$, transversely attached to W in V. Then there exists a C^0-small isotopy of D through transversely attached disks to a totally real disk D' which is J-orthogonal to ∂W.*

Proof. Let us denote by f the inclusion $D \hookrightarrow V$. There exists a homotopy of monomorphisms $\Phi_t : TD \to TV$, $t \in [0, 1]$, covering f such that $\Phi_0 = df$ and Φ_1 is totally real. We can assume without a loss of generality that

(a) $\Phi_1(T\partial D) \subset \xi$, and

(b) $\Phi_t(T\partial D) \subset T\partial W$

for all $t \in [0, 1]$. Indeed, by Lemma A.1 (a) we have

Argument changed, please check!

$$\pi_{k-1}(V^{\mathbb{C}}_{n,k}, V^{\mathbb{C}}_{n-1,k-1}) = 0 \quad \text{and} \quad \pi_k(V^{\mathbb{R}}_{2n,k}, V^{\mathbb{R}}_{2n-1,k-1}) = 0$$

for $k \le n$, where $V^{\mathbb{C}}_{n,k}$ and $V^{\mathbb{R}}_{n,k}$ are the complex resp. real Stiefel manifolds of k-frames in $\mathbb{C}^n$ resp. $\mathbb{R}^n$. Now fix outward pointing vector fields $\eta_{\partial D}, \eta_{\partial W}$ along ∂D and ∂W. Then these two vanishing homotopy groups are precisely the obstructions to achieving (a) and (b) together with the condition $\Phi_t(\eta_{\partial S}) = -\eta_{\partial W}$ for all $t \in [0, 1]$.

The restriction $\Phi_t|_{T(\partial D)}$ gives us a homotopy of monomorphisms $\widetilde{\Phi}_t : T(\partial D) \to T(\partial W)$ covering $f|_{\partial D}$. Now we use Proposition 6.11 to construct an isotopy $g_t : \partial D \to \partial W$ such that

(i) $g_0 = f|_{\partial D}$,

(ii) g_1 is isotropic, and

(iii) the path of homomorphisms $dg_t : T(\partial D) \to T(\partial W)$, $t \in [0,1]$ is homotopic to $\widetilde{\Phi}_t$ in the class of paths of monomorphisms beginning at dg_0 and ending at a totally real homomorphism $T(\partial D) \to \xi$.

Extend the isotopy g_t to an isotopy $f_t : D \to V \setminus \operatorname{Int} W$ of smooth embeddings transversely attached to W such that $f_0 = f$. According to Remark 6.13 we can assume that the disk $f_1(D)$ is J-orthogonal to ∂W. We claim that there exists a homotopy of monomorphisms $\Psi_t : TD \to TV$, $t \in [0,1]$ such that

a) $\Psi_0 = df_1 : TD \to TV$,

b) Ψ_1 is totally real, and

c) $\Psi_t = df_1$ on $TD|_{\partial D}$.

Indeed, consider first a homotopy

$$\tilde{\Psi}_t := \begin{cases} df_{1-2t}, & t \in [0, \frac{1}{2}]; \\ \Phi_{2t-1}, & t \in (\frac{1}{2}, 1]. \end{cases}$$

The homotopy $\tilde{\Psi}_t$ satisfies the above conditions a) and b), but not c). However, in view of property (iii) above the path $\tilde{\Psi}_t|_{TD|_{\partial D}}$ is homotopic through paths with fixed ends to a path of totally real monomorphisms and hence the homotopy $\tilde{\Phi}_t$ can be modified to a homotopy Ψ_t satisfying condition c) as well. More explicitly, property (iii) allows us to pick a continuous family of monomorphisms $\Gamma_t^s : T(\partial D) \to \xi$, $s, t \in [0,1]$, such that $\Gamma_t^0 = \tilde{\Psi}_t$, $\Gamma_t^1 = df_1|_{\partial D}$, $\Gamma_0^s = fd_1|_{\partial D}$, and Γ_1^s is totally real for all $s \in [0,1]$, see Figure [fig:???]. After rescaling in the unit disk D we may assume that $\tilde{\Psi}_t(x)$ is independent of the radius for $x \in D$ with
Please check this! $|x| \geq 1/2$. Then the desired homotopy Ψ can be defined by

$$\Psi_t(x) := \begin{cases} \tilde{\Psi}_t(2x), & |x| \in [0, \frac{1}{2}]; \\ \Gamma_t^{2|x|-1}(x), & |x| \in (\frac{1}{2}, 1]. \end{cases}$$

It remains to apply Gromov's h-principle for totally real embeddings 6.12. It provides an isotopy of embeddings $f_t : D \to V \setminus \operatorname{Int} W$, $t \in [1,2]$, fixed along ∂D together with its differential, such that $f_2 : D \to V \setminus \operatorname{Int} W$ is totally real and J-orthogonal to ∂W. Finally, note that all the isotopies provided by Propositions 6.11 and 6.12 can be chosen C^0-small. This concludes the proof of Theorem 6.14. □

6.6 The three-dimensional case

[to be added]

Chapter 7

Some complex analysis

7.1 Some complex analysis on Stein manifolds

There exist a number of equivalent definitions of a Stein manifold. We have already encountered two of them.

Affine definition. *A complex manifold V is Stein if it admits a proper holomorphic embedding into some $\mathbb{C}^N$.*

J-convex definition. *A complex manifold V is Stein if it admits an exhausting J-convex function $f : V \to \mathbb{R}$.*

The classical definition rests on the concept of holomorphic convexity. To a subset $K \subset V$ of a complex manifold associate its *holomorphically convex hull*

$$\hat{K} := \{x \in V \mid |f(x)| \leq \sup_K |f| \text{ for all holomorphic functions } f : V \to \mathbb{C}\}.$$

Call V *holomorphically convex* if $\hat{K}$ is compact for all compact subsets $K \subset V$.

Example 7.1. Let $B \subset \mathbb{C}^N$ be a closed ball around the origin. For $x \notin B$ the holomorphic function $f(z) := (z, x)$ satisfies $|f(z)| \leq |z|\,|x| < |x|^2 = |f(x)|$ for all $z \in B$. Hence $B = \hat{B}$ equals its own holomorphically convex hull.

Next consider a properly embedded complex submanifold $V \subset \mathbb{C}^{\mathbb{N}}$ and a compact subset $K \subset V$. Let $B \subset \mathbb{C}^N$ be a closed ball containing K. Then $\hat{K} \subset \widehat{(V \cap B)} \subset \hat{B} = B$, where the first two holomorphically convex hulls are taken in V and the third in $\mathbb{C}^N$. Since $\hat{K}$ is closed in V, it is compact. This shows that V is holomorphically convex.

Example 7.2 (Hartogs phenomenon). The *Hartogs domain* $\Omega := \mathrm{int}B^4(1) \setminus B^4(1/2) \subset \mathbb{C}^2$ has the holomorphically convex hull $\hat{\Omega} = \mathrm{int}B^4(1)$ (in particular, Ω is not holomorphically convex). To see this, let $f : \Omega \to \mathbb{C}$ be a holomorphic function. For fixed $z \in \mathbb{C}$, $|z| < 1$, the function $w \mapsto f(z, w)$ on the annulus (or disk) $A_z := \{w \in \mathbb{C} \mid 1/4 - |z|^2 < |w|^2 < 1 - |z|^2\}$ has a Laurent expansion

$$f(z,w) = \sum_{k=-\infty}^{\infty} a_k(z) w^k.$$

The coefficients $a_k(z)$ are given by

$$a_k(z) = \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(z,\zeta)}{\zeta^{k+1}} d\zeta$$

for any $r > 0$ with $1/4 - |z|^2 < r^2 < 1 - |z|^2$. In particular, $a_k(z)$ depends holomorphically on z with $|z| < 1$. Since A_z is a disk for $|z| > 1/2$, we have $a_k(z) = 0$ for $k < 0$ and $|z| > 1/2$, hence by unique continuation for all z with $|z| < 1$. Thus the Laurent expansion defines a holomorphic extension of f to the ball $\mathrm{int} B^4(1)$.

Classical definition. *A complex manifold V is Stein if it has the following 3 properties:*

(i) V is holomorphically convex;

(ii) for any $x \neq y \in V$ there exists a holomorphic function $f : V \to \mathbb{C}$ with $f(x) \neq f(y)$;

(iii) for every $x \in V$ there exist holomorphic functions $f_1, \dots, f_n : V \to \mathbb{C}$ which form a holomorphic coordinate system at x.

Clearly, the affine definition implies the other two (holomorphic convexity was shown in Example 7.1). The classical definition immediately implies that every compact subset $K \subset V$ can be holomorphically embedded into some $\mathbb{C}^N$. The implication "classical $\Longrightarrow$ affine" is the content of

Theorem 7.3. *[Remmert [55]] A Stein manifold V in the classical sense admits a proper holomorphic embedding into some $\mathbb{C}^N$.*

Remark 7.4. A lot of research has gone into finding the smallest N for given $n = \dim_{\mathbb{C}} V$. After intermediate work of Forster, the optimal integer $N = [3n/2] + 1$ was finally established by Eliashberg-Gromov [17] and Schürmann [57].

The implication "J-convex $\Longrightarrow$ classical" was proved by Grauert in 1958:

Theorem 7.5 (Grauert [28])**.** *A complex manifold which admits an exhausting J-convex function is Stein in the classical sense.*

In particular, Grauert's theorem solves what was known, for domains in $\mathbb{C}^n$, as "Levi's problem":

Corollary 7.6. *A relatively compact domain $U \subset V$ in a Stein manifold V with smooth J-convex boundary ∂U is Stein.*

Proof. By Lemma 2.4, there exists a J-convex function $\phi : W \to (0,2)$ on a neighborhood W of ∂U in V with $\partial U = \phi^{-1}(1)$. Let $\psi : V \to \mathbb{R}$ be a J-convex function with $\min_{\overline{U}} \psi > 0$. Pick a convex increasing diffeomorphism $f : (0,1) \to (0,\infty)$. Then a smoothing of $\max(f \circ \phi, \psi) : U \to \mathbb{R}$ is J-convex and exhausting, so U is Stein by Grauert's theorem. □

Remark 7.7. In fact, Grauert proves in [28] the following generalization of Levi's problem: A relatively compact domain $U \subset V$ in a complex (not necessarily Stein) manifold V with smooth J-convex boundary ∂U is holomorphically convex.

It is clear from any of the definitions that properly embedded complex submanifolds of Stein manifolds are Stein. We will refer to them as *Stein submanifolds*.

Two fundamental results about Stein manifolds are Cartan's Theorems A and B. They are formulated in the language of sheaves, see [10] for the relevant definitions and properties. Let V be a complex manifold and $\mathcal{O}$ the sheaf of holomorphic functions on V. For a nonnegative integer p, let $\mathcal{O}^p$ be the sheaf of holomorphic maps to $\mathbb{C}^p$. A sheaf $\mathcal{F}$ on V is called *analytic* if for each $x \in V$, $\mathcal{F}_x$ is a module over $\mathcal{O}_x$, and the multiplication $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ is continuous. A sheaf homomorphism $f : \mathcal{F} \to \mathcal{G}$ between analytic sheaves is called analytic if it is a module homomorphism. An analytic sheaf $\mathcal{F}$ is called *coherent* if every $x \in V$ has a neighborhood U such that $\mathcal{F}_U$ equals the cokernel of an analytic sheaf homomorphism $f : \mathcal{O}_U^p \to \mathcal{O}_U^q$, for some nonnegative integers p, q.

Oka's Coherence Theorem [53] states that a subsheaf $\mathcal{F}$ of $\mathcal{O}^p$ is coherent if and only if it is *locally finitely generated*, i.e., for every point $x \in V$ there exists a neighborhood U and finitely many sections f_i of $\mathcal{F}_U$ that generate $\mathcal{F}_y$ as an $\mathcal{O}_y$-module for every $y \in U$.

Example 7.8. Let $W \subset V$ be a properly embedded complex submanifold of a complex manifold V and $d \geq 0$ an integer. For an open subset $U \subset V$, let $\mathcal{I}_U$ be the ideal of holomorphic functions on U whose d-jet vanishes at all points of $U \cap W$. This defines an analytic sheaf $\mathcal{I}$ on V. We claim that $\mathcal{I}$ is coherent. To see this, let $x \in V$. If $x \notin W$ we find a neighborhood U of x with $U \cap W = \emptyset$ (since $W \subset V$ is closed), hence $\mathcal{I}_U = \mathcal{O}_U$. If $x \in W$ we find a small open polydisk $U \cong \text{int}\big(B^2(1) \times \cdots \times B^2(1)\big) \subset V$ around x with complex coordinates $(z_1, \ldots, z_n)$ in which $W \cap U = \{z_1 = \cdots = z_k = 0\}$. Then the ideal I_U is generated as an $\mathcal{O}_U$-module by the monomials of degree $(d+1)$ in $z_1, \ldots, z_k$, so by Oka's Coherence Theorem [53], $\mathcal{I}$ is coherent.

Remark 7.9. The coherence of the sheaf $\mathcal{I}$ in the preceding example can also be proved without Oka's theorem as follows. As above, let $(z_1, \ldots, z_n)$ be complex coordinates on a polydisk U in which $W \cap U = \{z_1 = \cdots = z_k = 0\}$. We claim that every $f \in \mathcal{I}_U$ has a unique representation

$$f(z) = \sum_I f_I(z) z^I,$$

where the summation is over all $I = (i_1, \ldots, i_k)$ with $i_1 + \cdots + i_k = d + 1$ and

$z^I = z_1^{i_1} \dots z_k^{i_k}$. The coefficient f_I is a holomorphic function of $z_\ell, \dots, z_n$, where $1 \le \ell \le k$ is the largest integer with $i_\ell \neq 0$.

We first prove the claim for $d = 0$ by induction over k. The case $k = 1$ is clear, so let $k > 1$. The function $(z_k, \dots, z_n) \mapsto f(0, \dots, 0, z_k, \dots, z_n)$ vanishes at $z_k = 0$, thus (as in the case $k = 1$) it can be uniquely written as $z_k f_k(z_k, \dots, z_n)$ with a holomorphic function f_k. Since the function $(z_1, \dots, z_n) \mapsto f(z_1, \dots, z_n)$ vanishes at $z_1 = \dots = z_{k-1} = 0$, by induction hypothesis it can be uniquely written as $z_1 f_1(z_1, \dots, z_n) + \dots + z_{k-1} f_{k-1}(z_{k-1}, \dots, z_n)$ with holomorphic functions $f_1, \dots, f_{k-1}$. This proves the case $d = 0$. The general case $d > 0$ follows by induction over d: Using the case $d = 0$, we write $f(z)$ uniquely as $z_1 f_1(z_1, \dots, z_n) + \dots + z_k f_k(z_k, \dots, z_n)$. Now note that the functions $f_1, \dots, f_k$ must vanish to order $d-1$ at $z_1 = \dots = z_k = 0$ and use the induction hypothesis. This proves the claim.

By the claim, $\mathcal{I}_U$ is the direct sum of copies of the rings $\mathcal{F}_U^\ell$ of holomorphic functions of $z_\ell, \dots, z_n$ for $1 \le \ell \le k$. Since $\mathcal{F}_U^\ell$ is isomorphic to the cokernel of the homomorphism $\mathcal{O}_U^{\ell-1} \to \mathcal{O}_U$, $f_1, \dots, f_{\ell-1} \mapsto z_1 f_1 + \dots + z_{\ell-1} f_{\ell-1}$, this proves coherence of $\mathcal{I}$.

Now we can state Cartan's Theorems A and B. Denote by $H^q(V, \mathcal{F})$ the cohomology with coefficients in the sheaf $\mathcal{F}$. In particular, $H^0(V, \mathcal{F})$ is the space of sections in $\mathcal{F}$. Every subsheaf $\mathcal{G} \subset \mathcal{F}$ induces a long exact sequence

$$\dots \to H^q(V, \mathcal{G}) \to H^q(V, \mathcal{F}) \to H^q(V, \mathcal{F}/\mathcal{G}) \to H^{q+1}(V, \mathcal{G}) \to \dots .$$

Theorem 7.10 (Cartan [10]). *Let V be a Stein manifold and $\mathcal{F}$ a coherent analytic sheaf on V. Then*

(A) for every $x \in V$, $H^0(V, \mathcal{F})$ generates $\mathcal{F}_x$ as an $\mathcal{O}_x$-module;

(B) $H^q(V, \mathcal{F}) = \{0\}$ for all $q > 0$.

We will only use the following two consequences of Cartan's Theorem B.

Corollary 7.11. *Let W be a Stein submanifold of a Stein manifold V. Then every holomorphic function $f : W \to \mathbb{C}$ extends to a holomorphic function $F : V \to \mathbb{C}$. More generally, let $f : U \to \mathbb{C}$ be a holomorphic function on a neighborhood of W and d a nonnegative integer. Then there exists a holomorphic function $F : V \to \mathbb{C}$ whose d-jet coincides with that of f at points of W.*

Proof. Let $\mathcal{I}$ be the analytic sheaf of holomorphic functions on V whose d-jet vanishes at points of W. By the example above, $\mathcal{I}$ is coherent. Thus by Cartan's Theorem B, $H^1(V, \mathcal{I}) = 0$, so by the long exact sequence the homomorphism $H^0(V, \mathcal{O}) \to H^0(V, \mathcal{O}/\mathcal{I})$ is surjective. Now $\mathcal{O}_x/\mathcal{I}_x = \{0\}$ for $x \notin W$, and for $x \in W$ elements of $\mathcal{O}_x/\mathcal{I}_x$ are d-jets of germs of holomorphic functions along W. So f defines a section in $\mathcal{O}/\mathcal{I}$, and we conclude that f is the restriction of a section F in $\mathcal{O}$. □

Corollary 7.12. *Every Stein submanifold W of a Stein manifold V is the common zero set of a finite number (at most $\dim_{\mathbb{C}} V + 1$) of holomorphic functions $f_i : V \to \mathbb{C}$.*

Proof. The argument is given in [11]. It uses some basic properties of analytic subvarieties, see e.g. [30]. An *analytic subvariety* of a complex manifold V is a closed subset $Z \subset V$ that is locally the zero set of finitely many holomorphic functions. Z is a stratified space $Z = Z_0 \cup \cdots \cup Z_k$, where Z_i is a (non-closed) complex submanifold of dimension i. Define the (complex) dimension of Z as the dimension k of the top stratum. If $Z' \subset Z$ are analytic subvarieties of the same dimension, then Z' contains a connected component of the top stratum Z_k of Z.

Now let $W \subset V$ be a Stein submanifold of a Stein manifold V. Pick a set $S_1 \subset V$ containing one point on each connected component of $V \setminus W$. Since S_1 is discrete, $W \cup S_1$ is a Stein submanifold of V. By Corollary 7.11, there exists a holomorphic function $f_1 : V \to \mathbb{C}$ which equals 0 on W and 1 on S_1. The zero set $W_1 := \{f_1 = 0\}$ is an analytic subvariety of V, containing W, such that $W_1 \setminus W$ has dimension $\leq n - 1$, where $n = \dim_{\mathbb{C}} V$. Pick a set $S_2 \subset W_1 \setminus W$ containing one point on each connected component of the top stratum of W_1 that is not contained in W. Since each compact set meets only finitely many components of W_1, the set S_2 is discrete, so $W \cup S_2$ is a Stein submanifold of V. By Corollary 7.11, there exists a holomorphic function $f_2 : V \to \mathbb{C}$ which equals 0 on W and 1 on S_2. The zero set $W_2 := \{f_1 = f_2 = 0\}$ is an analytic subvariety of V, containing W, such that $W_2 \setminus W$ has dimension $\leq n - 2$. Continuing this way, we find holomorphic functions $f_1, \dots, f_{n+1} : V \to \mathbb{C}$ such that $W \subset W_{n+1} := \{f_1 = \cdots = f_{n+1} = 0\}$ and $W_{n+1} \setminus W$ has dimension ≤ -1. Thus $W_{n+1} \setminus W = \emptyset$ and $W = \{f_1 = \cdots = f_{n+1} = 0\}$. □

7.2 Real analytic approximations

In order to holomorphically attach handles, we need to approximate smooth objects by real analytic ones. In this section we collect the relevant results.

A function $f : U \to \mathbb{R}^m$ on an open domain $U \subset \mathbb{R}^n$ is called *real analytic* if it is locally near each point given by a convergent power series. A *real analytic manifold* is a manifold with an atlas such that all transition functions are real analytic. A submanifold is called real analytic if it is locally the transverse zero set of a real analytic function. Real analytic bundles and sections are defined in the obvious way.

Remark 7.13. As a special case of the Cauchy-Kowalewskaya theorem (see e.g. [20]), the solution of an ordinary differential equation with real analytic coefficients depends real analytically on all parameters.

Complexification. There is a natural functor, called *complexification*, from the real analytic to the holomorphic category. First note that any real analytic function $f : U \to \mathbb{R}^m$, defined on an open domain $U \subset \mathbb{R}^n$, can be uniquely extended to a holomorphic function $f^{\mathbb{C}} : U^{\mathbb{C}} \to \mathbb{C}^m$ on an open domain $U^{\mathbb{C}} \subset \mathbb{C}^n$ with $U^{\mathbb{C}} \cap \mathbb{R}^n = U$. A bit less obviously, any real analytic manifold M can be complexified to a complex manifold $M^{\mathbb{C}}$ which contains M as a real analytic submanifold. This can be seen as follows (see [8] for details). Pick a locally finite covering of M by countably many real analytic coordinate charts $\phi_i : \mathbb{R}^n \supset U_i \to M$. So the transition functions

$$\phi_{ij} := \phi_j^{-1} \circ \phi_i : U_{ij} := \phi_i^{-1}\Big(\phi_i(U_i) \cap \phi_j(U_j)\Big) \to U_{ji}$$

are real analytic diffeomorphisms. Successively extend them to biholomorphic maps $\phi_{ij}^{\mathbb{C}} : U_{ij}^{\mathbb{C}} \to U_{ji}^{\mathbb{C}}$ such that $\phi_{ji}^{\mathbb{C}} = (\phi_{ij}^{\mathbb{C}})^{-1}$. Note that $U_{ii}^{\mathbb{C}} = U_i^{\mathbb{C}}$ and $\phi_{ii}^{\mathbb{C}} = \mathbb{1}$. Define $M^{\mathbb{C}}$ as the quotient of the disjoint union $\coprod_i U_i^{\mathbb{C}}$ by the equivalence relation $z_i \sim z_j$ iff $z_i \in U_{ij}^{\mathbb{C}}$ and $z_j = \phi_{ij}^{\mathbb{C}}(z_i) \in U_{ji}^{\mathbb{C}}$. (This is an equivalence relation because of the cocycle condition $\phi_{jk}^{\mathbb{C}} \circ \phi_{ij}^{\mathbb{C}} = \phi_{ik}^{\mathbb{C}}$.) The inclusions $U_i^{\mathbb{C}} \hookrightarrow \coprod_j U_j^{\mathbb{C}}$ induce coordinate charts $U_i^{\mathbb{C}} \hookrightarrow M^{\mathbb{C}}$ with biholomorphic transition functions. Finally, this construction needs to be slightly modified to ensure that $M^{\mathbb{C}}$ is Hausdorff (see [8]).

Similarly, one sees that a real analytic map $f : M \to N$ between real analytic manifolds extends to a holomorphic map $f^{\mathbb{C}} : M^{\mathbb{C}} \to N^{\mathbb{C}}$ between (sufficiently small) complexifications. It follows that the complexification $M^{\mathbb{C}}$ is unique in the sense that if V, W are complex manifolds, containing M as real analytic and totally real submanifolds, with $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W = \dim_{\mathbb{R}} M$, then some neighborhoods of M in V and W are biholomorphic. A corresponding uniqueness holds for complexifications of maps. As a real manifold, the complexification $M^{\mathbb{C}}$ is diffeomorphic to the tangent bundle TM.

Complexification has the obvious functorial properties. For example, if $N \subset M$ is a real analytic submanifold of a real analytic manifold M, then the (sufficiently small) complexification $N^{\mathbb{C}}$ is a complex submanifold of $M^{\mathbb{C}}$.

The crucial observation, due to Grauert [28], is that complexifications of real analytic manifolds are in fact Stein.

Proposition 7.14. *Let $M^{\mathbb{C}}$ be the complexification of a real analytic manifold M. Then M possesses arbitrarily small neighborhoods in $M^{\mathbb{C}}$ which are Stein.*

Proof. By Proposition 2.13, M possesses arbitrary small neighborhoods with exhausting J-convex functions. By Grauert's Theorem 7.5, these neighborhoods are Stein. □

A complexification $M^{\mathbb{C}}$ which is Stein is called a *Grauert tube* of M. Now the basic results about real analytic manifolds follow via complexifiction from corresponding results about Stein manifolds.

Corollary 7.15. *Every real analytic manifold admits a proper real analytic embedding into some $\mathbb{R}^N$.*

Proof. By Theorem 7.3, a Grauert tube $M^{\mathbb{C}}$ of M embeds properly holomorphically into some $\mathbb{C}^N$. Then restrict this embedding to M. □

Corollary 7.16. *Let N be a properly embedded real analytic submanifold of a real analytic manifold M. Then every real analytic function $f : N \to \mathbb{R}$ extends to a real analytic function $F : M \to \mathbb{R}$. More generally, let $f : U \to \mathbb{R}$ be a real analytic function on a neighborhood of N and d a nonnegative integer. Then there exists a real analytic function $F : M \to \mathbb{R}$ whose d-jet coincides with that of f at points of N.*

Proof. Let $M^{\mathbb{C}}$ be a Grauert tube of M. After possibly shrinking $M^{\mathbb{C}}$, we may assume that a complexification $N^{\mathbb{C}}$ of N is a properly embedded complex submanifold of $M^{\mathbb{C}}$, and f complexifies to a holomorphic function $f^{\mathbb{C}}$ on a neighborhood of $N^{\mathbb{C}}$ in $M^{\mathbb{C}}$. Corollary 7.11 provides a holomorphic function $G : M^{\mathbb{C}} \to \mathbb{C}$ whose d-jet agrees with that of $f^{\mathbb{C}}$ at points of $N^{\mathbb{C}}$. Then the restriction of the real part of G to M is the desired function F. □

Corollary 7.17. *Every properly embedded real analytic submanifold N of a real analytic manifold M is the common zero set of a finite number (at most $2\dim_{\mathbb{R}} M + 2$) of real analytic functions $f_i : M \to \mathbb{R}$.*

Proof. Complexify N to a properly embedded submanifold $N^{\mathbb{C}} \subset M^{\mathbb{C}}$ of a Grauert tube $M^{\mathbb{C}}$. By Corollary 7.12, $N^{\mathbb{C}}$ is the zero set of at most $n+1$ holomorphic functions $F_i : M^{\mathbb{C}} \to \mathbb{C}$, where $n = \dim_{\mathbb{R}} M$. The restrictions of $\operatorname{Re} F_i$ and $\operatorname{Im} F_i$ to M yield the desired functions f_i. □

Remark 7.18. H. Cartan [11] takes a slightly different route to prove Corollaries 7.16 and 7.17: Define coherent analytic sheaves on real analytic manifolds analogously to the complex analytic case. Cartan proves that for every coherent analytic sheaf $\mathcal{F}$ on M, there exists a coherent analytic sheaf $\mathcal{F}^{\mathbb{C}}$ on a complexification $M^{\mathbb{C}}$ such that $\mathcal{F}^{\mathbb{C}}|_M = \mathcal{F} \otimes \mathbb{C}$. From this he deduces the analogues of Theorems A and B in the real analytic category, which imply the corollaries as in the complex analytic case.

Corollary 7.15 implies that every C^k-function on a real analytic manifold M can be C^k-approximated by real analytic functions. To state the result, equip M with a metric and connection so that we can speak of k-th (covariant) derivatives of functions on M and their norms.

Corollary 7.19. *Let $f : M \to \mathbb{R}$ be a C^k-function on a real analytic manifold. Then for every compact subset $K \subset M$ and $\varepsilon > 0$ there exists a real analytic function $g : M \to \mathbb{R}$ which is ε-close to f together with its first k derivaties on K.*

Proof. Embed M real analytically into some $\mathbb{R}^N$. Pick any C^k-function $F : \mathbb{R}^N \to \mathbb{R}$ which coincides with f on K. By Weierstrass' theorem (see e.g. [22]), F can be C^k-approximated over K by a polynomial $G : \mathbb{R}^N \to \mathbb{R}$. Let g be the restriction of G to M. □

On the other hand, Corollary 7.16 shows that every real analytic function on a properly embedded real analytic submanifold N of a real analytic manifold M can be extended to a real analytic function on M, with prescribed normal d-jet along N. The following result combines the approximation and extension results.

Proposition 7.20. *Let $f : M \to \mathbb{R}$ be a C^k-function on a real analytic manifold. Let N be a properly embedded real analytic submanifold, $K \subset M$ a compact subset, d a nonnegative integer and $\varepsilon > 0$. Suppose that f is real analytic on a neighborhood of N. Then there exists a real analytic function $F : M \to \mathbb{R}$ with the following properties:*

- *F is ε-close to f together with its first k derivatives over K;*
- *the d-jet of F coincides with that of f at every point of N.*

The proof is based on the following

Lemma 7.21. *For every $d, k \in \mathbb{N}$ there exists a constant $C_{d,k}$ such that for all $p \in \mathbb{N}$, $D > \delta > 0$ and $\gamma > 0$ there exists a polynomial $P : \mathbb{R} \to \mathbb{R}$ with the following properties:*

- *$P(0) = 1$ and $P'(0) = \cdots = P^{(d)}(0) = 0$;*
- *$|P^{(l)}(x)| \leq \gamma$ for all $0 \leq l \leq k$ and $\delta \leq |x| \leq D$;*
- *$|P^{(l)}(x)| \leq C_{d,k}/\delta^l$ for all $0 \leq l \leq k$ and $|x| \leq \delta$.*

Proof. Let k be given. Pick a C^k-function $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

- $f(x) = 0$ near $x = 0$;
- $f(x) = 1$ for $|x| \geq 1$;
- $|(f-1)^{(l)}(x)| \leq C_k/2$ for $|x| \leq 1$ and $0 \leq l \leq k$,

with a constant C_k depending only on k. For $D > \delta > 0$ define $g(x) := f(x/\delta)$. Then $g : \mathbb{R} \to \mathbb{R}$ has the following properties:

- $g(x) = 0$ near $x = 0$;
- $g(x) = 1$ for $\delta \leq |x| \leq D$;
- $|(g-1)^{(l)}(x)| \leq C_k/(2\delta^l)$ for $|x| \leq \delta$ and $0 \leq l \leq k$.

By Weierstrass' theorem (see e.g. [22]), we find for every $\beta > 0$ a polynomial $Q : \mathbb{R} \to \mathbb{R}$ satisfying

- $Q(0) = 0$;

- $|(Q-1)^{(l)}(x)| \leq \beta$ for $\delta \leq |x| \leq D$ and $0 \leq l \leq k$;
- $|(Q-1)^{(l)}(x)| \leq C_k/\delta^l$ for $|x| \leq \delta$ and $0 \leq l \leq k$.

For $d \in \mathbb{N}$ consider the polynomial $P(x) := Q(x)^{d+1} - 1$. By the Leibniz rule,

$$(Q^{d+1})^{(l)}(x) = \sum_{i_1+\cdots+i_{d+1}=l} \binom{l}{i_1 \dots i_{d+1}} Q^{(i_1)}(x) \cdots Q^{(i_{d+1})}(x).$$

This shows $P(0) = 1$ and $P^{(l)}(0) = 0$ for $1 \leq l \leq d$. Now let $\gamma > 0$ be given. For $\delta \leq |x| \leq D$ and $1 \leq l \leq k$ the estimates on Q yield

$$\begin{aligned} |(P)^{(l)}(x)| &\leq \sum_{i_1+\cdots+i_{d+1}=l} \binom{l}{i_1 \dots i_{d+1}} \beta^l (1+\beta)^{d+1} \\ &= (d+1)^l \beta^l (1+\beta)^{d+1} \leq (d+1)^k \beta (1+\beta)^{d+1} \leq \gamma \end{aligned}$$

for β sufficiently small. For $\delta \leq |x| \leq D$ and $l = 0$ we find

$$|P^{(l)}(x)| = |Q(x) - 1|\, |1 + Q(x) + \cdots + Q^d(x)| \leq \beta(2d+1) \leq \gamma$$

for $\beta \leq 1$ sufficiently small. Similarly, for $|x| \leq \delta$ and $1 \leq l \leq k$ we get

$$|(P)^{(l)}(x)| \leq \sum_{i_1+\cdots+i_{d+1}=l} \binom{l}{i_1 \dots i_{d+1}} \frac{(C_k+1)^{d+1}}{\delta^l} \leq \frac{(d+1)^k (C_k+1)^{d+1}}{\delta^l},$$

and for $|x| \leq \delta$ and $l = 0$,

$$|P(x)| \leq| Q^{d+1}(x)| + 1 \leq (C_k+1)^{d+1} + 1.$$

Hence P satisfies the required estimates with $C_{d,k} := (d+1)^k (C_k+1)^{d+1} + 1$. □

Proof of Proposition 7.20. By Corollary 7.17, there exist real analytic functions $\phi_1, \dots, \phi_m : M \to \mathbb{R}$ such that $N = \{\phi_1 = \cdots = \phi_m = 0\}$. Then $\phi := \phi_1^2 + \cdots + \phi_m^2 : M \to \mathbb{R}$ is real analytic and $N = \phi^{-1}(0)$. Let dist_N be the distance from N with respect to some Riemannian metric on M. Since ϕ vanishes only to finite order in directions transversal to N, there exists an $r \in \mathbb{N}$ such that, after rescaling the metric, we have $\phi(x) \geq \mathrm{dist}_N(x)^r$ for all $x \in K$. Set $D := \max_K \phi$. For δ sufficiently small, $W := \{\phi \leq \delta\}$ is a tubular neighborhood of N over K.

For $\delta, \gamma > 0$ let $P : \mathbb{R} \to \mathbb{R}$ be the polynomial from Lemma 7.21. The real analytic function $\psi := P \circ \phi : M \to \mathbb{R}$ has the following properties:

- $\psi(x) =$ and $\psi'(x) = \cdots = \psi^{(d)}(x) = 0$ for $x \in N$;
- $|\psi^{(l)}(x)| \leq C_1 \gamma$ for all $0 \leq l \leq k$ and $x \in K \setminus W$;
- $|\psi^{(l)}(x)| \leq C_1 C_{d,k}/\delta^l$ for all $0 \leq l \leq k$ and $x \in W \cap K$,

with a constant C_1 depending only on ϕ (and not on δ and γ). Here and in the following we assume $M \subset \mathbb{R}^N$ and denote by $\psi^{(l)}$ any partial derivative of order l.

Without loss of generality, we may increase d until $d \geq r(k+1)$. By Corollary 7.16, there exists a real analytic function $h : M \to \mathbb{R}$ whose d-jet agrees with that of f at points of N. Hence there exists a constant C_2, depending only on f and h, such that

- $|(f-h)^{(l)}(x)| \leq C_2 \operatorname{dist}_N(x)^{d-l} \leq C_2\, \delta^{(d-l)/r}$ for all $0 \leq l \leq k$ and $x \in W \cap K$;
- $|(f-h)^{(l)}(x)| \leq C_2$ for all $0 \leq l \leq k$ and $x \in K \setminus W$.

We can estimate the product $\psi \cdot (f-h)$ for $0 \leq l \leq k$ and $x \in W \cap K$ by

$$\begin{aligned} |[\psi(f-h)]^{(l)}(x)| &\leq \sum_{i=0}^{l} |(f-h)^{(i)}(x)|\, |\psi^{(l-i)}(x)| \\ &\leq \sum_{i=0}^{l} C_2\, \delta^{(d-i)/r} \frac{C_1 C_{d,k}}{\delta^{l-i}} \leq (k+1) C_1 C_2 C_{d,k} \delta, \end{aligned}$$

since the exponent of δ satisfies $(d-i)/r + i - l \geq d/r - l \geq 1$ by the choice of d. Similarly, for $0 \leq l \leq k$ and $x \in K \setminus W$ we obtain

$$|[\psi(f-h)]^{(l)}(x)| \leq (k+1) C_1 C_2 \gamma.$$

Now let $\varepsilon > 0$ be given. By Corollary 7.19, there exists a real analytic function $g : M \to \mathbb{R}$ with $|(f-g)^{(l)}(x)| < \varepsilon'$ for all $0 \leq l \leq k$ and $x \in K$, with $\varepsilon' > 0$ to be determined later. Define the real analytic function

$$F := g + \psi \cdot (h-g) : M \to \mathbb{R}.$$

Since $\psi'(x) = \cdots = \psi^{(d)}(x) = 0$ for $x \in N$, the d-jet of F agrees with that of f at points of N. For $0 \leq l \leq k$ and $x \in K \setminus W$ we estimate

$$\begin{aligned} |(F-f)^{(l)}(x)| &\leq |[(1-\psi)(g-f)]^{(l)}(x)| + |[\psi(h-f)]^{(l)}(x)| \\ &\leq (1 + C_1\gamma)\varepsilon' + (k+1) C_1 C_2 \gamma. \end{aligned} \tag{7.1}$$

For $0 \leq l \leq k$ and $x \in W \cap K$ we find

$$\begin{aligned} |(F-f)^{(l)}(x)| &\leq |[(1-\psi)(g-f)]^{(l)}(x)| + |[\psi(h-f)]^{(l)}(x)| \\ &\leq \frac{C_1 C_{d,k}}{\delta^k}\varepsilon' + (k+1) C_1 C_2 C_{d,k} \delta. \end{aligned} \tag{7.2}$$

Now first choose $\gamma > 0$ small enough so that the second term on the right-hand side of (7.1) becomes $< \varepsilon/2$. Given γ, choose $\delta > 0$ small enough so that the second term on the right-hand side of (7.2) becomes $< \varepsilon/2$. Finally, choose $\varepsilon' > 0$ small enough so that the first terms on the right-hand sides of (7.1) and (7.2) become $< \varepsilon/2$. Then F has the desired properties. □

Proposition 7.20 clearly generalizes to sections in real analytic bundles $E \to M$. For this, view the total space of the bundle as a real analytic manifold and note that a map $M \to E$ that is C^0-close to a section is a section. Thus we have

Theorem 7.22. *Let $f : M \to E$ be a C^k-section in a real analytic fibre bundle $E \to M$ over a real analytic manifold M. Let $N \subset M$ be a properly embedded real analytic submanifold, $K \subset M$ a compact subset, d a nonnegative integer and $\varepsilon > 0$. Suppose that f is real analytic on a neighborhood of N. Then there exists a real analytic section $F : M \to E$ with the following properties:*

(i) F is ε-close to f together with its first k derivatives over K;

(ii) the d-jet of F coincides with that of f at every point of N.

Example 7.23. Every Riemannian metric on a real analytic manifold can be C^k-approximated by a real analytic metric. By Remark 7.13, the exponential map of a real analytic metric is real analytic. Now the standard proof yields real analytic tubular resp. collar neighborhoods of compact real analytic submanifolds resp. boundaries. In particular, this allows us to extend any compact real analytic manifold with boundary to a slightly larger open real analytic manifold.

Theorem 7.22 also has a version with parameters.

Corollary 7.24. *Let $E \to M$ be a real analytic fibre bundle over a real analytic manifold M, $K \subset M$ a compact subset, and $\varepsilon > 0$. Let $f_t : M \to E$ be a family of C^k-sections depending in a C^k fashion on a parameter t in a compact real analytic manifold T with boundary. Suppose that the f_t are real analytic for $t \in \partial T$ and depend real analytically on $t \in \partial T$. Then there exists a family of real analytic sections $F_t : M \to E$, depending real analytically on $t \in T$, with the following properties:*

(i) F_t is ε-close to f_t together with its first k derivatives over K for all $t \in T$;

(ii) $F_t = f_t$ for $t \in \partial T$.

Proof. By Example 7.23, we can include Λ in a larger open real analytic manifold $\tilde{\Lambda}$. Extend f_t to a C^k-family $\tilde{f}_t$ over $\tilde{\Lambda}$ and view $\tilde{f}_t$ as a C^k-section in the bundle $E \to \tilde{\Lambda} \times M$. Now apply Theorem 7.22 to this section, the compact set $\Lambda \times K$, and the properly embedded real analytic submanifold $\partial\Lambda \times M$. □

We conclude this chapter with a result on real analytic approximations of isotropic submanifolds in contact manifolds that will be needed later. See Chapter 5 for the relevant definitions.

Corollary 7.25. *Let Λ be a closed isotropic C^k-submanifold ($k \geq 1$) in a real analytic closed contact manifold (M, α) (i.e., the manifold M and the 1-form α are both real analytic). Then there exists a real analytic isotropic submanifold $\Lambda' \subset (M, \alpha)$ arbitrarily C^k-close to Λ.*

Similarly, let $(\Lambda_t)_{t\in[0,1]}$ be a C^k-isotopy of closed isotropic C^k-submanifolds in (M,α) such that Λ_0 and Λ_1 are real analytic. Then there exists a real analytic isotopy of real analytic isotropic submanifolds Λ'_t, arbitrarily C^k-close to Λ_t, with $\Lambda'_0=\Lambda_0$ and $\Lambda'_1=\Lambda_1$.

Proof. Let $\tilde\Lambda\subset M$ be a real analytic submanifold C^k-close to Λ, but not necessarily isotropic. Then $\Lambda=\phi(\tilde\Lambda)$ for a C^k-diffeomorphism $\phi:M\to M$ that is C^k-close to the identity. The contact form $\phi^*\alpha$ vanishes on $\tilde\Lambda$ but need not be real analytic. Thus $\phi^*\alpha$ induces a C^k-section in the real analytic vector bundle $T^*M|_{\tilde\Lambda}\to\tilde\Lambda$ which vanishes on the real analytic subbundle $T\tilde\Lambda\subset T^*M|_{\tilde\Lambda}$. Let $\nu\to\tilde\Lambda$ be the normal bundle to $T\tilde\Lambda$ in $T^*M|_{\tilde\Lambda}$ with respect to a real analytic metric and denote by $(\phi^*\alpha)^\nu$ the induced C^k-section in ν. Let β^ν be a real analytic section of ν that is C^k-close to $(\phi^*\alpha)^\nu$ and extend it to a real analytic section β of $T^*M|_{\tilde\Lambda}$ that vanishes on $T\tilde\Lambda$, and hence is C^k-close to $\phi^*\alpha$ along $\tilde\Lambda$. Extend β to a C^k one-form on M (still denoted by β) that is C^k-close to $\phi^*\alpha$. By construction, β is real analytic along $\tilde\Lambda$ and $\beta|_{\tilde\Lambda}=0$.

By Theorem 7.22 (with $d=0$), there exists a real analytic 1-form $\tilde\alpha$ that is C^k-close to β and coincides with β along $\tilde\Lambda$. In particular, $\tilde\alpha|_{\tilde\Lambda}=0$. By construction, $\tilde\alpha$ is C^k-close to α. Hence $\alpha_t:=(1-t)\tilde\alpha+t\alpha$ is a real analytic homotopy of real analytic contact forms. By Gray's Stability Theorem 5.24, there exists a diffeotopy $\phi_t:M\to M$ and positive functions f_t with $\phi_t^*\alpha=f_t\tilde\alpha$. Now in Moser's proof of Gray's Stability Theorem (see e.g. [9]), the ϕ_t are constructed as solutions of an ODE whose coefficients are real analytic and C^k-small in this case. Hence by Remark 7.13 the ϕ_t are real analytic, C^k-close to the identity, and depend real analytically on t. It follows that $\Lambda':=\phi_1(\tilde\Lambda)$ is real analytic, C^k-close to Λ, and $\alpha|_{\Lambda'}=0$. □

Remark 7.26. (1) Corollary 7.25 remains valid (with essentially the same proof) if the submanifold Λ is not closed, providing a real analytic approximation on a compact subset $K\subset\Lambda$.

(2) If Λ is Legendrian, then Λ' is Legendrian isotopic to Λ: By the Legendrian neighborhood theorem (Proposition 5.20), Λ' is the graph of the 1-jet of a function f in $J^1\Lambda$, and the functions tf provide the isotopy.

Chapter 8

Recollections from Morse theory

NEW CHAPTER (TENTATIVE)

Throughout this chapter, V denotes a smooth manifold of dimension m.

8.1 Critical points of functions

Let $\phi : V \to \mathbb{R}$ be a smooth function $p \in V$ be a critical point of ϕ, i.e. $d_p\phi = 0$. The Hessian $\mathrm{Hess}_p\phi$ defines a symmetric bilinear form on T_pV. The *nullity* of ϕ at p is the dimension of $\ker \mathrm{Hess}_p\phi := \{v \in T_pV \mid \mathrm{Hess}_p\phi(v,w) = 0 \text{ for all } w \in T_pV\}$. The *Morse index* at p is the maximal dimension of a subspace on which the quadratic form $v \mapsto \mathrm{Hess}_p(v,v)$ is negative definite. The critical point p is *nondegenerate* if its nullity is zero.

Lemma 8.1 (Morse Lemma [49]). *Near a nondegenerate critical point p of ϕ of index k there exist smooth coordinates $u \in \mathbb{R}^m$ in which ϕ has the form*

$$\phi(u) = \phi(p) - u_1^2 - \cdots - u_k^2 + u_{k+1}^2 \cdots + u_m^2. \tag{8.1}$$

More precisely, this means that for a function ϕ on a neighborhood of $0 \in \mathbb{R}^m$ there exists a diffeomorphism g between neighborhoods of 0 such that $g^*\phi$ has the form (8.1).

Remark 8.2. (1) If the function ϕ on a neighborhood of $0 \in \mathbb{R}^n$ already satisfies $\phi(x_1, \dots, x_k, 0, \dots, 0) = \phi(p) - x_1^2 - \cdots - x_k^2$, then we can choose the diffomorphism g to satisfy $g(x_1, \dots, x_k, 0, \dots, 0) = (x_1, \dots, x_k, 0, \dots, 0)$. To see this, apply the proof of the Morse lemma in [49] to find new coordinates $u_1, \dots, u_m$ near 0 in which $\phi(u) = \phi(p) - u_1^2 - \cdots - u_k^2 + u_{k+1}^2 \cdots + u_m^2$. Inspection of the proof shows that $u_i = x_i$ on $\mathbb{R}^k \times \{0\}$.

(2) The Morse lemma also holds with parameters as follows: For a compact manifold (possibly with boundary) K let $\phi_z : V \to \mathbb{R}$, $z \in K$ be a smooth family of functions with a nondegenerate critical of index k at p for all z. Then there exists a smooth family of diffeomorphisms $g_z : (U,0) \to (V_z, p)$ from a neighborhood $U \subset \mathbb{R}^m$ of 0 onto neighborhoods $V_z \in V$ of p such that for all $z \in K$,

$$\phi_z \circ g_z(u) = \phi_z(p) - u_1^2 - \cdots - u_k^2 + u_{k+1}^2 \cdots + u_m^2.$$

The next lemma shows that near a degenerate critical point one can always split off the nondegenerate directions.

Lemma 8.3. *Near a critical point p of ϕ index k and nullity ℓ there exist smooth coordinates $(x_1, \dots, x_{m-k-\ell}, y_1, \dots, y_k, z_1, \dots, z_\ell) \in \mathbb{R}^m$ in which ϕ has the form*

$$\phi(x,y,z) = x_1^2 - \cdots + x_{m-k-\ell}^2 - y_1^2 \cdots - y_k^2 + \psi(z)$$

with a smooth function $\psi(z)$.

Proof. Set $B := \mathrm{Hess}_p \phi$ and $n := m - \ell$. Identify a neighborhood of p in V with a neighborhood of 0 in $\mathbb{R}^m = \mathbb{R}^n \oplus \mathbb{R}^\ell$ such that $\mathbb{R}^\ell = \ker B$. Define a function F on a neighborhood of 0 in $\mathbb{R}^m$ by

$$F(w,z) := \frac{\partial \phi}{\partial w}(w,z).$$

Since $\frac{\partial F}{\partial w}(0,0) = \frac{\partial^2 \phi}{\partial w^2}(0,0)$ is invertible, the zero set $F^{-1}(0)$ is a graph $w = w(z)$ over $\mathbb{R}^\ell$. After applying a diffeomorphism near $0 \in \mathbb{R}^m$ we may assume $F^{-1}(0) = \mathbb{R}^\ell$. Consider the smooth family of functions $\phi_z = \phi(\cdot, z) : \mathbb{R}^n \to \mathbb{R}$, $z \in \mathbb{R}^\ell$ near 0. By construction, each ϕ_z has a nondegenerate critical point of index k at $w = 0$. Now Lemma 8.3 follows from the parametrized Morse Lemma in Remark 8.2 □

We say that a 1-parameter family ϕ_t, $t \in (-\varepsilon, \varepsilon)$ of functions near $p \in V$ is of *birth type* if after t-dependent coordinate changes on V near p and on $\mathbb{R}$ it is of the form

Spell out coordinate changes!

$$\phi_t(x,y,z) = |x|^2 - |y|^2 + z^3 - tz \tag{8.2}$$

for $(x,y,z) \in \mathbb{R}^{m-k} \otimes \mathbb{R}^{k-1} \otimes \mathbb{R}$. The family ϕ_{-t}, $t \in (-\varepsilon, \varepsilon)$ is said to be of *death type*. The critical point of ϕ_0 is called *embryonic*. Note that in a birth type family a pair of nondegenerate critical points of indices k and $k-1$ appears at $t = 0$ and in a death type family such a pair disappears.

Lemma 8.4. *In a generic 1-parameter family of functions only birth-death type degeneracies appear.*

Proof. Using Lemma 8.3 we can reduce the lemma to the case $m = 1$ of 1-parameter families of functions $\mathbb{R} \to \mathbb{R}$. In this case Lemma 8.4 is just Whitney's theorem proved in [66]. □

8.2 Zeroes of vector fields

Let X be a smooth vector field on V and $p \in V$ be a zero of X. The differential $D_pX : T_pV \to T_pV$ induces a splitting into invariant subspaces

$$T_pV = E_p^+ \oplus E_p^- \oplus E_p^0,$$

where E_p^+ (resp. E_p^-, E_p^0) is spanned by the generalized eigenvectors to eigenvalues with positive (resp. negative, vanishing) real part. The dimension of E_p^- is called the *Morse index*[1] of X at p. Denote by $X^s : V \to V$, $s \in \mathbb{R}$, the flow of X.

Theorem 8.5 (Center Manifold Theorem [1]). *Let $p \in V$ be a zero of a C^{r+1}-vector field X, $r \in \mathbb{N}$. Then there exist local invariant C^{r+1}-manifolds $W_p^\pm$ tangent to $E_p^\pm$ and a local invariant C^r-manifold W_p^0 tangent to E_p^0 at p. The $W_p^\pm$ are unique and smooth resp. real analytic if X is.*

W_p^- (resp. W_p^+, W_p^0) are called the *local stable (resp. unstable, center) manifold* at p. The center manifold is in general neither unique nor smooth, even if X is. By the center manifold theorem we can choose C^r-coordinates $Z = (x, y, z) \in E_p^+ \oplus E_p^- \oplus E_p^0$ in which $W_p^\pm$ and W_p^0 correspond to $E_p^\pm$ resp. E_0^p; in these coordinates X is of the form

$$X(x, y, z) = (A^+x, A^-y, A^0z) + O(|Z|^2) \tag{8.3}$$

with linear maps A^+ (resp. A^-, A^0) all of whose eigenvalues have positive (resp. negative, zero) real part.

A zero p is called *hyperbolic* if $E_p^0 = \{0\}$, i.e. all eigenvalues of D_pX have nonzero real part. In this case we have *global* stable and unstable manifolds characterized by

$$W_p^\pm = \{x \in V \mid \lim_{s \to \mp\infty} X^s(x) = p\}.$$

They are injectively immersed (but not necessarily embedded) in V.

We say that a 1-parameter family X_t, $t \in (-\varepsilon, \varepsilon)$ of vector fields near $p \in V$ is of *birth type* if in suitable coordinates $(x, y, z) \in \mathbb{R}^{m-k} \otimes \mathbb{R}^{k-1} \otimes \mathbb{R}$ near p it is of the form

$$X_t(x, y, z) = \Big(A_t^+x + O(|x|\,|Z|), A_t^-y + O(|y|\,|Z|), z^2 - t + O(|z^2 - t|\,|Z|)\Big) \tag{8.4}$$

with smooth families of linear maps $A_t^\pm$ all of whose eigenvalues have positive resp. negative real part. The family X_{-t}, $t \in (-\varepsilon, \varepsilon)$ is said to be of *death type*. The zero of X_0 is called *embryonic*. Note that in a birth type family a pair of hyperbolic zeroes of indices k and $k-1$ appears at $t = 0$ and in a death type family such a pair disappears. Is this right? Smoothness?

[1] Not to be confused with the topological index of a vector field at an isolated zero!

Lemma 8.6. *(a) A generic vector field has only hyperbolic zeroes.*

(b) In a generic 1-parameter family of vector fields without nonconstant periodic orbits only birth-death type degeneracies appear.

Proof. (a) follows from general transversality arguments.

(b) In a generic 1-parameter family of vector fields only two types of degeneracies appear (see [4] $\$\$32-33$): The first type corresponds to birth-death type; the second type corresponds to a Hopf bifurcation in which a nonconstant periodic orbit appears or disappears at $t = 0$, which is excluded by the hypothesis of (b). □

Proof of normal form!!!

Lemma 8.7. *Let p be an embryonic zero of a smooth vector field X. Then*

$$\hat{W}_p^{\pm} := \{x \in V \mid \lim_{s \to \mp\infty} X^s(x) = p\}$$

is an injectively immersed smooth manifold with boundary $W_p^{\pm}$.

Proof. Pick coordinates (x, y, z) on a neighborhood U of p in which X is of the form (8.4) with $t = 0$. Then $U \cap \hat{W}_p^- = \{(x, y, z) \in U \mid x = 0, z \leq 0\}$ is a smooth submanifold with boundary $U \cap W_p^- = \{(x, y, z) \in U \mid x = z = 0\}$ and the statement for $\hat{W}_p^-$ follows by invariance under the flow of X. The statement for $\hat{W}_p^+$ is proved analogously. □

Smoothness? Check!

8.3 Gradient-like vector fields

We call a smooth function $\phi : V \to \mathbb{R}$ *Lyapunov function* for a vector field X, and X *gradient-like* for ϕ, if $X \cdot \phi > 0$ outside the zeroes of X. We call $\phi : V \to \mathbb{R}$ *strong Lyapunov function* for X, and X *strongly gradient-like* for ϕ, if $X \cdot \phi \geq \delta |X|^2$ for some $\delta > 0$, where $|X|$ is the norm with respect to some Riemannian metric on V.

The space of (strong) Lyapunov functions for a given vector field X is a convex cone. In particular, if ϕ_0, ϕ_1 are (strong) Lyapunov functions for X then so is $(1-t)\phi_0 + t\phi_1$ for all $t \in [0, 1]$. Note that critical points of ϕ are also zeroes of a gradient-like vector field X but not necessarily vice versa.

The question of existence of a Lyapunov function for a vector field X separates into two issues: local existence near the zero set of X, and global existence. Assuming local existence near the zero set, Sullivan [61] gives a necessary and sufficient criterion for the existence of a global Lyapunov function in terms of foliation cycles. The simplest obstruction to a Lyapunov function is a nonconstant periodic orbit of X.

Lemma 8.8. *Let X be strongly gradient-like for ϕ. Then each nondegenerate zero of X is hyperbolic and also a nondegenerate critical point of ϕ.*

Proof. Consider a nondegenerate zero $p \in V$ of X. In coordinates Z near $p = \{Z = 0\}$ we have

$$X(Z) = AZ + O(|Z|^2), \qquad \phi(Z) = \phi(p) + LZ + O(|Z|^2)$$

with the linear map $A := D_pX$ and the linear form $L := d_p\phi$. Strong gradient-likeness

$$X \cdot \phi(Z) = LAZ + O(|Z|^3) \geq \delta|AZ|^2 + O(|Z|^3).$$

implies $LAZ \geq \delta|AZ|^2/2$, which is only possible for $L = 0$ since A is nondegenerate. So we have Check!

$$\phi(Z) = \phi(p) + \frac{1}{2}B(Z,Z) + O(|Z|^3)$$

with the symmetric bilinear form $B := \mathrm{Hess}_p\phi$. Again strong gradient-likeness

$$X \cdot \phi(Z) = B(Z, AZ) + O(|Z|^3) \geq \delta|AZ|^2 + O(|Z|^3)$$

yields

$$B(v, Av) \geq \delta|Av|^2/2.$$

Nondegeneracy of A implies nondegeneracy of B, so p is a nondegenerate critical point of ϕ.

To prove hyperbolicity of p, extend $A = D_pX$ $\mathbb{C}$-linearly to the complexified tangent space $T_pV \otimes \mathbb{C}$ and extend $B = \mathrm{Hess}_p\phi$ to $T_pV \otimes \mathbb{C}$ by

$$B(x + iy, x' + iy') := \Big(B(x, x') + B(y, y')\Big) + i\Big(B(x'y) - B(x, y')\Big).$$

Thus B is $\mathbb{C}$-linear in the first and $\mathbb{C}$-antilinear in the second argument, $B(v, w) = \overline{B(w, v)}$, and $\mathrm{Re}\, B(v, Av) \geq \delta|v|^2/2$. Let $0 \neq v \in T_pV \otimes \mathbb{C}$ be an eigenvector of A to the eigenvalue $0 \neq \lambda \in \mathbb{C}$, i.e. $Av = \lambda v$. Then

$$\lambda B(v, Av) = B(Av, Av) = \overline{B(Av, Av)} = \bar{\lambda}\overline{B(v, Av)}.$$

If λ were purely imaginary, this would imply $B(v, Av) = -\overline{(v, Av)}$, in contradiction to positivity of $\mathrm{Re}\, B(v, Av)$. □

Remark 8.9. Suppose that X is the gradient of ϕ with respect to a positive definite but not necessarily symmetric $(2, 0)$ tensor field g, i.e. $d\phi(v) = g(X, v)$ for all $v \in TV$ and $g(v, v) > 0$ for all $v \neq 0$. Then X is strongly gradient-like for ϕ and the zeroes of X coincide with the critical points of ϕ. At a zero p of X we have $\mathrm{Hess}_p(v, w) = g_p(D_pX \cdot v, w)$, so p is a nondegenerate zero of X iff it is a nondegenerate critical point of ϕ. If g is symmetric (i.e. a Riemannian metric), then so is the bilinear form $\mathrm{Hess}_p(\cdot, D_pX\cdot) = g_p(D_pX\cdot, D_pX\cdot)$ and all eigenvalues of D_pX are real.

Lemma 8.10. *(a) Near each hyperbolic zero a vector field admits a strong Lyapunov function.*

(b) For a birth or death type family X_t near p there exists a neighborhood U of p and a smooth family of strong Lyapunov functions $\phi_t : U \to \mathbb{R}$ for X_t.

Proof. (a) Consider coordinates in which X has the form (8.3) with $E_p^0 = \{0\}$. By [2] Theorem 22.3 there exist quadratic forms $Q^\pm$ on $E_p^\pm$ which are strongly Lyapunov for the linear maps $A^\pm$. Then $\phi(x,y) := Q^+(x) + Q^-(y)$ is a strong Lyapunov function for X.

(b) Consider coordinates in which X has the form (8.4). Let $Q_t^\pm$ be a smooth family of quadratic forms on $E_p^\pm$ as in (a) that are strongly Lyapunov for $A_t^\pm$. Then

$$\phi_t(x,y,z) := Q_t^+(x) + Q_t^-(y) + \frac{1}{3}z^3 - tz$$

is a smooth family of strong Lyapunov functions for X_t. □

The following result states that a Lyapunov function can be put into any prescribed form near a hyperbolic or birth-death type zero.

Proposition 8.11. *(a) Let X be a vector field on V with a hyperbolic or embryonic zero p. Let $\phi : V \to \mathbb{R}$ be a Lyapunov function for X and $\phi^{\rm loc} : U \to \mathbb{R}$ a Lyapunov function on a neighborhood U of p with $\phi(p) = \phi^{\rm loc}(p)$. Then there exists a Lyapunov function $\psi : V \to \mathbb{R}$ which agrees with ϕ outside U and with $\phi^{\rm loc}$ near p.*

(b) Let X_t, $t \in [-\varepsilon, \varepsilon]$ be a smooth family of vector fields on V with a birth or death type zero p. Let $\phi_t : V \to \mathbb{R}$ be a smooth family of Lyapunov functions for X_t and $\phi_t^{\rm loc} : U \to \mathbb{R}$ a smooth family of Lyapunov functions on a neighborhood U of p with $\phi_t(p) = \phi_t^{\rm loc}(p)$ for all t. Then there exists a smooth family of Lyapunov functions $\psi_t : V \to \mathbb{R}$, $t \in [-\varepsilon, \varepsilon]$ which agrees with ϕ_t outside U and with $\phi_t^{\rm loc}$ near p.

Remark 8.12. (1) In case (a), $\phi_u := (1-u)\phi + u\psi$, $u \in [0,1]$ is a smooth family of Lyapunov functions with $\phi_0 = \phi$, $\phi_u = \phi$ outside U, and $\phi_1 = \phi^{\rm loc}$ near p.

(2) By Lemma 8.10, in case (a) we can choose $\phi^{\rm loc}$ to be strongly Lyapunov, so ψ is strongly Lyapunov near p.

Analogous remarks apply to case (b).

The proof of Proposition 8.11 will occupy the remainder of this section. It is based on a smooth version of the J-convex surroundings in Chapter 9 to which we now proceed.

Consider a vector field X with a Lyapunov function $\phi : V \to \mathbb{R}$ and a hyperbolic zero p of index k and value $\phi(p) = c$. Pick a regular value $a < c$ such that $D_p^- := W_p^- \cap \{\phi \geq a\}$ is a smoothly embedded k-disk.

Lemma 8.13. *For every neighborhood $\mathcal{N}$ of D_p^- there exists a $b > c$ and a closed subset*

$$S^{m-k-1} \times D^k \times [a,b] \cong U \subset \mathcal{N} \setminus D_p^-$$

with the following properties:

(i) $\phi|_{S^{m-k-1} \times D^k \times \{t\}} \geq t$*;*

(ii) $\phi|_{S^{m-k-1}\times D^k\times\{t\}} = t$ *near* $(S^{m-k-1}\times\partial D^k\times[a,b])\cup(S^{m-k-1}\times D^k\times\{b\})$;

(iii) each hypersurface $S^{m-k-1}\times D^k\times\{t\}$ *is transverse to* X.

Proof. Pick $b > c$ and a neighborhood

$$\Sigma_1 \cong S^{m-k-1}\times B^k_2$$

of $S^+_p := W^+_p\cap\phi^{-1}(b)$ in the level set $\phi^{-1}(b)$ such that $S^+_p \subset S^{m-k-1}\times B^k_1$. Denote by x the coordinate on S^{m-k-1} and by $(\rho,y)\in[0,2]\times S^{k-1}$ polar coordinates on B^k_2. For $c < b' < b$ denote by

$$W_1 \cong S^{m-k-1}\times B^k_2\times[b',b]$$

the result of flowing Σ_1 by $-X$ until it hits the level $\phi = b'$, so that $\phi(x,\rho,y,\tau) = \tau$ for $\tau\in[b',b]$ and ρ is invariant under X. Denote by

$$W_0 \cong S^{m-k-1}\times[1,2]\times S^{k-1}\times[a,b]$$

the result of flowing $S^{m-k-1}\times[1,2]\times S^{k-1}\subset\Sigma_1$ by $-X$ until it hits the level $\phi = a$, so that $\phi(x,\rho,y,\tau) = \tau$ for $\tau\in[a,b]$ and ρ is invariant under X. By choosing $b > c$ and Σ_1 sufficiently small we can ensure that

$$W := W_0\cup W_1\subset\mathcal{N}.$$

Since $X\cdot\rho = 0$ and $X\cdot\tau > 0$, any graph $\tau = f(\rho)$ in W is transverse to X. Now set

$$T := ([1,2]\times[a,b])\cup([0,2]\times[b',b])\subset\mathbb{R}^2$$

and pick a region

$$[0,2]\times[a,b]\cong R\subset T$$

with the following properties:

(i) $\tau(\rho,t)\geq t$;

(ii) $\tau(\rho,t) = t$ near $(\{2\}\times[a,b])\cup([0,2]\times\{b\})$;

(iii) each hypersurface $[0,2]\times\{t\}$ is a graph $\tau = f_t(\rho)$ with f_t constant near $\rho = 0$

(see Figure ???). Then the region

$$U := \{(x,\rho,y,\tau)\in W \mid (\rho,\tau)\in R\}$$

has the desired properties. □

Next we prove an analogue of Lemma 8.13 for an embryonic zero. Consider a vector field X with a Lyapunov function $\phi : V\to\mathbb{R}$ and an embryonic zero p of index $k-1$ and value $\phi(p) = c$. Define $\hat{W}^\pm_p$ as in Lemma 8.7. Pick a regular value $a < c$ such that $\hat{D}^-_p := \hat{W}^-_p\cap\{\phi\geq a\}$ is a smoothly embedded half k-disk.

Lemma 8.14. *For every neighborhood $\mathcal{N}$ of $\hat{D}_p^-$ there exists a $b > c$ and a closed subset*

$$D^{m-1} \times [a,b] \cong U \subset \mathcal{N} \setminus \hat{D}_p^-$$

with the following properties:

(i) $\phi|_{D^{m-1}\times\{t\}} \geq t$;

(ii) $\phi|_{D^{m-1}\times\{t\}} = t$ *near* $(\partial D^{m-1} \times [a,b]) \cup (D^{m-1} \times \{b\})$;

(iii) each hypersurface $D^{m-1} \times \{t\}$ *is transverse to* X.

Proof. Pick $b > c$ and a neighborhood

$$\Sigma_1 \cong B_2^{m-1}$$

of the $(m-k)$-disk $\hat{S}_p^+ := \hat{W}_p^+ \cap \phi^{-1}(b)$ in the level set $\phi^{-1}(b)$ such that $\hat{S}_p^+ \subset B_1^{m-1}$. Denote by $(\rho, y) \in [0,2] \times S^{m-2}$ polar coordinates on B_2^{m-1}. For $c < b' < b$ denote by

$$W_1 \cong B_2^{m-1} \times [b', b]$$

the result of flowing Σ_1 by $-X$ until it hits the level $\phi = b'$, so that $\phi(\rho, y, \tau) = \tau$ for $\tau \in [b', b]$ and ρ is invariant under X. Denote by

$$W_0 \cong [1,2] \times S^{m-2} \times [a,b]$$

the result of flowing $[1,2] \times S^{m-2} \subset \Sigma_1$ by $-X$ until it hits the level $\phi = a$, so that $\phi(\rho, y, \tau) = \tau$ for $\tau \in [a,b]$ and ρ is invariant under X. By choosing $b > c$ and Σ_1 sufficiently small we can ensure that

$$W := W_0 \cup W_1 \subset \mathcal{N}.$$

Now pick $R \subset T \subset \mathbb{R}^2$ exactly as in the proof of Lemma 8.13. Then the region

$$U := \{(\rho, y, \tau) \in W \mid (\rho, \tau) \in R\}$$

has the desired properties. □

The next lemma states that we can interpolate between two Lyapunov functions near the stable manifold.

Lemma 8.15. *Let X be a vector field with Lyapunov function ϕ and hyperbolic (resp. embryonic) zero p. Suppose that $\phi^{\rm loc}$ is a Lyapunov function for X near p with $\phi(p) = \phi^{\rm loc}(p) = c$. For suitable $a < c$ define D_p^- (resp. $\hat{D}_p^-$) as above. Then there exists a Lyapunov function $\chi : \mathcal{N} \to [a, \infty)$ on a neighborhood $\mathcal{N}$ of D_p^- (resp. $\hat{D}_p^-$) which agrees with ϕ near $\mathcal{N} \cap \phi^{-1}(a)$ and with $\phi^{\rm loc}$ near p.*

Proof. Pick a sufficiently small $\delta > 0$. If p is hyperbolic X has no critical points on the set $D_p^- \cap \{\phi \geq a+\delta\} \cap \{\phi^{\rm loc} \leq c-\delta\}$ and is transverse to its boundary. If p is embryonic X has no critical points on the set $\hat{D}_p^- \cap \{\phi \geq a+\delta\} \cap \{\phi^{\rm loc} \leq c-\delta\}$, is transverse to the boundary components $\hat{D}_p^- \cap \{\phi = a+\delta\}$ and $\hat{D}_p^- \cap \{\phi^{\rm loc} = c-\delta\}$, and is tangent to the boundary component $D_p^- \cap \{\phi \geq a+\delta\} \cap \{\phi^{\rm loc} \leq c-\delta\}$. Hence in either case we can use the flow of X to construct a Lyapunov function χ on D_p^- (resp. $\hat{D}_p^-$) which agrees with ϕ for $\phi \leq a+\delta$ and with $\phi^{\rm loc}$ for $\phi^{\rm loc} \leq c-\delta$. Applying the same argument to a small neighborhood of D_p^- (resp. $\hat{D}_p^-$) yields the desired function χ. □

Proof of Proposition 8.11. We first prove part (a) for p hyperbolic. Let $\chi : \mathcal{N} \to \mathbb{R}$ be as in Lemma 8.15, and let

$$S^{m-k-1} \times D^k \times [a,b] \cong U \subset \mathcal{N} \setminus D_p^-$$

be as in Lemma 8.13. For $t \in [a,b]$ set

$$U_t := S^{m-k-1} \times D^k \times \{t\}.$$

Then the function

$$\theta : U \to \mathbb{R}, \qquad \theta(x,\rho,y,t) := t$$

has the following properties:

(i) $\phi \geq \theta$ on U;

(ii) $\phi = \theta$ near $(S^{m-k-1} \times \partial D^k \times [a,b]) \cup U_b$;

(iii) $X \cdot \theta > 0$.

After replacing χ by $f \circ \chi$ for a suitable function f with $f(t) = t$ near $t \leq c$ we may assume that $\sup_{\mathcal{N}} \chi < b$. Pick a small $\delta > 0$ such that $\sup_{\mathcal{N}} \chi < b-\delta$ and $\chi = \phi$ on $\mathcal{N} \cap \{a \leq \phi \leq a+\delta\}$. Interchange level sets of θ near $S^{m-k-1} \times \partial D^k \times [a,b]$ to obtain a function $\tilde{\theta}$ for which (ii) and (iii) still hold, but instead of (i) we have $\tilde{\theta}(\rho, a+\delta) > b-\delta$ for all $\rho \in [0, 2-\delta]$. This condition ensures $\tilde{\theta} \geq \chi$ on

$$U' := (S^{m-k-1} \times [0,2-\delta] \times S^{k-1} \times [a+\delta,b]) \cup (S^{m-k-1} \times [0,2] \times S^{k-1} \times [a,a+\delta]).$$

Extend $\tilde{\theta}$ to $\mathcal{N}$ by setting $\tilde{\theta} := a$ on $\mathcal{N} \setminus U$. Set

$$U'' := S^{m-k-1} \times [2-\delta, 2] \times S^{k-1} \times [a+\delta, b]$$

and define $\psi : V \to \mathbb{R}$ by

$$\psi := \begin{cases} \max(\tilde{\theta}, \chi) & \text{on } \mathcal{N} \setminus U'', \\ \phi & \text{outside } \mathcal{N} \setminus U''. \end{cases}$$

Since $\tilde{\theta} \geq \chi$ on U' this defines a continuous function. Since $\chi \geq a$ on $\mathcal{N}$, the function ψ agrees with χ on $\mathcal{N} \setminus U$, in particular $\psi = \phi^{\text{loc}}$ near p. According to Proposition 3.22, a suitable smoothing of ψ will be a Lyapunov function for X with the desired properties. This proves part (a) for p hyperbolic.

State better result to refer to!

Part (a) for p embryonic is proved analogously, using Lemma 8.13 and Lemma 8.15 for the embryonic case.

Finally, we prove part (b). Let $X_t, \phi_t, \phi_t^{\text{loc}}$ for $t \in [-\varepsilon, \varepsilon]$ be as in the statement of (b). Part (a) yields for each t a Lyapunov function for X_t which agrees with ϕ_t outside U and with ϕ_t^{loc} near p. Note that for each t the constructions in part (a) can be done smoothly for all parameters s sufficiently close to t. So for each $t \in [-\varepsilon, \varepsilon]$ we find an open subset $I_t \subset [-\varepsilon, \varepsilon]$ and a smooth family of Lyapunov functions ψ_s^t, $s \in I_t$, for X_s which agrees with ϕ_s outside U and with ϕ_s^{loc} near p. Since finitely many of the I_t cover $[-\varepsilon, \varepsilon]$, we find a partition $-\varepsilon = t_0 < t_1 < \cdots < t_N = \varepsilon$ and smooth families of Lyapunov functions ψ_t^i, $t \in [t_i, t_{i-1}]$, $i = 1, \ldots, N$ for X_t which agree with ϕ_t outside U and with ϕ_t^{loc} near p. Now for each $1 \leq i \leq N-1$ the functions $\psi_{t_i}^i$ and $\psi_{t_i}^{i+1}$ both agree with ϕ_{t_i} outside U and with $\phi_{t_i}^{\text{loc}}$ near p, so the same holds for the interpolating functions $(1-s)\psi_{t_i}^i + s\psi_{t_i}^{i+1}$, $s \in [0,1]$. Concatenating the families ψ_t^i with these interpolations and appropriately changing the parametrization yields the desired family ψ_t and concludes the proof of Proposition 8.11. □

8.4 Morse functions

8.5 Modifications of Morse functions

8.6 The h-cobordism theorem

Chapter 9

J-convex surroundings

9.1 J-convex surrounding problem

For a closed subset A of a complex manifold (V, J), consider the following

Surrounding problem. Does A possess arbitrarily small neighborhoods with smooth J-convex boundary? In Section 2.5 we have seen that the surrounding problem is solvable for

- totally real submanifolds;
- properly embedded complex hypersurfaces with negative normal bundle.

The main theorem of this chapter solves the surrounding problem for totally real balls suitably attached to J-convex domains. For a hypersurface Σ in an almost complex manifold (V, J), we say that a submanifold L with boundary $\partial L \subset \Sigma$ is attached *J-orthogonally* to Σ along ∂L if, for each point $p \in \partial L$, $JT_pL \subset T_p\Sigma$ and $T_pL \not\subset T_p\Sigma$. The first condition implies that ∂L is an integral submanifold for the maximal complex tangency ξ on Σ. If Σ is J-convex and $\dim_{\mathbb{R}} L = \dim_{\mathbb{C}} V = n$, then the second condition $T_pL \not\subset T_p\Sigma$ follows from the first one because integral submanifolds of the contact structure ξ have dimension at most $n-1$.

Theorem 9.1. *Let (V, J) be a complex manifold of complex dimension n and $W \subset V$ a compact domain with smooth J-convex boundary ∂W. Let $\Delta \subset V \setminus \mathrm{int}W$ be a a totally real k-ball attached J-orthogonally to ∂W along $\partial\Delta$. Let $V' \subset V$ be an open neighborhood of $W \cup \Delta$. Then $W \cup \Delta$ has a compact neighborhood $W' \subset V'$ with smooth J-convex boundary.*

Moreover, if $k < n$ and $f(D^k \times D^{n-k}) \subset V \setminus \mathrm{int}W$ is a totally real embedding extending $\Delta = f(D^k \times \{0\})$, attached J-orthogonally to ∂W along $f(\partial D^k \times D^{n-k})$, then W' can be chosen such that $\partial W'$ intersects $f(D^k \times D^{n-k})$ J-orthogonally.

Proof. We use the notation of Lemma 4.9. Thus for $k \le n$ we set

$$r := \sqrt{x_1^2 + \cdots + x_n^2 + y_{k+1}^2 + \cdots + y_n^2}, \quad R := \sqrt{y_1^2 + \cdots + y_k^2},$$

where $x_1 + iy_1, \ldots, x_n + iy_n$ are complex coordinates in $\mathbb{C}^n$. The notation D stands for the unit k-disc $\{R \le 1, r = 0\} \subset \mathbb{C}^n$. We denote by H_ε the k-handle

$$\{R \le 1 + \varepsilon, r \le \varepsilon\} \subset \mathbb{C}^n.$$

Let us consider a slightly bigger domain $\widehat{W} \subset V'$, $W \subset \operatorname{Int} \widehat{W}$, with a J-convex boundary $\partial\widehat{W}$ which J-orthogonally intersect Δ. Denote $\widetilde{\Delta} := \Delta \setminus \operatorname{Int} \widetilde{W}$. Let us parameterize Δ by a diffeomorphism $f : D_\varepsilon \to \Delta$ such that $f(D) = \widetilde{\Delta}$. The diffeomorphism f can be extended to a totally real embedding $\widetilde{f} : D_\varepsilon \times D^{n-k} \to V'$ such that $\widetilde{f}|_{D_\varepsilon \times 0} = f$ and $\widetilde{f}(D_\varepsilon \times D^{n-k})$ is J-orthogonal to $\partial\widetilde{W}$. The embedding $\widetilde{f}$ can be extended to a diffeomorphism $F : U_\varepsilon(D_\varepsilon \times D^{n-k}) \to V'$ such that the 2-jet of the pull-back complex structure $\widetilde{J} = f_* J$ coincides with the standard complex structure i along $D_\varepsilon \times D^{n-k} \subset \mathbb{C}^n$. In particular, for any $\delta > 0$ there is a $\sigma > 0$ such that in $U_\sigma(D_\varepsilon \times D^{n-k})$ the complex structures i and $\widetilde{J}$ are δ-close in the C^2-metric. Denote $\widetilde{\Sigma} := F^{-1}\widehat{\Sigma}$. Using Proposition 3.15 we find for any $a > 1$ a hypersurface $\widetilde{\Sigma}'$ which coincides with $\widetilde{\Sigma}$ outside $U_\sigma(D_\varepsilon \times D^{n-k})$, and with the hypersurface $\{S(r) = \sqrt{1 + ar^2}\}$ in $U_{\sigma'}(D_\varepsilon \times D^{n-k})$ for a sufficiently small positive $\sigma' < \sigma$. It can make this construction keeping the i-orthogonality condition between $D_\varepsilon \times D^{n-k}$ and the hypersurface $\widetilde{\Sigma}'$. Using Corollary 4.15 we can construct an i-convex hypersurface Σ' (given by $\{R = \varphi(r)\}$ for a suitable shape φ) which surrounds the disk D and coincides with $\widetilde{\Sigma}'$ outside $U_{\sigma'}(D_\varepsilon)$. Note that by Lemma 4.9 the hypersurface $\{R = \varphi(r)\}$, and hence Σ' is i-orthogonal to $D_\varepsilon \times D^{n-k}$. By Remark **??** the modulus of i-convexity of $\widetilde{\Sigma}'$ is bounded below by a constant independent of σ'. Hence, by Lemma **??** we conclude that if σ is chosen small enough then Σ' is also $\widetilde{J}$-convex. Let $\widetilde{W} \subset H$ be the region bounded by Σ' and containing D. Then $W' := \widehat{W} \cup F(\widetilde{W})$ is the desired neighborhood of $W \cup \Delta$.

□

Corollary 9.2. *Let (V, J) be a complex manifold and $W \subset V$ a compact domain with smooth J-convex boundary ∂W. Let $L \subset V \setminus \operatorname{int} W$ be a totally real compact submanifold attached J-orthogonally to ∂W along ∂L. Then $W \cup L$ has arbitrarily small neighborhoods with smooth J-convex boundary.*

Proof. Let $U \subset V$ be a given open neighborhood of $W \cup L$. Pick a Morse function $\phi : L \to \mathbb{R}$ with regular level set $\partial L = \phi^{-1}(0)$ and critical points p_i of values $0 < \phi(p_1) < \cdots < \phi(p_m)$ and Morse indices k_i. Consider the gradient flow of ϕ with respect to some Riemannian metric. The stable manifold $D^-(p_1)$ of p_1 is a totally real k_1-ball attached J-orthogonally to ∂W. By Theorem 9.1, we find a compact neighborhood $W_1 \subset U$ of $W \cup D^-(p_1)$ with smooth J-convex boundary. Moreover, due to the last statement in Theorem 9.1, we may assume that L intersects ∂W_1 J-orthogonally. In particular, $D^-(p_2)$ is attached J-orthogonally to W_1. Now continue by induction over the critical points. □

The preceding corollary extends to totally real immersions. We say that two totally real submanifolds L_1, L_2 of the same dimension in an almost complex manifold (V, J) *intersect J-orthogonally* at p if $JT_pL_1 = T_pL_2$.

Corollary 9.3. *Let (V, J) be a complex manifold and $W \subset V$ a compact domain with smooth J-convex boundary ∂W. Let $L \subset V \setminus \mathrm{int}W$ be a totally real immersion of a compact manifold, with finitely many J-orthogonal self-intersections away from ∂L and attached J-orthogonally to ∂W along ∂L. Then $W \cup L$ has arbitrarily small neighborhoods with smooth J-convex boundary.*

Proof. Let $U \subset V$ be a given open neighborhood of $W \cup L$. Let L_1, L_2 be the two local branches of L at a self-intersection point p. By J-orthogonality of the intersection, there exist local holomorphic coordinates in which $L_1 \subset \mathbb{R}^n$ and $L_2 \subset i\mathbb{R}^n$. Let $B(p) \subset U$ be the image in V of a small ball around the origin in $\mathbb{C}^n$. The boundary $\partial B(p)$ is J-convex and intersects L_1 and L_2 J-orthogonally. Construct such balls around all self-intersection points $p_1, \dots, p_m$, disjoint from each other and from ∂W. Then $W' := W \cup B(p_1) \cup \dots \cup B(p_m) \subset U$ has J-convex boundary, to which the totally real submanifold $L \setminus \mathrm{int}W'$ is attached J-orthogonally. Hence the result follows from Corollary 9.2. □

In particular, for $W = \emptyset$ we obtain

Corollary 9.4. *Let (V, J) be a complex manifold and $L \subset V$ a totally real immersion of a compact manifold with finitely many J-orthogonal self-intersections. Then L has arbitrarily small neighborhoods with smooth J-convex boundary.*

Remark 9.5. An alternative proof of the last corollary combines surroundings of totally real embeddings (Proposition 2.13) with the surroundings near the double points provided by Lemma 4.7 below.

9.2 J-convex surroundings and extensions

Lemma 9.6. *Let A be a closed subset of a complex manifold (V, J). If the surrounding problem is solvable for A, then given a bounded J-convex function ϕ on a neighborhood of A, A possess arbitrarily small neighborhoods U with smooth J-convex functions ψ such that $\psi = \phi$ near A and ∂U is a regular level set of ψ.*

Proof. By hypothesis, A possesses arbitrarily small neighborhoods U with smooth J-convex boundary. Let $c < \inf_A \phi$ and $C > \sup_A \phi$. By Lemma 2.4, there exists a J-convex surjective function $\tilde{\phi} : W \to [c, C + 1]$ on a neighborhood W of ∂U such that $\partial U = \tilde{\phi}^{-1}(C)$ is a regular level set. A smoothing of $\max(\phi, \tilde{\phi})$ is the desired function ψ. □

The main theorem of this chapter allows us to extend J-convex functions over handles with control over the critical points.

Theorem 9.7. *Let (V, J) be a complex manifold of complex dimension n and $W \subset V$ a compact domain. Let $\Delta \subset V \setminus \mathrm{int}W$ be an embedded totally real k-ball attached J-orthogonally to ∂W along $\partial\Delta$. Let $\phi : W \to \mathbb{R}$ be a J-convex function with regular level set $\partial W = \phi^{-1}(a)$ which is extended to a function on Δ such that $\phi > a$ on $\mathrm{int}\Delta$. Then given any open neighborhood $\tilde{V} \subset V$ of $W \cup \Delta$ and $b > \max_\Delta \phi$, there exists a compact neighborhood $\tilde{W} \subset \tilde{V}$ of $W \cup \Delta$ and a J-convex function $\psi : \tilde{W} \to \mathbb{R}$ with the following properties:*

(a) $\psi = \phi$ on $W' := \{\phi \le a'\}$ for some $a' < a$;

(b) $\psi^{-1}(b)$ is a regular level set that coincides with $\phi^{-1}(a)$ outside a neighborhood $U \subset V \setminus W'$ of $\partial\Delta$;

(c) $\psi = f \circ \phi$ on $W \setminus U$ for a smooth function $f : \mathbb{R} \to \mathbb{R}$;

(d) there exists an isotopy $h_t : \Delta' \to \Delta'$ (on an extension Δ' of Δ up to $\partial W'$) such that $h_t = \mathbb{1}$ on $\Delta' \setminus U$, $h_0 = \mathbb{1}$, and $h_1^*\phi = \psi$;

(e) the critical points of ψ agree with the critical points of $\phi|_\Delta$ and have positive definite Hessian transversely to Δ.

The following is the key result for the proof of the Theorem 9.7.

Proposition 9.8. *Let H be a standard k-handle and $\phi : U \to \mathbb{R}$ an i-convex function on a neighborhood of S such that $\phi|_S \equiv a$ and $d\phi = -2dR$ along S. Extend ϕ to a function $D \cup U \to \mathbb{R}$ such that $\phi > a$ on $\mathrm{int}D$. Let $\tilde{U}$ be a neighborhood of S in U and $b > \max_D \phi$. Then there exists a neighborhood $W \subset H$ of $\{\phi \le a\} \cup D$ and an i-convex function $\psi : W \to \mathbb{R}$ with the following properties:*

(a) $\psi = \phi$ on $\{\phi \le a'\}$ for some $a' < a$;

(b) $\psi^{-1}(b)$ is a regular level set that coincides with $\phi^{-1}(a)$ outside $\tilde{U}$;

(c) $\psi = f \circ \phi$ on $\{\phi \le a\} \setminus \tilde{U}$ for a smooth function $f : \mathbb{R} \to \mathbb{R}$;

(d) there exists an isotopy $h_t : D_{1+\varepsilon} \to D_{1+\varepsilon}$ such that $h_t = \mathbb{1}$ outside $D_{1+\varepsilon} \cap \tilde{U}$, $h_0 = \mathbb{1}$, and $h_1^*\phi = \psi$;

(e) the critical points of ψ agree with the critical points of $\phi|_D$ and have positive definite Hessian transversely to D.

Proof. Fix $A > 1$. By hypothesis, ϕ coincides together with its differential along $S = \{r = 0, R = 1\}$ with the i-convex function $Ar^2 - R^2 + 1 + a$. By Proposition 3.15, there exists an i-convex function $\tilde{\phi} : U \to \mathbb{R}$, C^1-close to ϕ, with $\tilde{\phi} = \phi$ outside $\tilde{U}$ and $\tilde{\phi} = Ar^2 - R^2 + 1 + a$ near S. Since $\tilde{\phi}$ and ϕ are C^1-close and have no critical points on (sufficiently small) $\tilde{U}$, there exists and isotopy $h_t : D_{1+\varepsilon} \to D_{1+\varepsilon}$ such that $h_t = \mathbb{1}$ outside $D_{1+\varepsilon} \cap \tilde{U}$, $h_0 = \mathbb{1}$, and $h_1^*\phi = \tilde{\phi}$.

Extend $\tilde{\phi}$ to an open neighborhood $\tilde{W}$ of $U \cup D$ by $\tilde{\phi}|_D + Ar^2$ outside U.

This function will be i-convex for A sufficiently large. Choose $\tilde{W}$ so small that $\sup_{\tilde{W}} \tilde{\phi} <b$.

By construction, the level set $\tilde{\phi}^{-1}(a)$ agrees with the hypersurface $\{R = \sqrt{1+Ar^2}\}$ on $r \leq \gamma$ for some $\gamma > 0$. By Corollary 4.15 and Lemma 4.9, there exists an i-convex hypersurface $\Sigma \subset \tilde{W}$ (given by $\{R = \varphi(r)\}$ for a suitable shape φ) which surrounds the disk D and coincides with $\tilde{\phi}^{-1}(a)$ along $r = \gamma$.

Choose a tubular neighborhood $\Sigma\tilde{[}-1,1]$ of $\Sigma = \Sigma\tilde{\{}0\}$ in $\tilde{W}$ such that the hypersurfaces $\Sigma_t := \Sigma\tilde{\{}t\}$ are i-convex, and outside $\tilde{U}$ they coincide with level surfaces of the function ϕ. By Lemma 2.4, there exists an i-convex function $\zeta : \Sigma\tilde{[}-1,1] \to \mathbb{R}$ with level sets Σ_t such that $\zeta|_{\Sigma_0} = b$ and $\zeta_{\Sigma_{-1}} = b' < \inf_{\tilde{W}} \tilde{\phi}$. Extend ζ to the domain bounded by Σ_{-1} as the constant b' and set

$$\tilde{\psi} := \max(\tilde{\phi}, \zeta)$$

on the domain $W := \{\zeta \leq b\} \subset \tilde{W}$ bounded by Σ. Note that $\tilde{\psi} = \tilde{\phi}$ in the region $\{\zeta = b'\}$ bounded by Σ_{-1}, hence $\tilde{\psi}$ is strictly i-convex (although the constant function b' is not) and $\tilde{\psi} = \tilde{\phi}$ near D. In particular, the critical points of $\tilde{\psi}$ on $\{\zeta = b'\}$ agree with the critical points of $\phi|_D$ and have positive definite Hessian transversely to D. On the other hand, $\tilde{\psi} = \zeta$ near Σ, and in particular $\Sigma = \tilde{\psi}^{-1}(b)$.

Observe that on $W \setminus \tilde{U}$ we have $\tilde{\psi} = f \circ \tilde{\phi}$ for a continuous convex function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ which is smooth except at one point (where $\zeta = \tilde{\phi}$) and satisfies $\tilde{f}(x) = x$ for $x \leq a'$ with some $a' < a$. Let f be a smooth convex function which agrees with $\tilde{f}$ for $x \leq a'$ and $x \geq a$. We can replace $\tilde{\psi}$ on $\{\tilde{\phi} \leq a\} \setminus \tilde{U}$ by the smooth function $f \circ \tilde{\phi}$, without changing it near $D_{1+\varepsilon}$ and keeping it i-convex. Let us denote the resulting function by $\hat{\psi}$. Finally, we smoothen the function $\hat{\psi}$, without changing it on $\{\tilde{\phi} \leq a\} \setminus \tilde{U}$ and near $D_{1+\varepsilon}$, to the desired i-convex function ψ.

It only remains to verify that the smoothing processes do not create new critical points. For the step from $\tilde{\psi}$ to $\hat{\psi}$ this is obvious. For the smoothing from $\hat{\psi}$ to ψ, after shrinking $\tilde{U}$ and $\tilde{W}$, we may assume that it takes place in the region where $0 < r < \gamma$ and $\tilde{\phi}(x,y) = \tilde{\phi}_{D_{1+\varepsilon}}(x_1, \dots, x_k) + Ar^2$. In this region we have $\nabla r \cdot \tilde{\phi} > 0$. Since ∇r is also transverse to the hypersurface $\Sigma = \{R^2 - Ar^2 = 1\}$, hence to each of the nearby hypersurfaces Σ_t, it satisfies $\nabla r \cdot \zeta > 0$ whenever $\zeta > b'$. Now it follows from Propositions 3.21 and 3.22 that the smoothing of $\max(\zeta, \tilde{\phi})$ does not create new critical points. □

Proof of Theorem 9.7. After a small perturbation near $\partial\Delta$, we may assume that ϕ is real analytic near $\partial\Delta$. Let ν_Δ be the unique vector field tangent to Δ along $\partial\Delta$ with $\nu_\Delta \cdot \phi = -2$. Thus ν_Δ is real analytic. By Lemma **??**, there exists a holomorphic embedding $F : H_\varepsilon \hookrightarrow V$ with $F(D) = \Delta$ whose differential along S maps ν to ν_Δ and $T(\partial^- H)|_S$ to $T(\partial W)$. The i-convex function $F^*\phi : U \cup D \to \mathbb{R}$, with U a neighborhood of S, satisfies $\phi|_S \equiv a$ and $d\phi = -2dR$ along S. Let $\tilde{\psi} : H \supset \tilde{W} \to \mathbb{R}$ be the i-convex function provided by Proposition 9.8. Property (c) allows us to extend $F_*\tilde{\psi}$ by $f \circ \phi$ to a J-convex function ψ on $W' :=$

$W \cup F(\tilde{W})$. The properties of ψ in Theorem 9.7 follow from the corresponding properties in Proposition 9.8. □

9.3 Surrounding by level-sets of a given J-convex function

Theorem 9.9. *Let $\phi : V \to \mathbb{R}$ be a J-convex function. Suppose that for a critical point p of ϕ and a real a the stable manifold of p intersects $\phi \geq a$ along a disc Δ. Then for any neighborhood U of $\{\phi \leq a\} \cup \Delta$ there exists diffeomorphisms $g : V \to V$ and $h : \mathbb{R} \to \mathbb{R}$ such that*

- *the function $\psi = h \circ \phi \circ g$ is J-convex;*
- *$g|_V \setminus U$ preserves the level sets of ϕ;*
- *$h|_{(-\infty,a)} = \mathrm{Id}$;*
- *there exists $a' > a$ such that the level set $\{\psi = a'\}$ surrounds $\{\phi \leq a\} \cup \Delta$ and is contained in U.*

TO BE CONTINUED

Chapter 10

Modifications of J-convex Morse functions

In this chapter we show how to modify critical points of J-convex Morse functions. This parallels the h-cobordism theory for ordinary Morse functions. Thus we wish to do the following modifications:

- moving the attaching spheres by isotopies;
- changing the order of critical levels;
- creation and cancellation of critical points;
- handle slides.

10.1 Moving the attaching spheres by isotopies

For a function $\phi : V \to \mathbb{R}$ we will use the notations

$$V^b := \phi^{-1}(b), \qquad V^{[a,b]} := \phi^{-1}([a,b]).$$

The goal in this section is to prove the following result.

Proposition 10.1. *Consider a complex manifold (V, J) and a proper J-convex function $\phi : V \to \mathbb{R}$ without critical values in the interval $[a, b]$. Let $\Lambda \subset V^b$ be an isotropic submanifold and $L \subset V$ its image under the flow of $-\nabla_\phi \phi$. Let $(\Lambda_t)_{t\in[0,1]}$ be an isotropic isotopy of $\Lambda_0 := L \cap V^a$ in V^a.*

Then, after composing ϕ with a sufficiently convex increasing function f : $[a, b] \to \mathbb{R}$, there exists a diffeotopy $h_t : V \to V$ with the following properties for all $t \in [0, 1]$:

(i) $h_t = \mathbb{1}$ outside $V^{[a,b]}$;

(ii) $\phi_t := \phi \circ h_t$ *is J-convex;*

(iii) the image L_t *of* Λ *under the flow of* $-\nabla_{\phi_t}\phi_t$ *intersects* V^a *in* Λ_t.

Remark 10.2. The corresponding result for ordinary functions ϕ is very easy: It just states that one can realize a smooth isotopy of spheres Λ_t descending spheres for a homotopy of gradient-like vector fields, keeping the function ϕ fixed. In contrast, Proposition 10.1 is more subtle because the gradient vector fields $\nabla_{\phi_t}\phi_t$ are determined by the functions ϕ_t themselves.

The proof requires some preparation. The following lemma is the main technical ingredient.

Remove assumption "real analytic"?

Lemma 10.3. *Let* Σ *be a J-convex hypersurface in a complex manifold* (V, J) *and* $X^\perp$ *a vector field near* Σ *with* $JX^\perp \in T\Sigma$. *Let* $\Lambda \subset \Sigma$ *be an isotropic submanifold and* X *be a vector field along* Λ *that is transverse to* Σ. *Suppose that* $\Sigma, \Lambda, X^\perp$ *are real analytic. Then for any compact subset* $K \subset \Lambda$ *there exists a J-convex hypersurface* Σ' *with the following properties:*

(i) $K \subset \Sigma'$ *and* $\xi \subset T\Sigma'$ *along* K*;*

(ii) Σ' *is transverse to* $X^\perp$ *and* $\Sigma' = \Sigma$ *outside a neighborhood of* K*;*

(iii) $JX(x) \in T_x\Sigma'$ *for all* $x \in K$.

Proof. Let $n = \dim_{\mathbb{C}} V$ and $k - 1 = \dim \Lambda$. We will only carry out the proof in the Legendrian case $k = n$, the case $k < n$ being analogous but notationally more involved. Note that the case $k < n$ formally follows from the Legendrian case provided that the symplectic normal bundle $(T\Lambda)^\omega/T\Lambda$ of Λ in the maximal complex tangency $\xi \subset T\Sigma$ is trivial. Indeed, in this case a neighborhood of Λ (after shrinking it) is isomorphic to a neighborhood of the zero section in $J^1\Lambda \oplus \mathbb{C}^{n-k}$ by a real analytic contactomorphism (see Chapter5). So we can extend Λ to a real analytic Legendrian submanifold $\tilde{\Lambda} \cong \Lambda \times \mathbb{R}^{n-k} \subset \Sigma$ and X to a vector field $\tilde{X}$ along $\tilde{\Lambda}$.

After possibly changing its sign, we may assume that $X^\perp$ is opposite to the coorientation of Σ. The flow of $X^\perp$ extends Λ (after shrinking Λ) to a real analytic submanifold $\Lambda \times [-1, 1] \subset V$. Thus a neighborhood of Λ in V is biholomorphic to a neighborhood of $\Lambda \oplus 0$ in $\Lambda^{\mathbb{C}} \oplus \mathbb{C}$. Here $\Lambda^{\mathbb{C}}$ is the complexification of Λ and $X^\perp$ generates the real line $0 \oplus i\mathbb{R}$. This implies that $T\Sigma = T\Lambda^{\mathbb{C}} \oplus \mathbb{R}$ with complex tangency $\xi = T\Lambda^{\mathbb{C}}$ along Λ. Denote coordinates on $\Lambda^{\mathbb{C}} \oplus \mathbb{C}$ by $(z, w) = (x, y, u + iv)$, where y are coordinates on Λ and x coordinates in the fibres of $\Lambda^{\mathbb{C}}$. In these coordinates, Σ can be written near Λ as the graph

$$\Sigma = \{v = \phi(x, y, u)\}$$

of a function ϕ with $\phi(0, y, 0) = 0$ and $d\phi(0, y, 0) = 0$. The choice of $X^\perp$ implies that Σ is J-convex cooriented from above. We will find Σ' as the graph $\Sigma' = \{v = \tilde{\phi}(x, y, u)\}$ of a function $\tilde{\phi}$ with $\tilde{\phi} = \phi$ outside a neighborhood of $K \oplus 0$ in $\Lambda^{\mathbb{C}} \oplus \mathbb{R}$. The condition $K \subset \Sigma'$ and $\xi \subset T\Sigma$ along K are equivalent to $\tilde{\phi}(0, y, 0) = 0$ and $d_z\tilde{\phi}(0, y, 0) = 0$ for $y \in K$. After rescaling and possibly changing its sign, we can write the given vector field X as $X = \partial_v - \tau(y)\partial_u + Y$ with Y tangent to $\Lambda^{\mathbb{C}}$ and τ some given function on Λ. Then $JX \in T\Sigma'$ along K is equivalent to $\tilde\phi_u(0, y, 0) = \tau(y)$ for $y \in K$.

Let $Q := \mathrm{dist}^2_\Lambda$ be the squared distance (with respect to some Hermitian metric) from the zero section in $\Lambda^{\mathbb{C}}$. By Lemma 2.13, Q is a J-convex function. Note that the hypersurface $\{v = Q(x, y)\}$ is tangent to Σ along Λ. Its Levi form at points of Λ is given by $-dd^{\mathbb{C}}\big(Q(x, y) - v\big)|_{\xi=T\Lambda^{\mathbb{C}}} = -dd^{\mathbb{C}}Q$, so $\{v = Q(x, y)\}$ is J-convex along Λ cooriented from above. Thus by Corollary 3.18, we can modify Σ near K, preserving J-convexity and the condition $\Lambda \subset \Sigma$, such that $\Sigma = \{v = Q(x, y)\}$ near K.

Now let a function $\tau(y)$ be given as above. Our task is to find a smooth function $\tilde\phi$ with J-convex graph such that

$$\tilde\phi(0, y, 0) = 0, \qquad d_z\tilde\phi(0, y, 0) = 0, \qquad \tilde\phi_u(0, y, 0) = \tau(y)$$

for $y \in K$ and $\tilde\phi(x, y, u) = Q(x, y)$ outside a neighborhood of K.

Pick a function $g(y, u)$ on $\Lambda \oplus \mathbb{R}$ with $g(y, 0) = 0$ and $g_u(y, 0) = \tau(y)$ for all $y \in K$, and such that $g(y, u) < -1$ outside $K' \times [-1, 1]$ for some compact neighborhood K' of K in Λ. For any $\varepsilon > 0$ let $g^\varepsilon(y, u) := \varepsilon g(y, u/\varepsilon)$. These functions satisfy $g^\varepsilon(y, 0) = 0$, $d_z g^\varepsilon(y, 0) = 0$ and $g^\varepsilon_u(y, 0) = \tau(y)$ for all $y \in K$, and $g^\varepsilon(y, u) < -\varepsilon$ outside $K' \times [-\varepsilon, \varepsilon]$. Moreover, we have

$$|g^\varepsilon(y, u)| \le C_0 \max(|u|, \varepsilon), \quad |g^\varepsilon_y|, |g^\varepsilon_{yy}| \le C_0\varepsilon, \quad |g^\varepsilon_u|, |g^\varepsilon_{yu}| \le C_0, \quad |g^\varepsilon_{uu}| \le C_0/\varepsilon$$

for $(y, u) \in K' \times [-\varepsilon, \varepsilon]$ with a constant C_0 not depending on ε. For $0 < a \le 1/2$ and $\varepsilon > 0$ consider the function

$$\psi(x, y, u) := aQ(x, y) + g^\varepsilon(y, u).$$

Our desired function $\tilde\phi$ will be a smoothing of

$$\tilde\psi := \max(Q - \varepsilon, \psi).$$

Let us first determine the region where $\psi < Q - \varepsilon$, or equivalently,

$$g^\varepsilon(y, u) + \varepsilon < (1 - a)Q(x, y). \tag{10.1}$$

For $|u| > \varepsilon$ or $y \notin K'$ this inequality holds because the left hand side is negative and the right hand side is nonnegative. Moreover, $1 - a \ge 1/2$ implies

$$g^\varepsilon(y, u) + \varepsilon \le (C_0 + 1)\varepsilon \le 2(C_0 + 1)\varepsilon(1 - a),$$

so inequality (10.1) holds if $Q(x, y) > C_1\varepsilon$ with the constant $C_1 := 2(C_0 + 1)$ not depending on ε and a. So we have $\psi \le Q - \varepsilon$ outside the compact region

$$W' := \{(x, y, u) \mid y \in K', |u| \le \varepsilon, Q(x, y) \le C_1\varepsilon\}.$$

On the other hand, in the region

$$W := \{(x, y, u) \mid y \in K', Q(x, y) + C_0|u| \le \varepsilon\}$$

we have the converse estimate

$$g^\varepsilon(y,u)+\varepsilon \geq \varepsilon\beta(y) - C_0|u|\beta(y) \geq Q(x,y) \geq (1-a)Q(x,y).$$

Hence $\psi \geq Q-\varepsilon$ on the neighborhood W of K.

We will show below that for a and ε sufficiently small the graph of ψ is J-convex on W'. Assuming this for the moment, note that the graph of $Q-\varepsilon$ is also J-convex. Thus by Corollary 3.26, we can C^0-approximate $\widetilde{\psi}$ by a smooth function ψ with J-convex graph which agrees with ψ on W and $Q-\varepsilon$ outside W'. (Note that in Corollary 3.26 the minimum appears rather than the maximum because the graphs are cooriented from below rather than above). Now on any fixed (i.e. independent of a,ε) compact neighborhood U of K', the function $Q-\varepsilon$ C^2-approaches Q as $\varepsilon \to 0$. Hence for small ε we can modify $\tilde{\Sigma}$ outside W' so that it agrees with Σ outside U. This yields the desired hypersurface Σ'.

It remains to prove J-convexity of the hypersurface $\{v=\psi(z,u)\}$ over W' for small a and ε. For this, cover K' by finitely many holomorphic coordinate charts in which Λ corresponds to $i\mathbb{R}^{n-1}$. Choose ε so small that the coordinate charts cover the region $\{(x,y) \mid y\in K', Q(x,y)\leq C_1\varepsilon\}$. According to Lemma **??**, in each such coordinate chart a sufficient condition for J-convexity of the hypersurface $\{v=\psi(z,u)\}$ is given by

$$\mathcal{L}^{\min}(\psi) := H^{\min}_\psi - 2|\psi_{uu}|\,|d_z\psi|^2 - 4|d_z\psi_u|\,|d_z\psi|(1+|\psi_u|) > 0.$$

By the J-convexity of the function Q, we have $H^{\min}_Q \geq \gamma$ for some constant $\gamma>0$. Moreover, $|Q_z|\leq C|x|$ and all derivatives of Q involving a u-derivative vanish. Here and in the following C denotes a generic constant that depends on C_0, C_1, γ but not on a,ε. The estimates for g^ε yield

$$H^{\min}_\psi \geq \gamma a - C\varepsilon, \quad |\psi_z| \leq Ca|x| + C\varepsilon, \quad |\psi_u|, |\psi_{zu}| \leq C, \quad |\psi_{uu}| \leq C/\varepsilon$$

for $(y,u)\in K'\times[-\varepsilon,\varepsilon]$. It follows that

$$\mathcal{L}^{\min}(\psi) \geq \gamma a - C\varepsilon - Ca|x| - Ca^2|x|^2/\varepsilon.$$

Now on W' we have $\gamma|x|^2 \leq Q(x,y) \leq C_1\varepsilon$, and hence

$$\mathcal{L}^{\min}(\psi) \geq \gamma a - C\varepsilon - Ca\sqrt{\varepsilon} - Ca^2.$$

Choosing $\varepsilon \leq a^2$, we obtain

$$\mathcal{L}^{\min}(\psi) \geq \gamma a - Ca^2,$$

which is positive for $a>0$ sufficiently small. This proves J-convexity of the hypersurface $\{v=\psi(z,u)\}$ and hence the lemma. □

Lemma 10.4. *Let ϕ be a proper J-convex function on the complex manifold V without critical values in $[a,b]$. Let $L\subset V^{[a,b]}$ be a totally real submanifold that intersects each level set J-orthogonally. Suppose that ϕ and L are real analytic.*

Then there exists a J-convex function ψ, C^1-close to ϕ, such that $\psi = \phi$ on L and $\nabla_\psi \psi$ is tangent to L.

Moreover, if $\nabla_\phi \phi$ is already tangent to L near $V^{[a,a']} \cup V^{[b',b]}$ for some $[a', b'] \subset (a, b)$, then we can choose $\psi = \phi$ on $V^{[a,a']} \cup V^{[b',b]}$.

Proof. Let X be the unique vector field tangent to L, orthogonal to the intersection of L with level sets of ϕ, with $d\phi(X) \equiv 1$. Then X is real analytic and JX is tangent to the level sets of ϕ. The flow of X defines a real analytic diffeomorphism $\Lambda \times i[a, b] \cong L$, where $\Lambda := L \cap V^a$. This diffeomorphism extends to a biholomorphic identification of a neighborhood of L in V with $\Lambda^{\mathbb{C}} \times \mathbb{C}$, where $\Lambda^{\mathbb{C}}$ is the complexification of Λ. Denote coordinates on $\Lambda^{\mathbb{C}}$ by z and on $\mathbb{C}$ by $u+iv$. Under this identification L corresponds to $\Lambda \times i[a, b]$, and $X = \partial_v$, $\phi = v$ along L. Since the level sets of ϕ are J-orthogonal to L, they are tangent to $T\Lambda^{\mathbb{C}} \oplus \mathbb{R}$ along L.

Define the function

$$\psi(z, u, v) := v + Q(z) + \frac{1}{2} f(v) u^2$$

on $\Lambda^{\mathbb{C}} \times \mathbb{C}$, where $Q := \mathrm{dist}^2_\Lambda$ for some Hermitian metric on $\Lambda^{\mathbb{C}}$ and f is a positive function. We compute

$$\begin{aligned} d\psi &= dv + dQ + f(v)u\,du + \frac{1}{2} f'(v)u^2 dv, \\ d^{\mathbb{C}}\psi &= du + dQ \circ J_{\Lambda^{\mathbb{C}}} - f(v)u\,dv + \frac{1}{2} f'(v)u^2 du, \\ \omega_\psi &= -dd^{\mathbb{C}}\psi = \omega_Q + f(v) du \wedge dv \text{ along } L. \end{aligned}$$

In particular, ψ is J-convex and $d\psi = dv = d\phi$ along L. Hence by Proposition 3.15, ψ can be extended to a J-convex function on V which is C^1-close to ϕ and agrees with ϕ outside a neighborhood of L. The gradient of ψ is determined by the equation

$$\omega_\psi(\nabla_\psi \psi, Y) = -d^{\mathbb{C}}\psi(Y)$$

for all $Y \in TV$. Now $d^{\mathbb{C}}\psi = du$ along L implies $\nabla_\psi \psi = f(v)\partial_v$ along L, so $\nabla_\psi \psi$ is tangent to L.

Finally, suppose that $\nabla_\phi \phi$ is already tangent to L near $V^{[a,a']} \cup V^{[b',b]}$. Pick a cutoff function $\beta : V \to [0, 1]$ which equals 0 outside $V^{[a',b']}$ and 1 where $\nabla_\phi \phi$ is not tangent to L. Construct ψ as above and set

$$\theta := (1 - \beta)\phi + \beta\psi.$$

This function agrees with ϕ on $V^{[a,a']} \cup V^{[b',b]}$, and by Lemma 3.16 , θ is J-convex for ψ sufficiently C^1-close to ϕ.

Consider a point x with $0 < \beta(x) < 1$. By construction, we have $\phi(x) = \psi(x)$ and $d\phi(x) = d\psi(x)$. Moreover, since $\nabla_\phi \phi(x)$ is tangent to L, the vector fields

X, $\nabla_\phi \phi$ and $\nabla_\psi \psi$ are parallel along L. By appropriate choice of the function f in the construction of ψ, we can therefore arrange $\nabla_\phi \phi = \nabla_\psi \psi$ along L. Since ϕ and ψ agree to first order, we have

$$\omega_\theta = (1-\beta)\omega_\phi + \beta\omega_\psi$$

at the point x. Hence for any $Y \in T_x V$,

$$\begin{aligned}
\omega_\theta(\nabla_\theta \theta, Y) &= -d^{\mathbb{C}}\theta(Y) \\
&= -(1-\beta)d^{\mathbb{C}}\phi - \beta d^{\mathbb{C}}\psi \\
&= (1-\beta)\omega_\phi(\nabla_\phi \phi, Y) + \beta\omega_\psi(\nabla_\psi \psi, Y) \\
&= (1-\beta)\omega_\phi(\nabla_\phi \phi, Y) + \beta\omega_\psi(\nabla_\phi \phi, Y) \\
&= \omega_\theta(\nabla_\phi \phi, Y).
\end{aligned}$$

This shows $\nabla_\theta \theta = \nabla_\phi \phi$ along L. In particular, $\nabla_\theta \theta$ is tangent to L, so θ is the desired function. □

Next we will prove a special case of Proposition 10.1.

Lemma 10.5. *Proposition 10.1 holds provided that the Λ_t are sufficiently C^2-close to Λ_0.*

Include complete proof!

Proof. (sketch) We will construct the h_t C^2-close to the identity. Then the ϕ_t will be C^2-close to ϕ and hence automatically J-convex. So we only have to show that by C^2-small variations of ϕ we can arrange L to meet V^a in any Legendrian embedding C^2-close to Λ_0.

Consider a variation $\phi_\varepsilon := \phi + \varepsilon\psi$ of ϕ in the direction of a function ψ with a small parameter ε. The new gradient field will be of the form

$$\nabla_{\phi_\varepsilon} \phi_\varepsilon = \nabla_\phi \phi + \varepsilon Y + O(\varepsilon^2).$$

For any function f let $\omega_f := -dd^{\mathbb{C}} f$. If this is nondegenerate the gradient $\nabla_f g$ of another function is determined by the equation

$$\omega_f(\nabla_f g, v) = -dg(Jv)$$

for all $v \in TV$. Using this, we find

$$\begin{aligned}
\omega_{\phi_\varepsilon}(\nabla_{\phi_\varepsilon} \phi_\varepsilon, v) &= -d\phi_\varepsilon(Jv) \\
&= -d\phi(Jv) - \varepsilon d\psi(Jv) \\
&= \omega_\phi(\nabla_\phi \phi, v) + \varepsilon\omega_\phi(\nabla_\phi \psi, v).
\end{aligned}$$

On the other hand, we have

$$\omega_{\phi_\varepsilon}(\nabla_{\phi_\varepsilon} \phi_\varepsilon, v) = \omega_\phi(\nabla_\phi \phi, v) + \varepsilon\omega_\phi(Y, v) + \varepsilon\omega_\psi(\nabla_\phi \phi, v) + O(\varepsilon^2).$$

Comparing the linear terms in ε, we find

$$i_{(Y-\nabla_\phi\psi)}\omega_\phi = i_{\nabla_\phi\phi}dd^{\mathbb{C}}\psi.$$

This equation uniquely determines $Y = \nabla_\phi\psi + Y'$ with $i_{Y'}\omega_\phi = i_{\nabla_\phi\phi}dd^{\mathbb{C}}\psi$. As in the proof of Proposition 2.13, consider (in the Lagrangian case) local complex coordinates (x, y) near Λ_0 in which $L = \{y = 0\}$ and $\nabla_\phi\phi = \partial_{x_1}$. Take ψ to vanish on L, so ψ is only a function of x. Then Y' is a vector field in the x-coordinates, i.e. tangent to L. By varying the derivative of ψ along L we can arrange for $\nabla_\phi\psi$ any vector field in the y-coordinates, i.e. transverse to L. Now the result follows from the implicit function theorem. □

Proof of Proposition 10.1. **Step 1.** We first prove the proposition under the hypothesis that ϕ and the isotopy Λ_t are real analytic.

Let $\Sigma := V^a$. The flow of the real analytic vector field $\nabla\phi/|\nabla\phi|$ defines a real analytic diffeomorphism

$$\Sigma \times [a, b] \cong V^{[a,b]}.$$

Under this identification, ϕ corresponds to the function $(x, r) \mapsto r$, $\nabla\phi/|\nabla\phi|$ to the vector field ∂_r, L to $\Lambda \times [a, b]$, and Λ_t to $\Lambda_t \times \{a\}$. In view of Lemma 12.9, $\Lambda_t \times \{r\}$ is isotropic for all $r \in [a, b]$.

Pick a C^2-function $g : [a, b] \to [0, 1]$ which is real analytic on an interval $[a', b'] \subset (a, b)$ and equals 1 on $[a, a']$ and 0 on $[b', b]$. For $t \in [0, 1]$ define

$$L_t := \bigcup_{r\in[a,b]} \Lambda_{tg(r)} \times \{r\} \subset \Sigma \times [a, b].$$

This is a totally ral submanifold, real analytic on $\Sigma \times [a', b']$, which intersects each level set $\Sigma \times \{r\}$ in the isotropic submanifold

$$\Lambda_{t,r} := \Lambda_{tg(r)} \times \{r\}.$$

Let $X_{t,r}$ be the unique vector field tangent to L_t along $\Lambda_{t,r}$ with $dr(X_{r,t}) = 1$. In particular, $X_{t,r}$ is transverse to the level sets $\Sigma \times \{r\}$. Hence by Lemma 10.3 there exist J-convex hypersurfaces $\Sigma_{t,r}$ transverse to ∂_r such that $\Lambda_{t,r} \subset \Sigma_{t,r}$, the contact structure ξ_r is contained in $T\Sigma_{t,r}$ along $\Lambda_{t,r}$, and $JX_{t,r} \in T\Sigma_{t,r}$. Note that the last two conditions say that L_t intersects $\Sigma_{t,r}$ J-orthogonally for all r. Moreover, we may choose $\Sigma_{t,r} = \Sigma \times \{r\}$ for r outside $[a', b']$.

Proof!

By construction, the $\Sigma_{t,r}$ for fixed t and varying r form a foliation near L_t. Thus by Proposition 3.28, we can modify the $\Sigma_{t,r}$ to a J-convex foliation, keeping them fixed near L_t and for r outside $[a', b']$. Let ψ_t be the function which equals r on the new hypersurfaces $\tilde{\Sigma}_{r,t}$. Pick a sufficiently convex increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ \psi_t$ is J-convex for all $t \in [0, 1]$. Now we apply Lemma 10.4 to the functions $f \circ \psi_t$ and the totally real submanifolds L_t over the set $\{r \in [a', b']\}$. We find J-convex functions ϕ_t, C^1-close to $f \circ \psi_t$ and agreeing with $f \circ \psi_t$ on L_t and for r outside $[a', b']$, such that $\nabla_{\phi_t}\phi_t$ is tangent to L_t. Thus L_t is the image of $\Lambda_{t,b} = \Lambda_0 \times \{b\} = \Lambda$ under the flow of $-\nabla_{\phi_t}\phi_t$, and by construction L_t intersects $\Sigma \times \{a\}$ in $\Lambda_{t,a} = \Lambda_t \times \{a\}$. This proves property (iii).

By construction, ϕ_t agrees with $f \circ \phi$ for r outside $[a', b']$. Moreover, since $L_0 = L$, we can arrange $\phi_0 = f \circ \phi$. It remains to find an isotopy h_t such that $\phi_t = f \circ \phi \circ h_t$. Define diffeomorphisms $g_t : V^{[a,b]} \to V^{[a,b]}$ on the level $\phi^{-1}(r)$ by following the flow of $-\nabla_\phi \phi$ down to V^a and then the flow of $\nabla_{\phi_t}\phi_t$ up to the level $\phi_t^{-1}(r)$. Then $g_0 = \mathbb{1}$ and $\phi_t = f \circ \phi \circ h_t$. Moreover, $g_t = \mathbb{1}$ on $V^{[a,a']}$ and

$$g_t : V^{[b',b]} \cong \Sigma \times [b', b] \to \Sigma \times [b', b], \qquad (x, r) \mapsto \big(\gamma_t(x), r\big)$$

with $\gamma_t := g_t|_{V^{b'}}$. Define h_t on the level $\phi^{-1}(r)$ as $g_{t\rho(r)}$ with a smooth function $\rho : [a, b] \to [0, 1]$ which equals 1 on $[a, b']$ and 0 near b. Then $h_t = \mathbb{1}$ near b and h_t is the desired isotopy.

Step 2. It remains to remove the hypothesis that ϕ and the isotopy Λ_t are real analytic.

Let ϕ, Λ, Λ_t be as in the proposition. Pick an interval $[a', b'] \subset (a, b)$. Let ψ be C^2-close to ϕ (hence J-convex), real analytic on $\psi^{-1}([a', b'])$, with $\psi = \phi$ near $\partial V^{[a,b]}$. Denote by $\tilde\Lambda_t \subset \psi^{-1}(a')$ the image of Λ_t under the flow of $\nabla_\psi \psi$. By Corollary 7.25, we can C^1-approximate $\tilde\Lambda_t$ by a real analytic isotropic isotopy Λ'_t in $\psi^{-1}(a')$.

The image of Λ under the flow of $-\nabla_\psi \psi$ intersects $\psi^{-1}(a')$ in an isotropic submanifold $\Lambda^{a'}$ that is C^1-close to Λ'_0. So by Lemma 10.5, we can modify ψ inside the region $\psi^{-1}([b', b])$ to achieve $\Lambda^{a'} = \Lambda'_0$. Similarly, again using Lemma 10.5, for every $t \in [0, 1]$ we can perturb ψ inside the region $\psi^{-1}([a, a'])$ to ψ_t such that the image of Λ'_t under the flow of $-\nabla_{\psi_t}\psi_t$ intersects $\psi^{-1}(a)$ in Λ_t.

Denote by $\Lambda^{b'} \subset \psi^{-1}(b')$ the image of Λ under the flow of $-\nabla_\psi \psi$. Now apply Step 1 to the restriction of ψ to $\psi^{-1}([a', b'])$ and the isotropic submanifolds $\Lambda^{b'}, \Lambda'_t$. Denote the resulting J-convex functions on $\psi^{-1}([a', b'])$ by ϕ_t and extend them to $V^{[a,b]}$ via ψ on $\psi^{-1}([b', b])$ and ψ_t on $\psi^{-1}([a, a'])$. By construction, these extensions are J-convex, coincide with ϕ for $t = 0$ and near $\partial V^{[a,b]}$ and satisfy property (iii). Now the same argument as in Step 1 provides the diffeotopy h_t with $\phi_t = \phi \circ h_t$. This concludes the proof of Proposition 10.1. □

10.2 Changing the order of critical levels

In this section we consider the following situation. Let ϕ be a J-convex function on an n-dimensional complex manifold V. Let q be a nondegenerate critical point of ϕ of index $k \le n$ with $\phi(q) = b$. Let $a < b$ and suppose that the stable manifold W_q^- does not meet any critical points of value $\ge a$. Define the *stable* *disk and sphere*

$$D_q^- := W_q^- \cap \{\phi \geq a\}, \qquad S_q^1 := \partial D_q^-.$$

Let a', b' be given with $a' < a < b < b'$. The following result states that we can move the level a above the critical level b by a J-convex deformation.

Proposition 10.6. *There exists a homotopy of J-convex functions ϕ_t such that $\phi_0 = \phi$ and $\phi_t = f_t \circ \phi$ outside a neighborhood $U \subset V^{[a',b']}$ of D_q^-, where $f_t : \mathbb{R} \to \mathbb{R}$ are increasing convex smooth functions with $f_0 = \mathbb{1}$, $f_t(x) = x$ for $x \leq a'$, and $f_1(a) > b$. Moreover, all ϕ_t have q as a nondegenerate index k critical point of value b and no other critical points in U.*

The following lemma reduces the statement of Proposition 10.6 to a model function on the standard handle $H_\varepsilon = D^k_{1+\varepsilon} \times D^{2n-k}_\varepsilon \subset \mathbb{C}^n$. Here $z_j = x_j + iy_j$ are complex coordinates such that $(y_1, \ldots, y_k)$ are coordinates on $D^k_{1+\varepsilon}$ and $(x_1, \ldots, x_n, y_{k+1}, \ldots, y_n)$ on D^{2n-k}_ε. As in Lemma 4.9, introduce the functions

$$r := \sqrt{x_1^2 + \cdots + x_n^2 + y_{k+1}^2 + \cdots + y_n^2}, \qquad R := \sqrt{y_1^2 + \cdots + y_k^2}.$$

Lemma 10.7. *Under the hypothesis of Proposition 10.6 and for any constant $A > B := b - a$, after a C^1-small J-convex deformation of ϕ, fixed outside a neighborhood of D_q^- and keeping q as the only critical point (preserving nondegeneracy and its value), the following holds: There exists a neighborhood of D_q^- biholomorphic to the standard handle H_ε in which D_q^- corresponds to the disk $D = D_1^k \subset D^k_{1+\varepsilon}$ and ϕ to the standard function*

$$\psi_{\rm st}(r, R) = Ar^2 - BR^2 + b.$$

Proof. By the Morse lemma, for $\delta > 0$ small there exists an embedding $f : D^k_\delta \to D_q^-$ onto a neighborhood of q such that $f^*\phi(R) = b - BR^2$. Using gradient-like vector fields, this extends to a diffeomorphism $f : D \to D_q^-$ with $f^*\phi(R) = b - BR^2$. Let $\tilde f : D \hookrightarrow V$ be a real analytic embedding C^2-close to f. Extend $\tilde f$ to a holomorphic embedding $F : H_\varepsilon \hookrightarrow V$. Then $F^*\phi|_D = \tilde f^*\phi$ is C^2-close to the function $R \mapsto b - BR^2$. Moreover, since the level sets of ϕ are J-orthogonal to D_q^-, the level sets of $F^*\phi$ are C^1-close to being i-orthogonal to D. Hence we find $\tilde\phi$, C^2-close to ϕ (thus J-convex) and equal to ϕ outside a neighborhood of D_q^-, such that $F^*\tilde\phi|_D(R) = b - BR^2$ and the level sets of $F^*\tilde\phi$ are i-orthogonal to D.

Now consider on H_ε the function

$$\psi_{\rm st}(r, R) := Ar^2 - BR^2 + b,$$

which is i-convex for $A > B$. The conditions on $F^*\tilde\phi$ above show that $F^*\tilde\phi$ agrees with $\psi_{\rm st}$ together with its derivative along D. Hence by Proposition 3.15, after a C^1-small J-convex deformation supported near D we may assume that $F^*\tilde\phi = \psi_{\rm st}$ near D. Since the stable and unstable manifolds of 0 with respect to $F^*\tilde{\phi}$ and $\psi_{\rm st}$ coincide, the critical point 0 remains nondegenerate during this deformation. C^1-closeness ensures that no new critical points are generated. Now shrink ε such that $F^*\tilde{\phi} = \psi$ on H_ε and the lemma is proved. □

Proof of Proposition 10.6. We may assume without loss of generality $a = 0$ and $b = 1$; the general case then follows by composing all functions with the affine function $x \mapsto (b-a)x + a$. After applying the deformation in Lemma 10.7 with $B = 1$ and $A = 64$, we may further assume that there exists a neighborhood U of D_q^- biholomorphic to the standard handle H_ε in which D_q^- corresponds to the disk $D = D_1^k \subset D_{1+\varepsilon}^k$ and ϕ to the standard function

$$\psi_{\rm st}(r, R) = 64r^2 - R^2 + 1.$$

Let $\psi_t : H_\varepsilon \to \mathbb{R}$, $t \in [0,1]$, be the family of i-convex functions from Proposition 4.21 with $\beta < -a'$ and any $\rho \in (0, \varepsilon)$. The ψ_t can be extended to smooth functions $\phi_t : V \to \mathbb{R}$ by $\phi_t := f_t \circ \phi$ outside $U \cong H_\varepsilon$, where the functions $f_t : [-\beta, 1 + 64\varepsilon^2] \to \mathbb{R}$ from Proposition 4.21 are extended to functions $f_t : \mathbb{R} \to \mathbb{R}$ with $f_t(x) = x$ for $x \leq -\beta$ and sufficiently convex for $x \geq 1 + 64\varepsilon^2$ such that $f_t \circ \phi$ is i-convex. Now the properties of the ϕ_t follow from the corresponding properties of ψ_t and f_t. □

10.3 Creation and cancellation of critical points

10.3.1 Main propositions

In this section we describe creation and cancellation of critical points of J-convex functions. We begin by recalling the relevant concepts from Morse theory.

A local model for the creation of a pair of critical points is given by the family of functions

$$\psi_t(x) = x_1^3 - tx_1 - \sum_{i=2}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$$

for $x \in \mathbb{R}^n$ and $t \in [-1, 1]$. Note that ψ_t has no critical points for $t < 0$, two nondegenerate critical points $(\pm\sqrt{t/3}, 0, \ldots, 0)$ of indices $k, k-1$ for $t > 0$, and a unique degenerate critical point at the origin for $t = 0$. Replacing ψ_t by ψ_{-t} gives a local model for cancellation of a pair of critical points. We call a critical point p of a function ϕ an *embryo critical point* if in a neighborhood of p the function is equivalent to the model function ψ_0 in the sense that $\phi = f \circ \psi_0 \circ g$ for diffeomorphisms f, g.

As before, W denotes a compact cobordism with boundary $\partial W = \partial_+ W \amalg \partial_- W$ and all functions on W will be assumed to have $\partial_\pm W$ as regular level sets. A family of functions $\phi_t : W \to \mathbb{R}$ and gradient-like vector fields X_t is called a *cancellation (resp. creation) family* if there is a $t_0 \in (0,1)$ such that for $t > t_0$ (resp. $t < t_0$) the function ϕ_t has no critical points, for $t > t_0$ (resp. $t > t_0$) it has exactly two critical points of index k and $k-1$, $k=1,\dots,n$ transversely connected by exactly one trajectory of X_t, and for $t=t_0$ it has a unique embryo critical point. For J-convex functions we always assume in addition that $X_t=\nabla_{g_t}\phi_t$, where g_t is the metric defined by the function ϕ_t (see Section **??** above). A deformation of functions $\phi_t : W\to\mathbb{R}$, $t\in[0,1]$, is called *weakly supported* in $U\subset W$ if there exists an isotopy $h_t:\mathbb{R}\to\mathbb{R}$ such that on $W\setminus U$ we have $\phi_t=h_t\circ\phi_0$.

With our restructuring it is not clear where to refer with this

The goal of this section is to prove the following Propositions 10.8 and 10.9.

Proposition 10.8. *Let (W,J,ϕ) be a Stein cobordism, where the J-convex function ϕ has no critical points. Then given any point $p\in\operatorname{Int}W$ and an integer $k=1,\dots,n$, there is a creation family ϕ_t of J-convex functions, weakly supported in $\mathcal{O}p\,p$, such that $\phi_0=\phi$ and ϕ_1 has a pair of critical points of index k and $k-1$.*

Note that in the usual Morse theory an analog of Proposition 10.8 is rather trivial: using an appropriate cut-off construction any local creation family can be implanted into a globally defined family, see Subsection 10.3.2 below. However, in the context of J-convex functions this scheme does not seem to work. In fact, we do not know whether the statement remains true if one drop the word "weakly" and tries to construct a locally supported creation family.

Proposition 10.9. *Let (W,J,ϕ) be an elementary Stein cobordism of type II. In other words, the J-convex function ϕ has exactly two critical points p,q of index k and $k-1$, respectively, which are transversely connected by a unique gradient trajectory. Denote $a_-:=\phi|_{\partial_-W}, b:=\phi(q), c:=\phi(p)$. Choose a regular value $a\in(a_-,b)$. Let Δ be the closure of the stable disc of the critical point p in $\{\phi\geq a\}$. Then there exists a cancellation family $\phi_t : W\to\mathbb{R}$, $t\in[0,1]$, of J-convex functions, weakly supported in $\mathcal{O}p\,\Delta$, such that $\phi_0=\phi$ and ϕ_1 has no critical points.*

10.3.2 Recollections from Morse theory

We first recall some facts from the h-cobordism theory for ordinary Morse functions. The basic reference in [50].

Move all this in separate section?

Let $\phi : W\to\mathbb{R}$ be a Morse function and X a gradient-like vector field. For a critical point p of ϕ denote by $D^\pm_p$ its stable resp. unstable manifold.

Lemma 10.10. *$D^\pm_p$ are smooth submanifolds. If ϕ and X are real analytic then so are $D^\pm_p$.*

Reference for real analyticity?

Consider a cobordism W with a Morse function $\phi : W\to\mathbb{R}$ which is constant on $\partial_\pm W$. Suppose ϕ has precisely two critical points p,q of index k and $k-1$, respectively, which are transversely connected by a unique trajectory of some gradient-like vector field X. Denote $a_-:=\phi|_{\partial_-W}, b:=\phi(q), c:=\phi(p)$. Choose a regular value $a\in(a_-,b)$. Let Δ be the closure of the stable disc of the critical point p in $\{\phi\geq a\}$.

Lemma 10.11. *Δ is a smoothly embedded half-disk with upper boundary $\partial_+\Delta = D^-_q \cap \{\phi \geq a\}$ and lower boundary $\partial_-\Delta = \Delta \cap \{\phi = a\}$. If ϕ and X are real analytic then so are Δ and $\partial_\pm\Delta$.*

To be added.

Proof. □

10.3.3 Carving one J-convex function with another one

Maybe move to Section 3.3
REWRITTEN

Let $\phi : U \to \mathbb{R}$ be J-convex function on an open set U and $\Sigma = \{\phi = a\}$ be a regular level set. Let us denote by U_- and U_+ the domains $\{\phi \leq a\}$ and $\{\phi \geq a\}$, respectively. Let $\psi : \Omega \to [c_-, c_+]$ be another J-convex function defined on a compact subdomain $\Omega \subset U$ with boundary $\partial\Omega = \partial_+\Omega \cup \partial_-\Omega \cup \partial_v\Omega$ such that $\psi|_{\partial_\pm\Omega} = c_\pm$ and $\partial_+\Omega \cup \partial_v\omega \subset \text{int}U_+$. For a small $\varepsilon > 0$ let us denote by Ω^ε the domain $\{c_- + \varepsilon \leq \psi \leq c_+ - \varepsilon\} \subset \Omega$, and by U^ε_- the domain $\{\phi \leq a - \varepsilon\} \subset U_-$. By composing ϕ, ψ with increasing weakly convex diffeomorphisms $g, h : \mathbb{R} \to \mathbb{R}$ we can arrange that the functions $\widetilde{\phi} = g \circ \phi$ and $\widetilde{\psi} = h \circ \psi$ satisfy the following conditions:

- $\widetilde{\psi} > \widetilde{\phi}$ on $U^\varepsilon_- \cap \Omega_\varepsilon$;
- $\widetilde{\phi} > \widetilde{\psi}$ on $(U_+ \cap \Omega) \cup \partial_-\Omega$.

To see this, first compose ψ with h such that $h(c_-) < \min_\Omega \phi$ and $h(c_- + \varepsilon) > \max_{\Omega^\varepsilon} \phi$, thus $\widetilde{\psi} > \phi$ on Ω^ε and $\widetilde{\psi} < \phi$ on $\partial_-\Omega$. Then compose ϕ with g such that $g(x) = x$ for $x \leq a - \varepsilon$ and $g(a) > \max_{U_+ \cap \Omega} \widetilde{\psi}$, thus $\widetilde{\phi} > \widetilde{\psi}$ on $(U^+ \cap \Omega) \cup \partial_-\Omega$ and $\widetilde{\psi} > \widetilde{\phi}$ on $U^\varepsilon_- \cap \Omega_\varepsilon$.

Take the function $\max(\widetilde{\phi}, \widetilde{\psi})$ and apply to it the smoothing procedure from Section 3.2. This operation we will call the *carving with ψ of the level set Σ of the function ϕ.* The resulting function will be denoted by $\mathfrak{m}_\psi(\phi, \Sigma)$. Though there are numerous ambiguities in the definition of this operation it is important that it can be done for families of functions smoothly dependent on the parameters, that ε can be chosen arbitrarily small and the smoothing can be chosen sufficiently close to $\max(\widetilde{\phi}, \widetilde{\psi})$ in the sense of Corollary 3.8. In particularly, everywhere below where we use the notation $\mathfrak{m}_\psi(\phi_t, \Sigma)$ we assume that ε is chosen sufficiently small and the approximation is good enough.

10.3.4 The notation and special shapes

To be rewritten below here.

Let $\mathbb{R}^k = \mathbb{R}^{k-1} \times \mathbb{R}$ be the space with coordinates $(x_1, \dots, x_k)$, D_t, $t > 0$, denotes the disc $\{\sum\limits_1^k x_j^2 \leq t^2\}$ of radius t, and we write D instead of D_1. We will also use the notation $D_t(p)$ for the disc of radius t centered at a point $p \in \mathbb{R}^k$. We further denote by D_- the lower half disc $D \cap \{x_k \leq 0\}$, and set $\partial_+ D_- = D_- \cap \{x_k = 0\}$ and $\partial_- D_- = \partial D \cap D_-$, so that we have $\partial D_- = \partial_- D_- \cup \partial_+ D_-$.

Viewing $\mathbb{R}^k$ as a coordinate subspace of $\mathbb{C}^n$ with complex coordinates $(x_1 + iy_1, \dots, x_n + iy_n)$ we will consider the splitting $\mathbb{C}^n = \mathbb{R}^k \times \mathbb{R}^{2n-k}$, and will write $z \in \mathbb{C}^n$ as $z = (x, u)$, where $x = (x_1, \dots, x_k)$, $u = (x_{k+1}, \dots, x_n, y_1, \dots, y_n)$. We will also denote $x' := (x_1, \dots, x_{k-1})$. Let us set

$$\rho = ||x'|| = \sqrt{\sum_1^{k-1} x_j^2},\ R = ||x|| = \sqrt{\sum_1^{k} x_j^2},\ r = ||u|| = \sqrt{\sum_{k+1}^{n} x_j^2 + \sum_1^{n} y_l^2}.$$

We will also introduce the vector fields

$$\vec{x} = \sum_1^k x_j \frac{\partial}{\partial x_j},\ \vec{u} = \sum_{k+1}^n x_j \frac{\partial}{\partial x_j} + \sum_1^n y_j \frac{\partial}{\partial y_j},$$

so that $R = ||\vec{x}||, r = ||\vec{u}||$.

Given a compact subset $K \subset \mathbb{C}^n$ and $\sigma > 0$ we will denote by $U_\sigma(K)$ its open metric σ-neighborhood in $\mathbb{C}^n$. If K is a subset of $\mathbb{R}^k$ then we denote

$$B_\sigma(K) := K \times \{r \leq \sigma\} \subset \mathbb{R}^k \times \mathbb{R}^{2n-k} = \mathbb{C}^n.$$

In Corollary 4.15 we constructed shapes $\phi^a_{\gamma,\delta}(r)$ for $a > 0$ and sufficiently small $\gamma, \delta > 0$, which define i-convex hypersurfaces $C^a_{\gamma,\delta} = \{R = \phi^a_{\gamma,\delta}(r)\} \subset \mathbb{C}^n$ surrounding the disc D. We can choose the family smoothly depending on parameters a, γ, δ. The i-convex hypersurface $C^a_{\gamma,\delta}$ satisfies the following conditions

Insert a in the notation in Corollary 4.15

a) $C^a_{\gamma,\delta} \setminus U_\gamma(D) = \{R^2 - ar^2 = 1\} \setminus U_\gamma(D)$;

b) $C^a_{\gamma,\delta} \cap B_\delta(D_{1-\gamma}) = \{r = \delta\} \cap B_\delta(D_{1-\gamma})$.

10.3.5 Proof of Proposition 10.9

Assuming that the disc Δ in the Proposition 10.9 and the function $\phi|_{\partial_+\Delta}$ are real analytic, we can parameterize Δ by a real analytic diffeomorphism $\alpha : D_- \to \Delta$ so that we have $\alpha(\partial_- D_-) = \partial_-\Delta = \Delta \cap \partial_- W$, while $\alpha(\partial_+ D_-) = \partial_+\Delta$ is the stable disc of the critical point q. We also may assume that

(i) $\phi \circ \alpha|_{\partial_+ D_-} = -k\rho^2 + b$, where $k = b - a$;

(ii) $d\alpha\left(\vec{R}\,|_{\partial_- D_-}\right) = -2k\nabla\phi|_{\partial_-\Delta}$.

The embedding $\alpha : D_- \to \Delta \hookrightarrow W$ extends to a biholomorphism A between a neighborhood $U = U_\sigma(D_-)$ for some $\sigma > 0$ and a neighborhood $\mathcal{O}p\,\Delta \subset W$.

Let $\psi = \phi \circ A : U \to \mathbb{R}$ be the pull-back of the function ϕ to U. The second condition above means that $\nabla\phi|_{\partial_- D_-} = -\nabla(kR^2)|_{\partial_- D_-}$.

We will construct the cancellation family of i-convex functions ψ_t for the function $\psi_0 = \psi$ which is weakly supported in U. Then the deformation $\psi_t \circ A^{-1}$ extends to the required cancellation deformation ϕ_t on W.

A family of i-convex $\psi_t : U \to \mathbb{R}$, $t \in [0,1]$, will be called *admissible* if the following two conditions are satisfied:

Stab. The intersection $\mathbb{R}^k \cap U$, and the disc Δ_- are invariant with respect to the gradient flow of all functions ψ_t, $t \in [0,1]$;

Loc. The deformation ψ_t is weakly supported in U.

We will call an admissible deformation *preliminary* if the critical points of ϕ_t remain fixed (but may change the critical values).

The required cancellation deformation will be constructed by concatenating several admissible deformations. All the deformations which we construct below will be preliminary, except the last one which will be of cancellation type. To simplify the notation we will denote all the deformations by ψ_t, and parameterize them sometimes by $t \in [0,1]$, and sometimes by different intervals, assuming that at any given moment the function ψ is the function constructed as the result of the previous step. We will also use the notation β for the restriction of the current function ψ to $\mathbb{R}^k \cap U$.

Let us recall that $\psi|_{\partial_- D_-} = a$. The critical values $\psi(q)$ and $\psi(p)$ may change during the deformation but we will nevertheless always denote them by b and c.

Step 1. Normalization near D_- and critical points.

Lemma 10.12. *For any $A > 0$ there exists a preliminary deformation ψ_t such that the function ψ_1 has the form $\beta(x) + Ar^2$ near D_- for some $A > 0$, and*

$$\beta(x) = -kR^2 + b \text{ near } \partial_- D \quad \text{and on } \partial_+ D_-,$$

where $k = b - a$.

We will choose $A = 64k$.

Proof. Thanks to the condition $\nabla\phi|_{\partial_- D_-} = \nabla(kR^2)|_{\partial_- D_-}$ we can apply Proposition 3.15 to modify $\beta = \psi|_{\mathbb{R}^k \cap U}$ to ensure the conditions for β. Next, we can apply 3.15 again along D_- to satisfy the first condition. □

Let us now restrict ψ to a smaller neighborhood $U_\sigma(D_-)$ such that in this neighborhood $\psi = \beta(x) + Ar^2$, and in $U_\sigma(\partial_- D_-)$ we have $\beta(x) = -R^2 + c_0$. We will keep the notation $U = U(\sigma)$ for this smaller neighborhood. All further deformations will be chosen admissible for that smaller neighborhood.

Step 2. Construction of special surroundings. Choose

$$R' = 1 - \frac{\sigma}{2}, R'' = 1 - \frac{\sigma}{4}.$$

For $t \in [R', 1]$ let us denote by Σ_t the level set of the function ψ which contains the hemisphere $S_t = \{||x|| = t\}$. Denote by D', D'', $D'' \subset D' \subset D$ the discs of radius R', R'', respectively, and set $D'_- = D_- \cap \{R \le R'\} = D' \cap \{x_k \le 0\}$. We will use the notation $\Sigma, \Sigma', \Sigma''$ instead of Σ_t for $t = 1, R', R''$, respectively.

Next, we will construct a special family of i-convex hypersurfaces $\widetilde{\Sigma}_t, t \in [R', 1]$, as follows. Choose a smooth family of i-convex hypersurfaces $C^{64}_{\gamma,\delta} = \{R = \phi^{64}_{\delta,\gamma}(r)\}$ introduced above Section 10.3.4, and consider a 1-parametric family $C^{64}_{\delta(t),\gamma(t)}$, $t \in [R', 1]$ where the decreasing functions $\delta(t), \gamma(t)$ are chosen in such a way that the following conditions are satisfied

- $\delta(R'), \gamma(R') < \frac{\sigma}{2}$;
- the hypersurfaces $\widetilde{\Sigma}_t = tC^{64}_{\gamma(t),\delta(t)}$, $t \in [R', 1]$, define a smooth foliation of the domain Ω bounded by $\widetilde{\Sigma}' = \widetilde{\Sigma}_{R'}$ and $\widetilde{\Sigma} == \widetilde{\Sigma}_1$, where $tC^{64}_{\delta(t),\gamma(t)}$ its image of $C^{64}_{\delta(t),\gamma(t)}$ under the homothety $z \mapsto tz$;
- $\delta(1), \gamma(1) < \varepsilon$, where $\varepsilon > 0$ is determined below in the Lemma 10.15.

We denote by N_t the domain in U bounded by $\widetilde{\Sigma}_t$ and will write $\widetilde{\Sigma}'$ and N' instead of $\widetilde{\Sigma}_{R'}$ and $N_{R'}$ Note that $\widetilde{\Sigma}_t \setminus U_{\gamma(t)}(D) = \Sigma_t \setminus U_{\gamma(t)}(D)$.

Step 3. Second preliminary deformation.

Lemma 10.13. *There exists a preliminary deformation ψ_t, $t \in [0, 1]$, such that*

a) There exists $d > a_0$ and $\eta > 0$ such that the level set $A = \{\psi_1 = d\} \cap U_{\frac{\sigma}{2}}(D_-)$ is contained in $N' \setminus U_\eta(D)$ and $A \cap (U \setminus U_{\frac{\sigma}{2}}(D_-))$ coincides with Σ'';

b) the angle between the gradient vector field $\nabla\psi_1$ and the vector field $-\overrightarrow{u}$ is $> \frac{\pi}{4}$ in $B \setminus \mathbb{R}^k$, where $B = \{\psi_1 \le d\} \cap U$ is the domain bounded in U by the level set A;

c) the angle between the gradient vector field $\nabla\psi_1$ and the cone spanned by the vector fields $-\overrightarrow{u}$ and $\overrightarrow{x}$ is $> \frac{\pi}{4}$ in $(B \cap U_\sigma(\partial_- D_-)) \setminus \mathbb{R}^k$;

d) the deformation is weakly supported in $N' \setminus (U_{\frac{\sigma}{8}}(\Sigma)$.

Proof. Let us first observe that the function ψ satisfies the conditions b) and c) in a stronger form:

b') $d\psi(\overrightarrow{u}) > 0$ in $U \setminus \mathbb{R}^k$;

c') $d\psi(\vec{x}) < 0$ in $U_\sigma(\partial_- D_-) \setminus \mathbb{R}^k$,

which inplies that the angle in question is $> \frac{\pi}{2}$. Let us first use Lemma 3.15 and adjust ψ via a C^1-small preliminary deformation to make it equal to $k(\rho^2 - 64R^2 - 64x_k^2)$ in $U_\varepsilon(\partial_+ D_-)$ for a sufficiently small $\varepsilon < \frac{\sigma}{2}$. This can be done preserving the above conditions b') and c'). Next, we apply Proposition 4.20 near the $(k-1)$-disk $\partial_+ D_-$ and construct a preliminary deformation ψ_t, $t \in [0,1]$, which is weakly supported in $U_\varepsilon(\partial_+ D_-)$, such that one of its level sets surrounds $\partial_+ D_-$, i.e. there exists a regular value $d_1 \in (\psi_1(q), \psi_1(p))$ such that $A_1 \setminus U_\varepsilon(\partial_+ D_-) = \Sigma_t \setminus U_\varepsilon(\partial_+ D_-)$ for $t \in (R', 1)$, where $A_1 = \{\psi_1 = d_1\}$. In addition we can ensure that the properties b') and c') still holds for ψ_1. Let us denote $B_1 := \{\psi_1 \leq d_1\}$.

Before further adjusting the function ψ we will rename, following our notational convention, the constructed function ψ_1 back to ψ. Note that $\mathcal{D} = D_- \setminus \operatorname{Int} B_1$ is the stable disc of the critical point p in $\{\psi \geq d_1\}$. The function $\psi|_{\mathcal{D}}$ has a unique non-degenerate maximum at p, and hence it is equivalent to the function $c - AR^2$ on the unit disc $D \subset \mathbb{R}^k \subset \mathbb{C}^n$ for some $A > 0$, where $c = \psi(p)$. After a posible C^∞-small adjustement we can assume that the conjugating diffeomorphism $D \to \mathcal{D}$, which we again denote by α, is real analytic, and hence extends to a biholomorphism $\widehat{\alpha} : \mathcal{O}p\, D \to \mathcal{O}p\, \mathcal{D}$. Let us denote by $\vec{x}_1$ and $\vec{u}_1$ the images of the vector fields $\vec{x}$ and $\vec{u}$ under the diffeomorphism $\widehat{\alpha}$. Note that $\vec{u}_1 = \vec{u} + o(r)$ and $\vec{x}_1 = \mu\vec{x} + \vec{\tau} + o(r)$, where $\mu \geq 0$ and $\vec{\tau}$ is tangent to the spheres S_t.

We can modify the function $\widetilde{\psi} = \psi \circ \widetilde{\alpha}$ to have the form $c - AR^2 + 64Ar^2$ in $U_\theta(D)$ for a sufficiently small $\theta > 0$. The necessary perturbation is C^1-small, and hence we can ensure that the angle between $\nabla\widetilde{\psi}$ and the cones generated by $\vec{x}_1$ and $\vec{u}_1$ is bounded below by $\frac{\pi}{4}$. Apply now Proposition 4.20 to get a deformation $\widetilde{\psi}_t$ weakly supported in $U_{\frac{\theta}{2}}(D)$ such that the level level set $\widetilde{A}_1$ of the function $\widetilde{\psi}_1$ which coincides with A_1 outside $U_{\frac{\theta}{2}}(D)$ corresponds to a regular value $\widetilde{d}_1 > \widetilde{\psi}_1(0)$ and surrounds D, i.e. $\left(\widehat{A}_1 \cap U_{\frac{\theta}{2}}(D)\right) \cap D = \varnothing$. The special function constructed in 4.20 has the property that $d\widetilde{\psi}_1(\vec{x}) < 0, d\widetilde{\psi}_1(\vec{u}) > 0$ and $d\psi(\vec{\tau}) = 0$. Hence for a sufficiently small θ we conclude that the angle between $\nabla\widetilde{\psi}$ and the cones generated by $\vec{x}_1$ and $\vec{u}_1$ is bounded below by $\frac{\pi}{4}$. Replanting the constructed deformation $\widetilde{\psi}_t$ back to $\mathcal{O}p\, \mathcal{D}$ via the biholomorphism $\widetilde{\alpha}$ we get the deformation ψ_t with the required properties. □

Step 4. Elimination along the stable disc. Let us choose $\eta > 0$ be the number defined in Lemma 10.13. We will also assume that $\eta < \frac{\sigma}{8}$. Then we have s $U_\eta(D_-) \subset B$. The following statement is the standard cancellation lemma in Morse theory (see [**?**], Lemma ???).

Lemma 10.14. *For any positive θ there exists a cancellation deformation β_t : $D \cap \{x_k < \eta\} \to \mathbb{R}$ with $\beta_0 = \beta$ which is supported in $D''' \cap \{x_k < \eta\}$.*

Let us choose $C > 0$ such that the function $\beta_t(x) + Cr^2$ is i-convex in $B_\eta(D \cap \{x_k < \eta\})$ for all $t \in [0,1]$.

Lemma 10.15. *There exists $\varepsilon < \eta$ and a preliminary deformation ψ_t which is supported in $B_\eta(D \cap \{x_k \leq \theta\})$, such that ψ_1 satisfies conditions b)-c) of Lemma 10.13 and, in addition, $\psi_1 = \beta(x) + Cr^2$ in $B_\varepsilon(D \cap \{x_k \leq \theta\})$.*

Proof. Apply Proposition 3.15 along the disc D_-. □

Step 5. Elimination of critical points.

Let Ψ be an i-convex function on the domain Ω bounded by $\widetilde{\Sigma}'$ and $\widetilde{\Sigma}$ which has $\widetilde{\Sigma}_t$, $t \in [R', 1]$, as its level sets. Let Ψ_t denotes the restriction of the function Ψ to the domain Ω_t bounded by $\widetilde{\Sigma}'$ and $\widetilde{\Sigma}_t$, $t \in [R', 1]$. Let us apply the defined above carving operation and consider the family of i-convex functions $\psi_t = \mathfrak{m}_{\Psi_t}(\psi, A)$, $t \in (R', 1]$. This family has the following properties.

Lemma 10.16. *(i) $\psi_t = \psi$ for $t > R''$;*

(ii) ψ_t is weakly supported in U;

(iii) ψ_t have no critical pointts for all $t \in [R', 1]$.

(iv) $\psi_1|_{U_{\eta(D_-)}} = \mathfrak{M}_{\widetilde{\Psi}}(\beta(x) + Cr^2, A)$, where the function $\widetilde{\Psi} = \Psi_1|_{U_{\eta(D_-)}}$ has the form $h(r), r \in [\delta(1), \eta]$.

Note that the first three properties just say that ψ_t is a preliminary deformation of the function ψ.

Proof. The property (i) follows from the fact that $B \cap \Omega_t = \varnothing$ if $t > R''$, while (ii) follows from the compactness of the intersection $\Omega \cap B$. The property (iii) follows from Proposition 3.21 taking into account properties b) and c) of Lemma 10.13. Finally (iv) is a corollary of the inequality $\gamma(t), \delta(t) < \varepsilon$ and the properties of special shapes. □

Before continuing, we again rename, following our notation convention the function ψ_1 into ψ. Let us define $\psi_t|_{U_{\eta(D_-)}} = \mathfrak{m}_{\widetilde{\Psi}}(\beta_t + Cr^2)$, where β_t, $t \in [0,1]$ is the cancellation deformation from Lemma 10.14. This deformation is weakly supported in $U_{\eta(D_-)}$, and hence can be extended to the whole U as the required cancellation deformation for the function $\psi : U \to \mathbb{R}$.

This completes the proof of Proposition 10.9. □

10.3.6 Proof of Proposition 10.8

Take a point $p \in U$ Assuming that ϕ is real analytic near p, we can choose holomorphic coordinates in $\mathcal{O}p\, p$, such that

- the point p has coordinates $(-1, 0, \dots, 0)$;
- $\nabla\phi(p) = \frac{\partial}{\partial x_1}$ and $\nabla\psi|_{U\cap\mathbb{R}^k}$ is tangent to $\mathbb{R}^k$.

Using Proposition 3.15 we can modify the function ϕ in a ball $U = U_\sigma(p)$ for a positive $\sigma < \frac{1}{2}$ to make it equal

$$\frac{1}{2}\left((x_1 + \frac{3}{2})^2 + \sum_2^k x_j^2 + Cr^2\right) + c - \frac{1}{8},$$

$c = \psi(p)$, where the constant $C > 0$ satisfies the following condition. Pick a $\theta \in (0, \frac{\sigma^2}{8})$ and choose any creation type deformation $\beta_t : D_\theta(p) \to \mathbb{R}$, $t \in [0,1]$, supported in $\operatorname{Int} D_\theta(p)$ which creates two critical points of index k and $k-1$, where $\beta_0(x) = \widehat{\phi}|_{D_\theta(p)} = \frac{1}{2}\left((x_1 + \frac{3}{2})^2 + \sum\limits_2^k x_j^2\right)$. Then we choose the constant $C > 0$ such that the function $\phi_t(x,u) = \beta_t(x) + Cr^2$ is i-convex f in the domain $G_\theta = \{(x,u) \in U; x \in D_\theta(p)\}$. Let us denote by S_t, $t \in [1-2\theta, 1+2\theta]$ the level set of the function $\phi : U = U_\sigma(p) \to \mathbb{R}$ which contains the point $p_t = (-t, 0, \dots, 0)$, and by T_t the domain $\{z \in U; \phi(z) \le \phi(p_t)\}$ bounded by S_t. Consider a family of i-convex surfaces $\Sigma_t = tC^1_{\gamma(t),\delta(t)} \subset \mathbb{C}^n$, $t \in [1-2\theta, 1+2\theta]$, where the decreasing functions $\gamma(t), \delta(t)$ are chosen in such a way that surfaces C_t form a smooth foliation of the domain Ω bounded by $\Sigma_{1-2\theta}$ and $\Sigma_{1+2\theta}$, and $\gamma(1+2\theta), \delta(1+2\theta) = \theta' < \theta$. Consider a i-convex function $\Psi : \Omega \to \mathbb{R}$ such that the hypersurfaces Σ_t serve as the level sets for Ψ. Denote by Ω_t, $t \in (1-2\theta, 1+2\theta]$ the domain bounded by $\Sigma_{1-2\theta}$ and Σ_t. Denote $\Psi_t := \Psi|_{\Omega_t}$. We have $\Omega_{1+2\theta} = \Omega$ and $\Psi_{1+2\theta} = \Psi$.

Remark 10.17. Note that in view of property b) of special shapes $C^a\gamma, \delta$ we have $\Sigma_{1+2\theta} \cap G_\theta = \{r = \theta'\} \cap G_\theta$ and the restriction $\psi_{1+2\theta}$ of the function $\Psi_{1+2\theta}$ to G_θ has cylinders $\{r = \delta(t)\} \cap G_\theta$ for $t \in [1-\theta, 1+2\theta]$ as its level sets.

Consider a family of functions $\Phi_t = \mathfrak{m}_{\Psi_t}(\phi, S_{1+\theta}), t \in (1-2\theta, 1+2\theta]$.

Lemma 10.18. *The family ϕ_t, $t \in [1-\theta, 1+2\theta]$, has the following properties.*

(i) $\phi_t = \phi$ *for* $t < 1-\theta$*;*

(ii) the deformation Φ_t *is weakly supported in* $U_{\frac{\sigma}{2}}(p)$

(iii) ϕ_t *has no critical points for all* t*.*

(iv) $\phi_1|_{G_\theta} = \mathfrak{m}_{\psi_{1+2\theta}}(\phi|_{G_\theta}, S_{1-\theta} \cap G_\theta)$.

Proof. For $t < 1-\theta$ we have $\Omega_t \cap T_{1-\theta} = \varnothing$ which implies 1). On the other hand, we have $\Omega_{1+2\theta} \cap T_{1-\theta} \subset U_{\frac{\sigma}{2}}(p)$, which implies 2). Both functions have positive derivatives along the vector field $\vec{x} - \vec{u}$, and hence Proposition 3.21 implies 3). Finally, 4) follows from Remark 10.17. $\square$

Before constructing the next and final step of the deformation we rename the function $\phi_{1+2\sigma}$ back into ϕ. According to 10.18.4) we have $\phi|_{G_\theta} = \mathfrak{m}_{\psi_{1+2\theta}}(\phi|_{G_\theta}, S_{1-\theta} \cap G_\theta)$. Let us define ϕ_t on G_θ for $t \in [0,1]$ as follows. Let $\widetilde{\phi}_t(x,u) = \beta_t(x) + Cr^2$ be the creation family considered above. Then we set

$$\phi_t := \mathfrak{m}_{\psi_{1+2\theta}}(\widetilde{\phi}|_{G_\theta}, S_{1-\theta} \cap G_\theta).$$

This deformation is weakly compactly supported in G_θ, and hence can be extended to U as equal to the function ϕ (possibly, rescaled in the target) on $U \backslash G_\theta$. It remains to notice that the functions ϕ_t are non-singular in the complement of the disc $D_\theta(p)$. Indeed, both functions, $\widetilde{\psi}_t = \beta_t + cr^2$ and $\psi_{1+2\theta}$ have a positive derivative along the vector field $\vec{u}$ on $G_\theta \cap \Omega$, and hence the claim follows from Proposition 3.21 implies 3). This completes the proof of Proposition 10.8. $\square$

Chapter 11

Proof of the existence theorems

Move this chapter?

11.1 Existence of Stein structures on cobordisms

A *cobordism* is a compact oriented manifold W with oriented boundary $\partial W = \partial_+ W \amalg \partial_- W$, where the orientation agrees with the boundary orientation for $\partial_+ W$ and is opposite to it for $\partial_- W$. We allow one or both of $\partial_\pm W$ to be emtpy. A *Morse cobordism* (W, ϕ) is a cobordism W with a Morse function $\phi : W \to \mathbb{R}$ having $\partial_\pm W = \phi^{-1}(c_\pm)$ as regular level sets.

Theorem 11.1. *Let (W, ϕ) be a Morse cobordism of dimension $2n$ with $\partial_\pm W = \phi^{-1}(c_\pm)$. Let J an integrable complex structure on W such that $\partial_- W$ is real analytic and ϕ is J-convex near $\partial_- W$. Suppose that $n > 2$ and all critical points of ϕ have index $\leq n$. Then there exist*

- *a diffeotopy $h_t : W \to W$ with $h_0 = \mathbb{1}$ and $h_t = \mathbb{1}$ near ∂W;*
- *a homotopy of convex increasing functions $g_t : \mathbb{R} \to \mathbb{R}$ with $g_0 = \mathbb{1}$ and $g_t = \mathbb{1}$ near $(-\infty, c_-]$;*
- *a homotopy of regular values c_t of $\phi_t := g_t \circ \phi \circ h_t^{-1}$ with $c_0 = c_+$;*

such that $\partial_+ W_1$ is real analytic and $\phi_1|_{W_1}$ is J-convex, where $W_t := \phi_t^{-1}\big([c_-, c_t]\big)$.

Remark 11.2. Note that the conclusions of Theorem 11.1 imply that ϕ_t has no critical points with values in $[c_t, c_+]$.

For the proof we will decompose a cobordism into elementary ones: We call a Morse cobordism (W, ϕ) *elementary* if ϕ admits a gradient-like vector field X

such that no two critical points of ϕ are connected by an X-trajectory. An *admissible partition* of a Morse cobordism (W, ϕ) with $\phi^{-1}(\partial_\pm W) = c_\pm$ is a finite sequence $c_- = c_0 < c_1 < \cdots < c_N = c_+$ of regular values of ϕ such that each subcobordism $W_k = \phi^{-1}([c_{k-1}, c_k])$, $k = 1, \dots, N$ is elementary. The following lemma is proved e.g. in [50].

Lemma 11.3. *Every Morse cobordism admits an admissible partition.*

Proposition 11.4. *Theorem 11.1 holds for an elementary Morse cobordism.*

Proof. Let (W, ϕ, J) be as in Theorem 11.1 and X a gradient-like vector field such that no two critical points of ϕ are connected by an X-trajectory.

If ϕ has no critical points set $h_t := \mathbb{1}$, $g_t := \mathbb{1}$ and let c_t be a decreasing homotopy from $c_0 = c_+$ to $c_1 > c_0$ such that ϕ is J-convex on $\phi^{-1}([c_-, c_1])$.

Otherwise consider a critical point p. Since its stable disk D_p^- meets no other critical points, it meets the J-convex hypersurface $\partial_- W$ transversely along a sphere S_p^-. Moreover, $n > 2$ and $\dim D_p^- \leq n$. Thus by Theorem 6.14 there exists a C^0-small homotopy of disks D_t transversely attached to $\partial_- W$ with $D_0 = D_p^-$ and such that D_1 is totally real and J-orthogonal to $\partial_- W$. Moreover, by Theorem 7.22 and Corollary 7.25 we may assume that D_1 is real analytic.

Let $g_t : W \to W$ be a diffeotopy with $g_0 = \mathbb{1}$, $g_t = \mathbb{1}$ near $\partial_+ W$ and $D_t = g_t(D_p^-)$. Since the D_t are transversely attached to $\partial_- W$, we can choose the g_t to preserve levels of ϕ on $\phi^{-1}([c_-, d_-])$ for some small $d_- > c_-$. Define a new diffeotopy $h_t : W \to W$ by $h_t := g_t$ on $\{\phi \geq d_-\}$ and $h_t := g_{\beta(c)t}$ on $\phi^{-1}(c)$, $c \in [c_-, d_-]$, where $\beta : [c_-, d_-] \to [0, 1]$ is a smooth increasing function with $\beta = 0$ near c_- and $\beta = 1$ near d_-. Thus $h_t = \mathbb{1}$ near $\partial_- W$ and $\psi_t := \phi \circ h_t^{-1} = \phi \circ g_t^{-1}$. It follows that the vector field $X_t := g_{t*} X$ is gradient-like for ψ_t and the stable disk of the critical point $h_t(p)$ of ψ_t with respect to X_t equals D_t. (Note, however, that $X_t \neq h_{t*} X$).

Consider a slightly smaller disk $\tilde{D}_1 := D_1 \cap \{\phi \geq \tilde{c}_-\}$ for a small $\tilde{c}_- > c_-$. Since $\tilde{D}_1$ is real analytic and totally real, it has a neighborhood biholomorphic to a standard k-handle $H_\varepsilon \subset \mathbb{C}^n$ such that $\tilde{D}_1$ corresponds to the unit disk $D \subset i\mathbb{R}^k$. In view of Proposition 3.15, we can C^1-perturb ψ_1 near ∂D to a J-convex function $\tilde{\phi}$ which corresponds to the standard function $\psi_{\rm st}$ near ∂D.

To be continued... □

Proof of Theorem 11.1. Let (W, ϕ, J) be as in Theorem 11.1. By Lemma 11.3 (W, ϕ) admits an admissible partition $c_- = c_0 < c^1 < \cdots < c^N = c_+$. We will prove by induction on i that the statement of the theorem holds for $W^i := \phi^{-1}([c^0, c^i])$. For $i = 0$ (with c^0 moved slightly above c_-) this is trivially true. Now suppose that the statement of the theorem holds for W^i. Let h_t^i, g_t^i, c_t^i be the corresponding homotopies so that $\partial_+ W_1^i$ is real analytic and $\phi_1^i|_{W_1^i}$ is J-convex, where $\phi_t^i = g_t^i \circ \phi \circ (h_t^i)^{-1}$ and $W_t^i = (\phi_t^i)^{-1}([c_-, c_t^i])$.

Since $\phi_t^i = g_t^i \circ \phi$ near $\partial_+ W^i$, we can extend ϕ_1^i to a function $\tilde\phi^{i+1} : W^{i+1} \to \mathbb{R}$ by

$$\tilde\phi^{i+1} := \begin{cases} \phi_1^i & \text{on } W^i \\ g_1^i \circ \phi & \text{on } W^{i+1} \setminus W^i. \end{cases}$$

Since ϕ_1^i has no critical points in $W^i \setminus W_1^i$, the Morse cobordism

$$(\tilde W^{i+1} := W^{i+1} \setminus W_1^i, \tilde\phi^{i+1})$$

is elementary, see Figure [fig]. Since $\partial_- \tilde W^{i+1} = \partial_+ W_1^i$ is real analytic and $\tilde\phi^{i+1}$ is J-convex near $\partial_- \tilde W^{i+1} = \partial_+ W_1^i$, we can apply Proposition 11.4 to this cobordism. Let $\tilde h_t^{i+1}, \tilde g_t^{i+1}, \tilde c_t^{i+1}$ be the corresponding homotopies so that $\partial_+ \tilde W_1^{i+1}$ is real analytic and $\tilde\phi_1^{i+1}|_{\tilde W_1^{i+1}}$ is J-convex, where $\tilde\phi_t^{i+1} = \tilde g_t^{i+1} \circ \phi \circ (\tilde h_t^{i+1})^{-1}$ and $\tilde W_t^{i+1} = (\tilde\phi_t^{i+1})^{-1}\big([c_1^i, \tilde c_t^{i+1}]\big)$. Fig!!!

We extend h_t^i and $\tilde h_t^{i+1}$ to diffeotopies of W^{i+1} via the identity on $W^{i+1} \setminus W^i$ resp. W_1^i and define homotopies on W^{i+1} by

$$h_t^{i+1} := \begin{cases} h_{2t}^i & \text{for } 0 \le t \le 1/2, \\ \tilde h_{2t-1}^{i+1} \circ h_1^i & \text{for } 1/2 \le t \le 1, \end{cases}$$

$$g_t^{i+1} := \begin{cases} g_{2t}^i & \text{for } 0 \le t \le 1/2, \\ \tilde g_{2t-1}^{i+1} \circ g_1^i & \text{for } 1/2 \le t \le 1, \end{cases}$$

$$c_t^{i+1} := \begin{cases} g_{2t}^i(c^{i+1} & \text{for } 0 \le t \le 1/2, \\ \tilde c_{2t-1}^{i+1} \circ h_1^i & \text{for } 1/2 \le t \le 1. \end{cases}$$

Note that c_t^{i+1} is continuous at $t = 1/2$ because $\tilde c_0^{i+1} = \tilde\phi^{i+1}(\partial_+ \tilde W^{i+1}) = g_1^i(c^{i+1})$. The corresponding homotopies $\phi_t^{i+1} := g_t^{i+1} \circ \phi \circ (h_t^{i+1})^{-1}$ and $W_t^{i+1} = (\phi_t^{i+1})^{-1}\big([c_-, c_t^{i+1}]\big)$ are given by

$$\phi_t^{i+1} = \begin{cases} g_{2t}^i \circ \phi \circ (h_{2t}^i)^{-1} & \text{for } 0 \le t \le 1/2, \\ \tilde g_{2t-1}^{i+1} \circ \tilde\phi^{i+1} \circ (\tilde h_{2t-1}^{i+1})^{-1} & \text{for } 1/2 \le t \le 1, \end{cases}$$

$$W_t^{i+1} = \begin{cases} W^{i+1} & \text{for } 0 \le t \le 1/2, \\ W_1^i \cup \tilde W_{2t-1}^{i+1} & \text{for } 1/2 \le t \le 1. \end{cases}$$

In particular, we see that ϕ_t^{i+1} has regular value c_t^{i+1}. Moreover, ϕ_1^{i+1} is given by

$$\phi_1^{i+1} = \begin{cases} \phi_1^i & \text{on } W_1^i, \\ \tilde\phi_1^{i+1} & \text{on } \tilde W^{i+1} \end{cases}$$

and therefore J-convex. Hence the homotopies $h_t^{i+1}, g_t^{i+1}, c_t^{i+1}$ have the desired properties and Theorem 11.1 is proved. □

The following version of Theorem 11.1 for open manifolds was first pointed out by R. Gompf [26].

Theorem 11.5. *Let (V, J) be an open complex manifold of complex dimension n and $\phi : V \to \mathbb{R}$ an exhausting Morse function. Suppose that $n > 2$ and all critical points of ϕ have index $\leq n$. Then there exist*

- *a diffeotopy $h_t : W \to W$ with $h_0 = \mathbb{1}$;*
- *a homotopy of convex increasing functions $g_t : \mathbb{R} \to \mathbb{R}$ with $g_0 = \mathbb{1}$ and $g_t = \mathbb{1}$ near $(-\infty, \min\phi]$;*
- *a homotopy of smooth embeddings $f_t : V \hookrightarrow V$ with $f_0 = \mathbb{1}$;*

such that $g_1 \circ \phi \circ h_1^{-1}$ is $f_1^* J$-convex.

Proof. **To be done.** □

To be rewritten: Proof of the existence results (abstract, in ambient manifold, with holo function as in Forstneric).

11.2 Handles in the holomorphic category

For the purposes of this section, let us slightly modify the definition of an attaching map. Let W be a manifold with boundary and extend it to a slightly larger manifold $\tilde{W}$. An *attaching map* is an embedding $F : H \supset U \hookrightarrow \tilde{W}$ such that $F(S) \subset \partial W$ and the differential dF along S maps $\partial^- H|_S$ to ∂W and the outward pointing vector field η to an inward pointing vector field η_F. Then for $\varepsilon > 0$ small let

$$W \cup_F H := W \amalg H / H \cap F^{-1}(W) \ni x \sim F(x) \in W \cap F(H)$$

Note that $F^{-1}(\partial W)$ is a graph over $\partial^- H$ near S, so $W \cup_F H$ describes indeed the attaching of a handle for ε small.

Remark 11.6. The following facts are seen as in the previous section.

(1) If J, J_H are almost complex structures on W, H and dF is complex linear along S, then $W \cup_F H$ carries a natural homotopy class of almost complex structures $J \cup_F H$ that agree with J on W and with J_H along D.

(2) An isotopy of attaching maps F_t, induces a canonical family of diffeomorphisms $\phi_t : W \cup_{F_0} H \to W \cup_{F_t} H$. Moreover, if the differentials dF_t are complex linear along S for almost complex structures J, J_t on W, H, then $\phi_t^*(J \cup_{F_t} J_t)$ is a continuous homotopy of almost complex structures on $W \cup_{F_0} H$.

(3) If J is an (integrable) complex structure on W and the attaching map $F : (U, i) \to (\tilde{W}, J)$ is holomorphic, then $W \cup_F H$ carries a natural (integrable) complex structure. We will consider below the case that ∂W is J-convex. Since $\partial^- H$ is Levi-flat for the complex structure i, the attaching map cannot map $\partial^- H$ to ∂W in that case. This explains our modified definition of "attaching map".

Lemma 11.7. *Let (W, J) be a complex manifold with real analytic J-convex boundary. Let $F_0 : H_\varepsilon \supset U_\varepsilon \hookrightarrow \tilde{W}$ be an attaching map such that $dF_0 : (TH|_S, i) \to (TW, J)$ is complex linear. Then (after shrinking ε) there exists a family of attaching maps $F_t : U_\varepsilon \hookrightarrow \tilde{W}$, $t \in [0, 1]$, C^∞-close to F_0, such that F_1 is holomorphic and*

$$dF_t : (TH|_S, i) \to (TW, J)$$

is complex linear for all $t \in [0, 1]$.

Proof. Without further mention, we will shrink ε whenever necessary. Moreover, all homotopies will be chosen C^∞-close to the original data.

As J integrable and ∂W is real analytic and J-convex, the maximal tangency ξ on ∂W is a real analytic contact structure. Set $P_\varepsilon := \partial D_1^k \times D_\varepsilon^{n-k}$ and consider the Legendrian embdding $g_0 := (F_0)|_{P_\varepsilon} : P_\varepsilon \hookrightarrow \partial W$. By Corollary 7.24 and the remark following it, there exists a Legendrian isotopy $g_t : P_\varepsilon \hookrightarrow \partial W$ such that g_1 is real analytic.

By hypothesis, dF_0 maps the vector field v along S to a vector field v_0 on ∂W transverse to ξ. By Theorem 7.22, there exists a family v_t of transverse vector fields on ∂W such that v_1 is real analytic. Set $\eta_t := Jv_t$. Again by Theorem 7.22, we can extend g_1 to a real analytic embedding $f_1 : (V_\varepsilon := D_{1+\varepsilon}^k \setminus \mathrm{int} D_1^k) \times D_\varepsilon^{n-k} \to \tilde{W}$ with $df_1 \cdot \eta = \eta_1$. Connect $f_0 := (F_0)|_{V_\varepsilon}$ to f_1 by a smooth isotopy of totally real embeddings $f_t : V_\varepsilon \to \tilde{W}$ with $df_t \cdot \eta = \eta_t$.

Complexify the (totally real) differentials df_t along S to complex linear isomorphisms $d^{\mathbb{C}} f_t : (TH|_S, i) \to (TW|_{f_t(S)}, J)$. Complexify the totally real embedding $f_1 : V_\varepsilon = U_\varepsilon \cap \mathbb{R}^n \hookrightarrow \tilde{W}$ to a holomorphic embedding $F_1 : U_\varepsilon \hookrightarrow \tilde{W}$. Note that $dF_0 = d^{\mathbb{C}} f_0$ and $dF_1 = d^{\mathbb{C}} f_1$ along S. Connect F_0 to F_1 by an isotopy of smooth embeddings $F_t : U_\varepsilon \hookrightarrow \tilde{W}$ with $dF_t = d^{\mathbb{C}} f_t$ along S. By construction, $dF_t = d^{\mathbb{C}} f_t$ maps $T(\partial^- H)|_S$ to $T(\partial W)$ and η to the inward pointing vector field η_t. Thus the F_t are attaching maps with $dF_t : (TH|_S, i) \to (TW, J)$ complex linear for all $t \in [0, 1]$. □

Proposition 11.8. *Let (W, J) be a compact almost complex manifold of complex dimension $n > 2$ with boundary $\partial W = \partial^- W \cup \partial^+ W$ (we allow $\partial^- W = \emptyset$). Suppose W carries a function which is constant on the boundary components and has a unique critical point of index $k \le n$. Suppose that near $\partial^- W$, J is integrable and $\partial^- W$ is J-convex.*

Then there exists an integrable complex structure $\tilde{J}$ on W such that $\tilde{J} = J$ near $\partial^- W$ and $\tilde{J} \sim J$ rel $\partial^- W$.

Proof. Let $W' \subset W$ be a collar neighborhood of $\partial^- W = \partial^- W'$ with real analytic J-conxev boundary $\partial^+ W'$. By Morse theory [50], there exists an embedding $f : H \hookrightarrow W$ of a k-handle, with attaching map $f_0 := f|_U : U \hookrightarrow W'$, such that W smoothly deformation retracts onto a smoothing of $W' \cup f(H)$. Let $J_0 := f^* J$ on H. By Proposition [prop:ac-attaching???] and Lemma 11.7, Reference?
there exists a family of almost complex structures J_t on H and an isotopy

of attaching maps $f_t : U \hookrightarrow \tilde{W}'$ such that $J_1 = i$, f_1 is holomorphic, and $df_t : (TH|_S, J_t) \to (TW|_{f_t(S)}, J)$ is a complex isomorphism for all t. By Lemma [lem:ac-homotopy???], this gives rise to a homotopy of almost complex structures J'_t on $W' \cup f(H)$, fixed near $\partial^- W$, such that $J'_0 = J$ and $\tilde{J}_1 =: J'$ is integrable.

Reference?

It only remains to extend J' to all of W. For this, let $\tilde{W} \subset W' \cup f(H)$ be a tubular neighborhood of $W' \cup f(D)$. Let $g_t : \tilde{W} \hookrightarrow W$ be an isotopy of embeddings such $g_t = 1\!\!1$ near $W' \cup f(D)$, g_0 is the inclusion, and g_1 is a diffeomorphism. Now $\tilde{J} := g_{1*}J'$ is an integrable complex structure on W which coincides with J' on W'. Moreover, $\tilde{J}_t := g_{1*}g_t^* J'$ provides a homotopy rel W' from $\tilde{J}_0 = \tilde{J}$ to $\tilde{J}_1 = J'$. Since J' was homotopic rel $\partial^- W$ to J, this concludes the proof. □

11.3 Extension of Stein structures over handles

Theorem 11.9. *Let (W, J) be a compact almost complex manifold of complex dimension $n > 2$ with boundary $\partial W = \partial^- W \cup \partial^+ W$ (we allow $\partial^- W = \emptyset$). Let $\phi : W \to [a, b]$ be a function with $\partial^- W = \phi^{-1}(a)$, $\partial^+ W = \phi^{-1}(b)$ and a unique critical point in W of index $k \le n$. Suppose that near $\partial^- W$, J is integrable and ϕ is J-convex.*

Then there exists an integrable complex structure $\tilde{J}$ on W such that $\tilde{J} = J$ near $\partial^- W$, $\tilde{J} \sim J$ rel $\partial^- W$, and ϕ is $\tilde{J}$-convex.

Proof. Let $\partial^- W \times [0, 1]$ be a collar neighborhood of $\partial^- W = \partial^- W \times \{0\}$ on which J is integrable and ϕ is J-convex with level sets $\partial^- W \times \{t\}$. Let ϕ' be C^2-close to ϕ, real analytic near $\partial^- W \times \{1/2\}$, with $\phi' = \phi$ outside $\partial^- W \times [1/4, 3/4]$. Then ϕ' is J-convex and $\phi' = f^*\phi$ for a diffeomorphism f isotopic to the identity rel $W \setminus \partial^- W \times [1/4, 3/4]$. Thus it suffices to prove the theorem for ϕ' and $J' := f^* J$. Denoting ϕ', J' again by ϕ, J, we may hence assume that ϕ is real analytic near a level set $\phi^{-1}(a')$, $a' > a$, and J-convex on $W' := \phi^{-1}([a, a'])$.

By Proposition 11.8, J is homotopic rel $\partial^- W$ to an integrable complex structure J'. Perturb the gradient vector field $\nabla_{g_\phi}\phi$, fixed near $\partial^+ W$, to a C^1-close vector field X. Then X is gradient-like for ϕ and has a nondegenerate zero at the critical point p of ϕ. Let $\Delta \subset W \setminus \mathrm{int} W'$ be the stable disk of p for X. Then Δ is totally real and real analytic. Moreover, since $X = \nabla_{g_\phi}\phi$ near $\partial^+ W'$, Δ is attached J'-orthogonally to $\partial^+ W'$ along $\partial\Delta$.

By Theorem 9.7, there exists a surjective J'-convex function $\psi : \tilde{W} \to [a, b]$ on a neighborhood $\tilde{W}$ of $W' \cup \Delta$ with $\psi = \phi$ on W' and a unique index k critical point at p. Moreover, there exists an isotopy $h_t : \Delta \to \Delta$, fixed near $\partial\Delta$ and p, with $h_0 = 1\!\!1$ and $h_1^*\phi = \psi$. Now we argue as in the proof of Proposition 11.8. Extend h_t to an isotopy of embeddings $\tilde{h}_t : \tilde{W} \hookrightarrow W$ such that $\tilde{h}_t|_{W'} = 1\!\!1$, $\tilde{h}_t|_\Delta = h_t$, $\tilde{h}_0$ is the inclusion, and $\tilde{h}_1$ is a diffeomorphism. Then the Morse functions $\tilde{h}_1^*\phi, \psi : \tilde{W} \to [a, b]$ coincide on $W' \cup \Delta$. By Lemma 13.8, there exists a diffeotopy $g_t : \tilde{W} \to \tilde{W}$, fixed on $W' \cup \Delta$, with $g_1^*\tilde{h}_1^*\phi = \psi$. Hence the embeddings

$f_t := \tilde{h}_t \circ g_t$ satisfy: f_0 is the inclusion, $f_t|_{W'} = \mathbb{1}$, and f_1 is a diffeomorphism with $f_{1*}\psi = \phi$. Now $\tilde{J} := f_{1*}J'$ is an integrable complex structure on W, which coincides with J' on W', such that ϕ is $\tilde{J}$-convex. Moreover, $J_t := f_{1*}f_t^*J'$ provides a homotopy rel W' from $J_0 = \tilde{J}$ to $J_1 = J'$. Since J' was homotopic rel $\partial^- W$ to J, this concludes the proof of Theorem 11.9 □

Now we are ready to prove the existence theorem for Stein structures stated in the introduction.

Theorem 11.10 (Eliashberg [14]). *Let V^{2n} be an open smooth manifold of dimension $2n > 4$ with an almost complex structure J and an exhausting Morse function ϕ without critical points of index $> n$. Then V admits a Stein structure. More precisely, J is homotopic through almost complex structures to an integrable complex structure $\tilde{J}$ such that ϕ is $\tilde{J}$-convex.*

Proof. Let $c_1 < c_2 < \dots$ be the critical levels of ϕ (possibly infinitely many). For simplicity, suppose that each critical level c_k carries a single critical point p_k; the obvious modifications for several critical points on one level are left to the reader. Let d_k be regular levels with

$$c_1 < d_1 < c_2 < d_2 < \dots$$

and set $V_k := \{\phi \le d_k\}$. We will inductively construct almost complex structures J_k, $k \in \mathbb{N}$, and homotopies J_k^t, $t \in [0,1]$, on V with the following properties:

- $J_k|_{V_k}$ is integrable and $\phi|_{V_k}$ is J_k-convex;
- $J_k^0 = J_{k-1}$, $J_k^1 = J_k$, and $J_k^t|_{V_{k-1}} = J_{k-1}$ for all $t \in [0,1]$.

Here we have set $J_0 := J$ and $V_0 := \emptyset$. The case $k = 1$ follows directly from Theorem 11.9 with $\partial^- W = \emptyset$. For the induction step, supppose that J_{k-1} and J_{k-1}^t have already been constructed. After replacing d_{k-1} by a slightly higher level in the preceding step, we may assume that J_{k-1} is integrable on a neighborhood of V_{k-1}. Applying Theorem 11.9 to $W := V_k \setminus \mathrm{int} V_{k-1}$ and the almost complex structure J_{k-1}, we find a homotopy of almost complex structures $\tilde{J}_k^t$ on V_k such that $\tilde{J}_k^t|_{V_{k-1}} = J_{k-1}$ for all t, $\tilde{J}_k^0 = J_{k-1}$, $\tilde{J}_k = \tilde{J}_k^1$ is integrable, and $\phi|_{V_k}$ is $\tilde{J}_k$-convex. Let $\partial V_k \times [0,1]$ be a collar neighborhood of $\partial V_k \cong \partial V_k \times \{0\}$ in $V \setminus \mathrm{int} V_k$ and extend $\tilde{J}_k^t$ to V by

$$J_k^t := \begin{cases} \tilde{J}_k^t & \text{on } V_k, \\ \tilde{J}_k^{t(1-s)} & \text{on } \partial V_k \times \{s\}, \\ J_{k-1} & \text{on } V \setminus (V_k \cup \partial V_k \times [0,1]). \end{cases}$$

This proves the induction step.

Now let sequences J_k, J_k^t as above be given. Since $J_k|_{V_{k-1}} = J_{k-1}$, the J_k fit together to an integrable complex structure $\tilde{J}$ on V with $\tilde{J}|_{V_k} = J_k$, and it follows that ϕ is $\tilde{J}$-convex. Define a homotopy of almost complex structures J^t, $t \in [0,1]$, on V as the concatenation of the homotopies J^t_k, $k \in \mathbb{N}$, carried out over the successively shorter time intervals $[1-2^{1-k}, 1-2^{-k}]$. Continuity of J^t for $t < 1$ follows from $J^1_{k-1} = J^0_k$. Continuity at $t = 1$ holds because $J^t|_{V_k} = J_k$ for all $t \geq 1-2^{-k}$, so near every point J^t becomes independent of t for t close to 1. In particular, we have $J^1 = \tilde{J}$ and $J^0 = J^0_1 = J_0 = J$. This concludes the proof of Theorem 11.10. □

Proposition 11.11. *Let J_0 and J_1 are two Stein cobordism structures on a manifold W with boundary $\partial W = \partial_- W \cup W_+$, where the complex structures J_0 and J_1 share the same concave and convex parts of the boundary, $\partial_- W$ and $\partial_+ W$. Suppose that there exist J_k-convex functions ϕ_k, $k = 0, 1$, such that the following conditions are satisfied:*

- *(i) $\phi_0 = \phi_1 = \phi$ on $\mathcal{O}p\,(\partial_- W)$;*
- *(ii) the functions ϕ_0 and ϕ_1 have a unique and common critical point p, and the stable manifolds of p for the gradient vector fields $X_0 = X_{\phi_0, J_0}$ and $X_1 = X_{\phi_1, J_1}$ coincide;*
- *(iii) J_0 and J_1 coincide on $\mathcal{O}p\,(\partial_- W \cup D)$, where D is a common stable manifold for the gradient vector fields X_0 and X_1.*
- *(iv) the functions ϕ_0 and ϕ_1 are equivalent: there exist a diffeomorphism and $g : W \to W$ which is fixed on $\mathcal{O}p\,(\partial_- W)$ and leaves D invariant, and a diffeomorophism $h : \mathbb{R} \to \mathbb{R}$ such that $\phi_1 = h \circ \phi_0 \circ g$.*

Then there exists a homotopy of Stein cobordism structures J_t on W and a family of J_t-convex functions $\phi_t : W \to \mathbb{R}$ such that

- *$J_t = J_0$ on $\mathcal{O}p\,(\partial_- W \cup D)$, $t \in [0,1]$;*
- *the stable disk of p for gradient fields $X_t = X_{J_t, \phi_t}$ coincides with D for all $t \in [0,1]$.*
- *$\phi_t = h_t \circ \phi_0 \circ g_t$ for isotopies $g_t : W \to W$ fixed on W' and $h_t : \mathbb{R} \to \mathbb{R}$ as in item (iv) above.*

Proof. TO BE ADDED □

Part III

From Stein to Weinstein and back

Chapter 12

Weinstein structures

12.1 Convex symplectic manifolds

We review in this section some notions introduced in [16].

Let (V, ω) be an exact symplectic manifold of dimension n. A primitive λ such that $d\lambda = \omega$ is called a *Liouville form* on V. The vector field X that is ω-dual to λ, i.e. such that $i_X\omega = \lambda$, is called *Liouville vector field*. Note that the equation $d\lambda = \omega$ is equivalent to $L_X\omega = \omega$. If X integrates to a flow $X^t : V \to V$ then $(X^t)^*\omega = e^t\omega$, i.e. the Liouville field X is (symplectically) *expanding*, while $-X$ is *contracting*. By a *Liouville manifold* we will mean a triple (V, ω, X) where X is an expanding vector field for ω. Note that

$$i_X\lambda = 0, \qquad i_X d\lambda = \lambda, \qquad L_X\lambda = \lambda, \tag{12.1}$$

so the flow of X also expands the Liouville form, $(X^t)^*\lambda = e^t\lambda$. A map $\psi : (V_0, \omega_0, X_0) \to (V_1, \omega_1, X_1)$ between Liouville manifolds with Liouville forms λ_i is called *exact symplectic* if $\psi^*\lambda_1 - \lambda_0$ is exact.

A Liouville manifold (V, ω, X) is called (symplectically) *convex* if the expanding vector field X is *complete* and there exists an exhaustion $V = \bigcup_{k=1}^{\infty} V^k$ by compact domains $V^k \subset V$ with smooth boundaries along which X is outward pointing (so the V^k are invariant under the contracting flow $X^{-t}, t > 0$).[1] The set

$$\mathrm{Core(V,\omega,X)} := \bigcup_{\mathrm{k=1}}^{\infty} \bigcap_{\mathrm{t>0}} \mathrm{X^{-t}(V^k)}$$

is independent of the choice of the exhausting sequence of compact sets V_k and is called the *core* of the convex Liouville manifold (V, ω, X). We have

[1] This notion of symplectic convexity is slightly more restrictive than one given in [16]. However, the authors do not know any examples of symplectic manifolds that are convex in one sense but not the other.

Lemma 12.1. $\operatorname{Int}\operatorname{Core}(\mathrm{V},\omega,\mathrm{X})=\varnothing$.

Proof. For each compact set V^k we have

$$\mathrm{Volume}\big(\mathrm{X}^{-\mathrm{t}}(\mathrm{V}^{\mathrm{k}})\big)=\mathrm{e}^{-\mathrm{t}}\frac{1}{\mathrm{n}!}\int\limits_{\mathrm{V}^{\mathrm{k}}}\omega^{\mathrm{n}}\underset{\mathrm{t}\to\infty}{\longrightarrow}0\,,$$

and hence $\mathrm{Volume}\big(\bigcap_{\mathrm{t}>0}\mathrm{X}^{-\mathrm{t}}(\mathrm{V}^{\mathrm{k}})\big)=0$ for all $k\in\mathbb{N}$. □

We say that a convex Liouville manifold (V,ω,X) has *cylindrical end* if X has no zeros outside a compact set. In this case, let $\Omega\subset V$ be a compact domain with smooth boundary $\Sigma=\partial\Omega$ along which X is outward pointing and such that X has no zeros outside Ω (e.g. $\Omega=V^k$ for large k). Then $V\setminus\operatorname{Int}\Omega$, splits as $\Sigma\times[0,\infty)$ and the Liouville form $\lambda=i_X\omega$ can be written as $e^t(\alpha)$, where $t\in\mathbb{R}$ is the parameter of the flow and $\alpha:=\lambda|_\Sigma$. The form α is contact, and thus $(V\setminus\operatorname{Int}\Omega,\omega)$ can be identified with the positive half of the symplectization of the contact manifold $(\Sigma,\xi=\ker\alpha)$. In fact, the whole symplectization of (Σ,ξ) sits in V as $\bigcup_{\mathrm{t}\in\mathbb{R}}X^t(\Sigma)$ and this embedding is canonical in the sense that the image is independent of the choice of Σ: Its complement $V\setminus\bigcup_{\mathrm{t}\in\mathbb{R}}X^t(\Sigma)$ equals the core $\mathrm{Core}(\mathrm{V},\omega,\mathrm{X})$ defined above. The Liouville manifold (V,ω,X) defines the contact manifold (Σ,ξ) canonically. We will write $(\Sigma,\xi)=\partial(V,X)$ and call it the *ideal contact boundary* of the Liouville manifold (X,ω) with cylindrical end.

We do not know whether the ideal boundary depends on the choice of the Liouville field X which satisfies the cylindrical end property. The answer depends on the following open problem:

Equivalent?

Problem 12.1. Does symplectomorphism of symplectizations imply contactomorphism of contact manifolds?

Note that all known invariants of contact manifolds (e.g contact homology and other SFT-invariants) depend only on their symplectizations (since symplectomorphism of the symplectizations yields symplectic cobordisms both ways whose composition is homotopic to product cobordism). and hence cannot distinguish contact manifolds with the same symplectization.

Is this correct? Don't we need exactness?

Contact manifolds which arise as ideal boundaries of Liouville symplectic manifolds with cylindrical end are called *strongly symplectically fillable.*

Reference? Not used so far.

12.2 Deformations of convex sympletic structures

A homotopy (V,ω_s,X_s), $s\in[0,1]$, of convex Liouville manifolds is called an *elementary homotopy of compact type* if there exists a smooth family of exhaustions $V=\bigcup_{k=1}^\infty V_s^k$ by compact domains $V_s^k\subset V$ with smooth boundaries along which X_s is outward pointing. A homotopy (V,ω_s,X_s), $s\in[0,1]$, is called of *compact type* if it is a composition of finitely many elementary homotopies of compact type.

Proposition 12.2. *Let (V, ω_s, X_s), $s \in [0,1]$, be a compact type homotopy of convex Liouville manifolds with Liouville forms λ_s. Then there exists a diffeotopy $h_s : V \to V$ such that $\lambda_0 - h_s^*\lambda_s$ is exact for all $s \in [0,1]$.*

Proof. It suffices to consider the case of an elementary homotopy (V, ω_s, X_s). Denote by Σ_s^k the boundary ∂V_s^k, by λ_s the Liouville form dual to X_s, and by ξ_s^k the contact structure induced on Σ_s^k by the contact form $\lambda_s|_{\Sigma_s^k}$, $s \in [0,1], k \in \mathbb{N}$. By Gray's Stability Theorem 5.24 there are families of contactomorphisms

$$\psi_s^k : (\Sigma_0^k, \xi_0^k) \to (\Sigma_s^k, \xi_s^k),$$

so that $(\psi_s^k)^*\lambda_s = e^{f_s^k}\lambda_0$ for a smooth family of functions $f_s^k : \Sigma_0^k \to \mathbb{R}$. (We denote the restriction of λ_s to the various hypersurfaces by the same letter). For $c \in \mathbb{R}$ set $\Sigma_s^{k,c} := X_s^c(\Sigma_s^k)$ and define the diffeomorphisms

$$\psi_s^{k,c} := X_s^c \circ \psi_s^k \circ X_0^{-c} : \Sigma_0^{k,c} \to \Sigma_s^{k,c}.$$

By equation (12.1) we have $(\psi_s^{k,c})^*\lambda_s = e^{f_s^k \circ X_0^{-c}}\lambda_0$. For a sequence of real numbers d_k (which will be determined later) set

$$\widetilde{\Sigma}_s^k := \Sigma_s^{k,d_k}, \quad \widetilde{\psi}_s^k := \psi_s^{k,d_k}, \quad \widetilde{V}_s^k := X_s^{d_k}(V_s^k), \quad \widetilde{f}_s^k := f_s^k \circ X_0^{-d_k} \circ (\widetilde{\psi}_s^k)^{-1}.$$

A short computation using equation (12.1) shows that the map $\Psi_s^k := X_s^{-\widetilde{f}_s^k} \circ \widetilde{\psi}_s^k : \widetilde{\Sigma}_0^k \to V$ satisfies $(\Psi_s^k)^*\lambda_s = \lambda_0$ and hence canonically extends to a map, still denoted by $\Psi_s^k : \mathcal{O}p\, \widetilde{\Sigma}_0^k \to \mathcal{O}p\, (X_s^{-\widetilde{f}_s^k}\widetilde{\Sigma}_s^k)$, which maps trajectories of X_0 to trajectories of X_s and satisfies $(\Psi_s^k)^*\lambda_s = \lambda_0$.

Now we choose the constants d_k such that for each $s \in [0,1]$ the hypersurfaces $\widetilde{\Sigma}_s^k$, $k \in \mathbb{N}$, are mutually disjoint and the hypersurfaces $X_s^{-\widetilde{f}_s^k}(\widetilde{\Sigma}_s^k)$, $k \in \mathbb{N}$, are mutually disjoint. We achieve the first condition by choosing the d_k nondecreasing. The second condition holds if we have

$$\min_x\big(d_k - \widetilde{f}_s^k(x)\big) \geq \max_x\big(d_{k-1} - \widetilde{f}_s^{k-1}(x)\big)$$

for all $s \in [0,1]$ and $k \geq 2$. So we can achieve both conditions by defining the d_k inductively by $d_1 := 0$ and

$$d_k := d_{k-1} + \max\big\{0, \max_{s,x} f_s^k(x) - \min_{s,x} f_s^{k-1}(x)\big\}.$$

These conditions ensure that the Ψ_s^k induce a diffeomorphism

$$\Psi_s : \mathcal{O}p\left(\bigcup_{k=1}^{\infty} \widetilde{\Sigma}_0^k\right) \to \mathcal{O}p\left(\bigcup_{k=1}^{\infty} X_s^{-\widetilde{f}_s^k}\widetilde{\Sigma}_0^k\right)$$

satisfying $\Psi_s^*\lambda_s = \lambda_0$. Let us extend Ψ_s anyhow to a diffeomorphism $\Psi_s : V \to V$. Now we apply Corollary 5.8 to each of the open domains $\operatorname{Int} \widetilde{V}_0^{k+1} \setminus \widetilde{V}_0^k$

and the family of exact symplectic forms $\Psi_s^*\omega_s = d(\Psi_s^*\lambda_s$ whose primitives are s-independent near the boundary $\widetilde{\Sigma}_0^{k+1} \cup \widetilde{\Sigma}_0^k$. This yields a family of diffeomorphisms $\phi_s : V \to V$ which are the identity on $\mathcal{O}p\left(\bigcup_{k=1}^\infty \widetilde{\Sigma}_0^k\right)$ and such that the composition $h_s := \Psi_s \circ \phi_s$ is the required exact symplectomorphism $(V, \omega_0, X_0) \to (V, \omega_s, X_s)$. □

In particular, Proposition 12.2 implies

Corollary 12.3. *A family (V, ω_s, X_s) of Liouville manifolds with cylindrical ends consists of exactly symplectomorphic manifolds if the closure* $\overline{\bigcup_{s\in[0,1]} \mathrm{Core(V, \omega_s, X_s)}}$ *of the union of their cores is compact.*

12.3 Weinstein manifolds

A *Weinstein manifold* (V, ω, X, ϕ) is a symplectic manifold (V, ω) with a complete Liouville field X which is gradient-like for an exhausting Morse function $\phi : V \to \mathbb{R}$. The triple (ω, X, ϕ) is called a *Weinstein structure* on V.

Remark 12.4. (1) Any Weinstein manifold (V, ω, X, ϕ) induces a convex Liouville manifold (V, ω, X). However, not every convex Liouville manifold arises from a Weinstein manifold, see [46, 23].

(2) Later on, in deformations of Weinstein structures we will also allow ϕ and X to have death-birth (or cusp) singularities; in this section we restrict ourselves to the Morse case.

A *Weinstein domain* (W, ω, X, ϕ) is a compact symplectic manifold (W, ω) with boundary ∂W with a Liouville vector field X which is outward pointing along the boundary and gradient-like for a Morse function $\phi : W \to \mathbb{R}$ which is constant on the boundary. Thus any Weinstein manifold (V, ω, X, ϕ) can be exhausted by Weinstein domains $W_k = \{\phi \le d_k\}$, where $d_k \nearrow \infty$ is a sequence of regular values of the function ϕ.

A Weinstein manifold (V, ω, X, ϕ) is said to be of *finite type* if X has only finitely many critical points. Note that by attaching a cylindrical end any Weinstein domain (W, ω, X, ϕ) can be completed to a finite type Weinstein manifold, called its *completion* and denoted by $\mathrm{Compl}(W, \omega, X, \phi)$. Conversely, any finite type Weinstein manifold can be obtained by attaching a cylindrical end to a Weinstein domain. The contact manifolds which appear as ideal boundaries of finite type Weinstein manifolds, or equivalently as boundaries of Weinstein domains, are called *Weinstein fillable*. In view of Theorem [???], this is equivalent to being *Stein fillable*. Reference?

An important example of a Weinstein structure is provided by the cotangent bundle $V = T^*Q$ of a closed manifold Q with the standard symplectic form $\omega = d\lambda$, $\lambda = pdq$. To define a Weinstein structure, take any Riemannian metric on Q and a Morse function $f : Q \to \mathbb{R}$. Note that the Hamiltonian vector field X_F of the function $F(q,p) := \langle p, \nabla f(q)\rangle$ (or in a more invariant notation $F = \lambda(\nabla f)$) coincides with ∇f along the zero-section of T^*Q. Thus the vector field $X := p\frac{\partial}{\partial p} + X_F$ is Liouville and gradient-like for the Morse function $\phi(q,p) := \frac{1}{2}|p|^2 + f(q)$ if f is small enough.

Exercise 12.5. Find explicitly a Weinstein structure on T^*Q if Q is not compact and describe its ideal contact boundary.

The product of two Weinstein manifolds $(V_1, \omega_1, X_1, \phi_1)$ and $(V_2, \omega_2, X_2, \phi_2)$ has a canonical Weinstein structure $(V_1 \times V_2, \omega_1 \oplus \omega_2, X_1 \oplus X_2, \phi_1 \oplus \phi_2)$. In particular, the product

$$(V, \omega, X, \phi) \times \left(\mathbb{R}^{2k}, \sum dx_i \wedge dx_i, \frac{1}{2}\sum\left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right), \sum\left(x_i^2 + y_i^2\right)\right)$$

is called the *k-stabilization* of the Weinstein manifold (V, ω, X, ϕ).

Recall from Section 5.1 that a subspace W of a symplectic vector space (V, ω) (and similarly for manifolds) is called isotropic resp. coisotropic if $W \subset W^\omega$ resp. $W^\omega \subset W$, where W^ω denotes the ω-orthogonal complement.

Proposition 12.6. *Let (V, ω) be a symplectic manifold with an expanding vector field X, and let p be a hyperbolic zero of X. Then*

(a) the stable manifold $W^-(p)$ is isotropic, and

(b) the unstable manifold $W^+(p)$ is coisotropic.

Proof. Let $\phi_t : V \to V$ be the flow of X. Abbreviate $W^+ := W^+(p)$ and $W^- := W^-(p)$, so $T_pV = T_pW^+ \oplus T_pW^-$. All eigenvalues of the linearization of X at p have negative real part on T_pW^- and positive real part on T_pW^+. It follows that the differential $T_p\phi_t : T_xV \to T_{\phi_t(x)}V$ satisfies

$$\lim_{t\to\infty} T_x\phi_t(v) = 0 \text{ for } x \in W^-, v \in T_xW^-,$$
$$\lim_{t\to-\infty} T_x\phi_t(v) = 0 \text{ for } x \in W^+, v \in T_pW^+.$$

(a) Let $x \in W^-$ and $v, w \in T_xW^-$. Since $\phi_t(x) \to p$ as $t \to \infty$, the preceding discussion shows

$$e^t\omega(v,w) = (\phi_t^*\omega)(v,w) = \omega_{\phi_t(x)}(T_x\phi_t \cdot v, T_x\phi_t \cdot w) \to 0$$

as $t \to \infty$. This implies $\omega(v,w) = 0$.

(b) Let $x \in W^+$ and $v \in (T_xW^+)^\omega \subset T_xV$. Suppose $v \notin T_xW^+$. Take a sequence $t_k \to -\infty$ and let $x_k := \phi_{t_k}(x)$. Pick $\lambda_k > 0$ such that $v_k := \lambda_k T_x\phi_{t_k} \cdot v$ has norm 1 with respect to some metric on V. Note that $v_k \in (T_{x_k}W^+)^\omega$ for all k. Pass to a subsequence so that $v_k \to v_\infty \in T_pV$. Since $T_x\phi_{t_k}$ contracts the component of v tangent to W^+ and expands the transverse component, we

Reference: lambda-lemma?

find $0 \neq v_\infty \in T_pW^-$.

We claim that $v_\infty \in (T_pW^+)^\omega$. Otherwise, there would exist a $w_\infty \in T_pW^+$ with $\omega(v_\infty, w_\infty) \neq 0$. But then $\omega(v_k, w_k) \neq 0$ for k large and some $w_k \in T_{x_k}W^+$, contradicting $v_k \in (T_{x_k}W^+)^\omega$. Hence v_∞ is ω-orthogonal to T_pW^+. Since T_pW^- is isotropic by part (a), v_∞ is also ω-orthogonal to T_pW^-. But this contradicts the nondegeneracy of ω because $T_pV = T_pW^+ \oplus T_pW^-$. □

Corollary 12.7. *Let (V, ω) be a symplectic manifold of dimension $2n$ with an expanding vector field X, and let p be a hyperbolic zero of X. Then the stable manifold satisfies $\dim W^-(p) \leq n$.*

Remark 12.8. In view of Lemma 8.8, Proposition 12.6 and Corollary 12.7 apply in particular to a zero p of the expanding vector field X in a Weinstein manifold (V, ω, X, ϕ). Thus its core, which is the union of all stable manifolds, consists of isotropic manifolds. Under suitable technical assumptions (X Morse-Smale and (X, ϕ) standard near critical points), the core is in fact an isotropic embedded CW complex, see [6].

Note that in a Weinstein manifold (V, ω, X, ϕ) any regular level set $\Sigma_c := \phi^{-1}(c)$ carries a canonical contact structure ξ_c defined by the contact form $\alpha_c := (i_X\omega)|_{\Sigma_c}$.

Lemma 12.9. *Let (V, ω, X, ϕ) be a Weinstein manifold.*

(a) If c is a regular value of ϕ then for any critical point $p \in V$ with $\phi(p) > c$ the intersection $W^-(p) \cap \Sigma_c$ is isotropic in the contact sense, i.e. it is tangent to ξ_c.

(b) Suppose ϕ has no critical values in $[a, b]$. Let $\Lambda^a \subset \Sigma_a = \phi^{-1}(a)$ be an isotropic submanifold. Then the image of Λ^a under the flow of X intersects Σ_b in an isotropic submanifold Λ^b.

Proof. (a) Since X is tangent to $W^-(p)$ and $W^-(p)$ is isotropic by Proposition 12.6 and Lemma 8.8, the Liouville form $\lambda = i_X\omega$ satisfies $\lambda|_{W^-(p)} = (i_X\omega)|_{W^-(p)} = 0$.

(b) Let $f > 0$ be the function such that $L_{fX}\phi \equiv 1$ on $\phi^{-1}([a, b])$. Denote by ψ_t the flow of fX, thus $\Lambda^b = \psi_{b-a}(\Lambda^a)$. By equation (12.1) the 1-form $\lambda = i_X\omega$ satisfies $L_X\lambda = \lambda$, hence $L_{fX}\lambda = f\lambda$, so the flow ψ_t only rescales λ and the lemma follows. □

Lemma 12.9 shows that every Weinstein structure on V provides a handlebody decomposition of V where cells are attached along isotropic (in the contact senss) spheres. The core disks of the handles are isotropic in the symplectic sense. We will discuss this handlebody decomposition picture with more details
New lemma in Chapter 13 below.

Lemma 12.10. *Let $\mathfrak{W} = (W, \omega, X, \phi)$ be a Weinstein cobordism structure such that ϕ has no critical points. Denote by ξ_t, $t \in [m = \min\phi, M = \max\phi]$, the* *induced contact structure on the level set* $V_t = \{\phi = t\}$. *Let* $g : ((V_m, \xi_m) \to (V_M \xi_M)$ *be the holonomy contactomorphism along* X. *and* $h_s : (V_m, \xi_m) \to (V_m, \xi_m)$, $s \in [0,1]$, $h_0 = \mathrm{Id}$, *be any contact diffeotopy. Then there exists a family of Weinstein structures* $\mathfrak{W}_s = (W, \omega_s, X_s, \phi)$,$\mathfrak{W}_0 = \mathfrak{W}$, *such that* $\mathfrak{W}_s$ *coincides with* $\mathfrak{W}$ *near* ∂W, *and the holonomy map* $g_s : V_m \to V_M$ *along* X_s *is equal to* $g \circ h_s$, $s \in [0,1]$.

12.4 Weinstein structure of a Stein manifold

Proposition 12.11. *[(see EliGro91] Let* (V, J) *be a Stein manifold and* $\phi : V \to \mathbb{R}$ *a completely exhausting (see Section 2.3 above)* J*-convex Morse function. Then*

$$(\omega_\phi := -d^{\mathbb{C}}\phi, X_\phi := \nabla_\phi \phi, \phi)$$

is a Weinstein structure on V. *The symplectic manifold* (V, ω_ϕ) *is independent, up to symplectomorphism isotopic to the identity, of the choice of completely exhausting* J*-convex Morse function* ϕ.

Proof. By definition of J-convexity, $\omega_\phi := -dd^{\mathbb{C}}\phi$ is a symplectic form, i.e., a closed nondegenerate 2-form. Denote $X_\phi := \nabla\phi$ the gradient of ϕ taken with respect to the metric $\langle X, Y\rangle := \omega_\phi(X, JY)$. Then X_ϕ is Liouville. Indeed, for any $Y \in TV$ we have

$$d^{\mathbb{C}}\phi(Y) = \langle \nabla\phi, JY\rangle = -\omega_\phi(\nabla\phi, Y) = -i_{X_\phi}\omega_\phi(Y).$$

Hence

$$i_{X_\phi}\omega_\phi = -d^{\mathbb{C}}\phi, \qquad L_{X_\phi}\omega_\phi = \omega_\phi.$$

To prove the second part of the proposition consider two completely exhausting J-convex functions $\phi_0, \phi_1 : V \to \mathbb{R}_+$. Using Lemma 3.19 we find smooth functions $h_0, h_1 : \mathbb{R}_+ \to \mathbb{R}_+$ with $h_0', h_1' \to \infty$ and $h_0'', h_1'' > 0$, a completely exhausting function $\psi : V \to \mathbb{R}_+$, and a sequence of compact domains V^k, $k = 1, \dots$, with smooth boundaries $\Sigma^k = \partial V^k$, such that

- $V^k \subset \mathrm{Int}\, V^{k+1}$ for all $k \geq 1$;
- $\bigcup_k V^k = V$;
- Σ^{2j-1} are level sets of the function ϕ_1 and Σ^{2j} are level sets of the function ϕ_0 for $j = 1, \dots$;
- $\psi = h_1 \circ \phi_1$ on $\mathcal{O}p\left(\bigcup_{j=1}^\infty \Sigma^{2j-1}\right)$ and $\psi = h_0 \circ \phi_0$ on $\mathcal{O}p\left(\bigcup_{j=1}^\infty \Sigma^{2j}\right)$.

Let us construct now a compact type homotopy between the Weinstein structures $(\omega_{\phi_0}, X_{\phi_0}, \phi_0)$ and $(\omega_{\phi_1}, X_{\phi_1}, \phi_1)$ on V. Then Proposition 12.2 will imply that the symplectic manifolds (V, ω_{ϕ_0}) and (V, ω_{ϕ_1}) are symplectomorphic via a diffeomorphism isotopic to the identity.

The required compact type homotopy can now be be constructed as a composition of four *elementary* compact type homotopies. First, note that for any function $h:\mathbb{R}_+\to\mathbb{R}_+$ such that $h'\to\infty$ and $h''>0$ the linear combination $h^s(x)=(1-s)x+sh(x)$ has the same properties for any $s\in(0,1]$, and hence the Weinstein structures which correspond to the family of completely exhausting J-convex functions $h_i^s\circ\phi_i$ provide elementary compact type homotopies between the Weinstein structures $(\omega_{\phi_i},X_{\phi_i},\phi_i)$ and $(\omega_{h_i\circ\phi_i},X_{h_i\circ\phi_i},h_i\circ\phi_i)$, $i=0,1$. On the other hand, for each $i=0,1$ the family $\phi_i^s=(1-t)h_i\circ\phi_i+t\psi$, $s\in[0,1]$, consists of exhausting J-convex functions which concide near boundaries of an exhausting sequence of compact domains. In view of Proposition 2.7 (by choosing the h_i sufficiently convex) we can also assume that these functions are completely exhausting. Hence the Weinstein structures which they generate provide elementary homotopies between $(\omega_{h_i\circ\phi_i},X_{h_i\circ\phi_i},h_i\circ\phi_i)$ and $(\omega_\psi,X_\psi,\psi)$. □

Note that the contact structure ξ_c defined on a regular level set $\Sigma_c=\phi^{-1}(c)$ by the form $d\phi^{\mathbb{C}}|_{\Sigma_c}$ is formed in this case by the distribution of complex tangent hyperplanes to the J-convex hypersurface Σ_c.

Remark 12.12. Let (V,J) be any almost complex manifold which admits an exhausting J-convex Morse function $\phi:V\to\mathbb{R}$. Then even if the symplectic form $\omega_\phi=-dd^{\mathbb{C}}\phi$ is not compatible with J one still gets a Weinstein structure $(V,\omega_\phi,X_\phi,\phi)$, similar to the one defined in Proposition 12.11. The only difference in this case is that the Liouville vector field X_ϕ should be defined directly as ω_ϕ-dual to $-d^{\mathbb{C}}\phi$, i.e. by

$$-d^{\mathbb{C}}\phi=i_{X_\phi}\omega_\phi.$$

Applying both sides to a tangent vector JZ we find

$$d\phi(Z)=\omega_\phi(X_\phi,JZ),$$

so X_ϕ is gradient-like for ϕ with respect to the positive definite (but in general non-symmetric) $(2,0)$ tensor field $g_\phi:=\omega_\phi(\cdot,J\cdot)$. Completeness of X_ϕ can be achieved similarly to the integrable case.

Proposition 12.11, Remark 12.12 and Corollary 12.7 imply

Corollary 12.13. *The indices of critical points of any J-convex Morse function on a $2n$-dimensional almost complex manifold are $\le n$.*

12.5 Weinstein structures near critical points

In this section we prove that a Weinstein structure can be arbitrarily altered near a hyperbolic or birth-death type critical point. The precise formulation is given in the following two propositions.

Proposition 12.14. *Let p be a hyperbolic critical point of ϕ_0 in a Weinstein manifold (V, ω, X_0, ϕ_0). Let $(\omega, X_{\rm loc}, \phi_{\rm loc})$ be a Weinstein structure on a neighborhood $V_{\rm loc}$ of p such that p is a hyperbolic critical point of $\phi_{\rm loc}$ of value $\phi_{\rm loc}(p) = \phi_0(p)$ and index ${\rm ind}_p(\phi_{\rm loc}) = {\rm ind}_p(\phi_0)$. Then there exists a homotopy of Weinstein structures (ω, X_t, ϕ_t) on V such that $(X_t, \phi_t) = (X_0, \phi_0)$ outside $V_{\rm loc}$, X_t has a unique hyperbolic zero at p in $V_{\rm loc}$ for all $t \in [0, 1]$, and $(X_1, \phi_1) = (X_{\rm loc}, \phi_{\rm loc})$ near p.*

Proposition 12.15. *State and prove analogue for birth-death case.*

We first prove Proposition 12.14 in the special case that $X_{\rm loc} = X_0$ and ϕ_0 is a strong Lyapunov function.

Lemma 12.16. *Let p be a hyperbolic critical point of ϕ in a Weinstein manifold (V, ω, X_0, ϕ) such that $X_0 \cdot \phi \geq \delta |X_0|^2$ for some $\delta > 0$. Let X_{loc} be a vector field on a neighborhood $V_{\rm loc}$ of p with a hyperbolic zero at p such that $L_{X_{loc}} \omega = \omega$ and $X_{loc} \cdot \phi \geq \delta |X_{\rm loc}|^2$. Then there exists a homotopy of strong Weinstein structures (ω, X_t, ϕ) on V such that $X_t = X_0$ outside $V_{\rm loc}$, X_t has a unique hyperbolic zero at p in $V_{\rm loc}$ for all $t \in [0, 1]$, and $X_1 = X_{\rm loc}$ near p.*

Define strong Weinstein structure.

Proof. Pick local coordinates $\{Z\}$ near $p = \{Z = 0\}$. Hyperbolicity of X_0 implies $|X_0(Z)| \geq \gamma |Z|$ for some $\gamma > 0$, and similarly for $X_{\rm loc}$. So the function ϕ satisfies $d\phi(Z) \leq c|Z|$ and $X_0 \cdot \phi(Z) \geq \delta |X_0(Z)|^2 \geq \beta |Z|^2$ for positive constants c and $\beta = \gamma\delta$, and similarly $X_{\rm loc} \cdot \phi(Z) \geq \beta |Z|^2$. It follows that the vector fields $\bar{X}_t := X_0 + t(X_{\rm loc} + X_0)$ satisfy $L_{\bar{X}_t} \omega = \omega$ and $\bar{X}_t \cdot \phi(Z) \geq \beta |Z|^2$ for all $t \in [0, 1]$. Cut the interval $[0, 1]$ into $0 = t_0 < t_1 < \cdots < t_N = 1$ such that

$$|\bar{X}_{t_{i+1}}(Z) - \bar{X}_{t_i}(Z)| \leq \alpha |Z|$$

for all i, with an arbitrarily small constant $\alpha = \alpha(c, \beta)$ to be chosen later. Thus by induction over i it suffices to prove the statement under the assumption

$$|X_{\rm loc}(Z) - X_0(Z)| \leq \alpha |Z|.$$

The 1-form $\lambda := i_{(X_0 - X_{\rm loc})} \omega$ is closed. So there exists a unique function H with $H(0) = 0$ and $dH = \lambda$, i.e. $X_0 - X_{\rm loc} = -X_H$, where X_H denotes the Hamiltonian vector field of H. Pick $\varepsilon > 0$ and a cutoff function $g : [0, \epsilon] \to [0, 1]$ with $g \equiv 1$ near 0, $g \equiv 0$ near ϵ, and $|g'| \leq 2/\epsilon$. Define

$$f(Z) := g(|Z|^2), \qquad H_t := tfH, \qquad X_t := X_0 + X_{H_t}.$$

The vector fields X_t satisfy $X_t = X_0$ for $|Z|^2 \geq \epsilon$ and $X_1 = X_{\rm loc}$ near p. Since

$$\begin{aligned} |H(Z)| &= |\int_0^1 \frac{d}{ds} H(sZ) ds| \\ &\leq \int_0^1 |Z|\,|dH(sZ)| ds \\ &\leq \int_0^1 |Z|\,|X_{\rm loc}(sZ) - X_0(sZ)| ds \\ &\leq \frac{\alpha}{2}|Z|^2, \end{aligned}$$

we can estimate for $|Z|^2 \leq \epsilon$ and $\alpha \leq \frac{\beta}{4c}$:

$$\begin{aligned} X_t \cdot \phi &= (X_0 + tfX_H)\cdot\phi + tHX_f\cdot\phi \\ &\geq \beta|Z|^2 - t|H(Z)|\,|df(Z)|\,|d\phi(Z)| \\ &\geq \beta|Z|^2 - \frac{\alpha}{2}|Z|^2\,\frac{4}{\epsilon}|Z|\,c|Z| \\ &\geq (\beta - 2c\alpha)|Z|^2 \\ &\geq \frac{\beta}{2}|Z|^2. \end{aligned}$$

This shows that X_t is strongly gradient-like for ϕ and has a hyperbolic zero at p for all t, so Lemma 12.16 is proved. □

Next we discuss linear Liouville vector fields.

Lemma 12.17. *The space of hyperbolic linear Liouville vector fields on a symplectic vector space (V,ω) with fixed unstable and stable subspaces $E^\pm$ is path connected.*

Proof. Recall that E^- is isotropic, E^+ is coisotropic and $V = E^- \oplus E^+$. Hence

Explain this?

we can identify (V,ω) with $(\mathbb{C}^n, \omega_{\rm st})$ with coordinates $z_j = x_j + iy_j$ such that E^- corresponds to $\mathbb{R}^k$ spanned by $x = (x_1,\dots,x_k)$ and E^+ corresponds to $\mathbb{R}^{2n-k}$ spanned by $y = (y_1,\dots,y_k)$ and $z = (z_{k+1},\dots,z_n)$. Now consider a hyperbolic linear Liouville vector field X on $(\mathbb{C}^n,\omega_{\rm st})$ with unstable and stable subspaces $E^\pm$. As the flow of X is conformally symplectic and preserves $E^\pm$ it preserves the splitting $\mathbb{C}^n = \mathbb{R}^k \oplus i\mathbb{R}^k \oplus \mathbb{C}^{n-k}$, thus X is of the form $X(x,y,z) = (Ax, -By, Cz)$ for matrices A, B, C all of whose eigenvalues have positive real parts. Now note that the spectrum $\sigma(M)$ of a matrix M satisfies for $a, t \in \mathbb{R}$

$$\sigma\Big((1-t)M + ta\mathbb{1}\Big) = (1-t)\sigma(M) + ta,$$

so if all eigenvalues of M have positive real part the same holds for $(1-t)M + ta\mathbb{1}$ for any $a > 0$ and $t \in [0,1]$. Hence the linear Liouville vector fields

$$X_t := (1-t)X + tX_{\rm st}, \qquad X_{\rm st}(x,y,z) = (2x, -y, \frac{1}{2}z)$$

are hyperbolic with unstable and stable subspaces $E^\pm$ for all $t \in [0,1]$. This shows that any X can be connected to the standard field $X_{\rm st}$ and thus the lemma. □

Proof of Proposition 12.14. Let (V,ω,X_0,ϕ_0), p and $(X_{\rm loc},\phi_{\rm loc})$ be as in Proposition 12.14. In view of Proposition 8.11 we may assume without loss of generality that ϕ_0 and $\phi_{\rm loc}$ are strong Lyapunov functions for X_0 resp. $X_{\rm loc}$. We will modify (X_0,ϕ_0) near p in 3 steps. Let us call a homotopy (Y_t,ϕ_t) *admissible* if (ω,X_t,ϕ_t) is a Weinstein structure, $(X_t,\phi_t)=(X_0,\phi_0)$ outside $V_{\rm loc}$, and X_t has a unique hyperbolic zero at p in $V_{\rm loc}$ for all $t\in[0,1]$.

Step 1: *There exists an admissible homotopy (X_t,ϕ_t) such that the unstable (resp. stable) subspace $E_p^\pm(X_1)$ of X_1 at p agrees with the unstable (resp. stable) subspace $E_p^\pm(X_{\rm loc})$ of $X_{\rm loc}$.*

To see this, pick a homotopy of symplectomorphisms f_t such that $f_0=\mathbb{1}$, $f_t=\mathbb{1}$ outside $V_{\rm loc}$, and the differential d_pf_1 maps $E_p^\pm(X_{\rm loc})$ to $E_p^\pm(X_1)$. Then $X_t := f_t^*X_0$ has the desired properties. After applying Step 1 and changing notation, we may thus assume that $E_p^\pm(X_{\rm loc}) = E_p^\pm(X_0) =: E_p^\pm$. Fix Darboux coordinates $\{Z\}$ near p and denote by $Y_0^{\rm lin}$ and $\phi_0^{\rm quad}$ the linear resp. quadratic parts in the Taylor expansion near p.

Step 2: *There exists an admissible homotopy (Y_t,ϕ_t) such that $Y_1=Y_0^{\rm lin}$ and $\phi_1=\phi_0^{\rm quad}$.*

The linear resp. quadratic parts satisfy

$$L_{X_0^{\rm lin}}\omega=\omega, \qquad X_0^{\rm lin}\cdot\phi_0(Z)\geq\delta|Z|^2, \qquad X_0^{\rm lin}\cdot\phi_0^{\rm quad}(Z)\geq\delta|Z|^2$$

for some $\delta>0$. Therefore we may first apply Lemma 12.16 to homotope X_0 to $X_0^{\rm lin}$ (fixing ϕ_0) and then Proposition 8.11 to homotope ϕ_0 to $\phi_0^{\rm quad}$ (fixing $X_0^{\rm lin}$). After applying Step 1 to (X_0,ϕ_0) and in the converse direction to $(X_{\rm loc},\phi_{\rm loc})$ we may thus assume that $X_0, X_{\rm loc}$ are linear and $\phi_0,\phi_{\rm loc}$ are quadratic in the same Darboux coordinates $\{Z\}$ near p.

Step 3: *There exists an admissible homotopy (X_t,ϕ_t) with $(\underline{X}_1,\phi_1)=(X_{\rm loc},\phi_{\rm loc})$ near p.*

By Lemma 12.17 there exists a homotopy of hyperbolic linear Liouville vector fields $\bar X_t$ near p from $\bar X_0=X_0$ to $\bar X_1=X_{\rm loc}$. By ??? there exists a homotopy of strong quadratic Lyapunov functions $\bar\phi_t$ for $\bar X_t$ near p from $\bar\phi_0=\phi_0$ to $\bar\phi_1=\phi_{\rm loc}$. Since the strong Lyapunov property is stable under C^2-small perturbations, there exists a partition $0=t_0<t_1<\cdots<t_N=1$ such that for all i the following hold:

- $\bar\phi_{t_i}$ is a strong Lyapunov function for $\bar X_t$ for all $t\in[t_i,t_{i+1}]$;
- $\bar\phi_t$ is a strong Lyapunov function for $\bar X_{t_{i+1}}$ for all $t\in[t_i,t_{i+1}]$.

Therefore we can inductively for each i apply Lemma 12.16 to homotope $\bar X_{t_i}$ to $\bar X_{t_{i+1}}$ (fixing $\bar\phi_{t_i}$) and then Proposition 8.11 to homotope $\bar\phi_{t_i}$ to $\bar\phi_{t_i+1}$ (fixing $\bar X_{t_{i+1}}$). This concludes the proof of Proposition 12.14. □

12.6 Weinstein normal forms

Section to be rewritten.

Proposition 12.18. *Let $\mathfrak{W}_j = (\omega_j, X_j, \phi)$ $j = 0, 1$, be two Weinstein cobordism structures on the manifold W with boundary $\partial W = \partial_+ W \cup \partial_- W$, which share the same Lyapunov function ϕ. Suppose that $\mathfrak{W}_0$ and $\mathfrak{W}_1$ coincide on $\mathcal{O}p\,(\partial W_-)$, that ϕ has a unique critical point $p \in W$, and that the stable discs D_j of p for vector fields X_j, $j = 0, 1$, coincide: $D_0 = D_1 = D$.Then the Weinstein structures $\mathfrak{W}_0$ and $\mathfrak{W}_1$ are strongly homotopic via a homotopy which is fixed on $\mathcal{O}p\,(\partial_- W)$ and leaves invariant the stable disc D.*

Proof. TO BE ADDED □

Remark 12.19. An analog of Proposition 12.18 holds also for Stein domains. See Proposition 11.11 below.

By Proposition 5.20, near an isotropic submanifold in a level set of a Weinstein manifold (V, ω, X, ϕ) we can put (ω, X) and *one* level set of ϕ into normal form. However, even in a neighborhood of a point there is no hope to find a normal form for the whole structure (ω, X, ϕ) since rescaling ϕ yields non-equivalent local data. The following results describe normal forms for Weinstein structures up to homotopy.

Recall that the core of a Weinstein manifolds is the union of all stable manifolds of critical points.

Proposition 12.20. *Let Δ be the core of a Weinstein manifold $(V_0, \omega_0, X_0, \phi_0)$. Let $(\omega_{\rm loc}, X_{\rm loc}, \phi_{\rm loc})$ be a Weinstein structure on a neighborhood $V_{\rm loc}$ of Δ such that $\phi_{\rm loc}$ has the same critical points as ϕ_0 of the same values and $X_{\rm loc}$ is tangent to Δ. Then there exists a homotopy of Weinstein structures (ω_t, X_t, ϕ_t) on V such that $(\omega_t, X_t, \phi_t) = (\omega_0, X_0, \phi_0)$ outside $V_{\rm loc}$ for all $t \in [0, 1]$ and $(\omega_1, X_1, \phi_1) = (\omega_{\rm loc}, X_{\rm loc}, \phi_{\rm loc})$ near Δ.*

If $(\omega_{\rm loc}, X_{\rm loc}, \phi_{\rm loc}) = (\omega_0, X_0, \phi_0)$ on a neighborhood of a closed subset $A \subset \Delta$, then we can achieve that $(\omega_t, X_t, \phi_t) = (\omega_0, X_0, \phi_0)$ near A for all $t \in [0, 1]$.

Corollary 12.21. *Let (ω_i, X_i, ϕ_i), $i = 0, 1$, be Weinstein structures on V having the same critical points of corresponding values and indices and the same core. Then (ω_0, X_0, ϕ_0) and (ω_1, X_1, ϕ_1) are Weinstein homotopic.*

Proof. By Proposition 12.20, after a Weinstein homotopy of (ω_0, X_0, ϕ_0) we may assume that (ω_0, X_0, ϕ_0) and (ω_1, X_1, ϕ_1) agree on a neighborhood U of their common core Δ. By Lemma [???], after shrinking U we may assume that ∂U is transverse to X_0. Now for $i = 0, 1$ the flow of X_i defines a Weinstein homotopy from $(U, \omega_i, X_i, \phi_i)$ to $(V, \omega_i, X_i, \phi_i)$. □

Reference? Make more precise!

Proof of Proposition 12.20. To be done. □

Chapter 13

Weinstein handlebodies

13.1 Handles in the smooth category

For integers $0 \leq k \leq m$ and a number $\varepsilon > 0$ consider the *m-dimensional k-handle*

$$H := H^k_\varepsilon := D^k_{1+\varepsilon} \times D^{m-k}_\varepsilon,$$

where D^k_r denotes the closed k-disk of radius r. We will use the following notations (see Figure [fig:handle]):

- the *core disk* $D := D^k_1 \times \{0\}$ and the *core sphere* $S := \partial D$;
- the *lower boundary* $\partial^- H := \partial D^k_1 \times D^{m-k}_\varepsilon$;
- the *upper boundary* $\partial^+ H := D^k_1 \times \partial D^{m-k}_\varepsilon$;
- the normal bundle $\nu := T(\partial^- H)|_S = S \times \mathbb{R}^{m-k}$ to S in $\partial^- H$;
- the outward normal vector field η along $S \subset D$;
- the *attaching region* $U := H \setminus D^k_1 \times D^{m-k}_\varepsilon$.

We are not fixing the "width" ϵ of the handle and allow us to choose it as small as it is convenient.

Now let W be a compact m-manifold with boundary ∂W. An *attaching map* for a k-handle is an embedding $f : \partial^- H \hookrightarrow \partial W$. Extend f to an embedding $F : (U, U \cap \partial^- H) \hookrightarrow (W, \partial W)$ by mapping η to an inward pointing vector field along ∂W. Then we can *attach a k-handle* to W by the map f to get a manifold

$$W \cup_f H := W \amalg H /_{H \ni x \sim F(x) \in W}.$$

Different extensions F give rise to manifolds that are *canonically diffeomorphic*, i.e., related by a diffeomorphism that is unique up to isotopy. Moreover, the diffeomorphism can be chose to be the identity on a *shrinking of* W, i.e., the complement of a tubular neighborhood of ∂W.

Remark 13.1. Note that the boundary of $W \cup_f H$ has a corner along $f(\partial D^k_1 \times \partial D^{m-k}_\varepsilon)$. But this corner can be smoothed in a canonical way as follows (cf. Chapter 4): Introduce the norms

$$R := \sqrt{x_1^2 + \cdots + x_k^2} \quad \text{and} \quad r := \sqrt{x_{k+1}^2 + \cdots + x_m^2}.$$

Pick a concave curve γ in the first quadrant of the (r, R)-plane as in Figure [fig:corner] which equals the curve $R \equiv 1$ near $(\varepsilon, 1)$ and $r \equiv \delta$ near $(\delta, 0)$ for some $0 < \delta < \varepsilon$. Denote by $H_\gamma \subset H$ the region bounded by the hypersurface $\{(r, R) \in \gamma\}$ and containing the core disk. Then $W \cup_f H_\gamma$ is a smooth manifold with boundary which is easily seen to be independent of the curve γ, up to canonical diffeomorphism fixed on a shrinking of W. Therefore, we will suppress γ from the notation and denote the resulting smooth manifold with boundary again by $W \cup_f H$.

In particular, this argument shows independence of the "width" ε.

Remark 13.2. The boundary of $W \cup_f H$ is obtained from ∂W by *surgery of index* k, i.e., by cutting out a copy of $\partial D^k \times D^{m-k}$ and gluing in $D^k \times \partial D^{m-k}$ along the common boundary $\partial D^k \times \partial D^{m-k}$. The manifold $(W \cup_f H) \setminus W'$, where $W' \subset W$ is the complement of a tubular neighborhood of ∂W, provides a canonical cobordism between ∂W and $\partial(W \cup_f H)$. This cobordism carries a Morse function which is constant on the boundaries and has a unique critical point of index k in the center of the handle, see [50] and Section 13.2 below.

Remark 13.3. By the tubular neighborhood theorem (see [42]), the attaching map $f : \partial^- H \hookrightarrow W$ is uniquely determined, up to isotopy, by the following two data:

(i) the embedding $f|_S : S \cong S^{k-1} \hookrightarrow \partial W$ (the *attaching sphere*);

(ii) the trivialization $df : \nu \cong S \times \mathbb{R}^{m-k} \to \nu_f$ of the normal bundle to f in ∂W (the *normal framing*).

Lemma 13.4. *An isotopy of attaching maps $f_t : \partial^- H \hookrightarrow \partial W$, $t \in [0,1]$, induces a canonical family of diffeomorphisms $\phi_t : W \cup_{f_0} H \to W \cup_{f_t} H$.*

Proof. By the isotopy extension theorem (see [42]), (after possibly shrinking ε) there exists a diffeotopy $\psi_t : \partial W \to \partial W$ such that $f_t = \psi_t \circ f_0$. Let $\partial W \times [-1, 0]$ be a collar neighborhood of $\partial W \cong \partial W \times \{0\}$ and define for each t a diffeomorphism

$$\Psi_t : \partial W \times [-t, 0] \to \partial W \times [-t, 0], \qquad (x, \tau) \mapsto \big(\psi_{\tau+t}(x), \tau\big).$$

Then Ψ_t fits together with the identity on $W \setminus (\partial W \times [-t, 0])$ and H to a diffeomorphism $\phi_t : W \cup_{f_0} H \to W \cup_{f_t} H$, see Figure [fig:???]. □

Example 13.5. In general, the diffeomorphism type of $W_f H$ depends on the normal framing. It also generally depends on the particular parametrization $f : S \to f(S)$ of the embedded sphere $f(S) \subset \partial W$. For example, attaching an m-handle to the m-ball D^m via a diffeomorphism $f : S^{m-1} \to S^{m-1}$ yields a manifold $D^m \cup_f H$ that is easily seen to be homeomorphic to S^m. However, it is in general not diffeomorphic to S^m. Indeed, by Lemma 13.4, $f \mapsto D^m \cup_f H$ defines a map from isotopy classes of diffeomorphisms of S^{m-1} to smooth structures on S^m (up to diffeomorphism). This map is known to be surjective for all $m \neq 4$ (see [42]; the remaining case $m = 4$ amounts to the 4-dimensional smooth Poincaré conjecture). For example, all the 28 smooth structures on S^7 arise in this way.

Morse theory. For a function $\phi : V \to \mathbb{R}$ on a manifold and $c < d$ we introduce the following self-explanatory notations:

$$V^c := \phi^{-1}(c), \qquad V^{\leq c} := \phi^{-1}\big((-\infty, c]\big), \qquad V^{[c,d]} := \phi^{-1}([c,d]) \qquad \text{etc.}$$

The main result of Morse theory can now be formulated as follows (see [50]):

Proposition 13.6. *Let $\phi : V \to \mathbb{R}$ be a proper function on a manifold such that $V^{[a,b]}$ contains a unique nondegenerate critical point p on level $c \in (a, b)$. Then $V^{\leq b}$ is obtained from $V^{\leq a}$ by attaching a k-handle, where $k = \mathrm{ind}(p)$.*

Since every (paracompact) manifold admits an exhausting Morse function with distinct critical levels (i.e., every level contains at most one critical point) (see [50]), this implies

Corollary 13.7. *Every manifold is obtained from a ball by successive attaching of at most countably many handles.*

We will later need the following lemma about equivalence of Morse functions.

Lemma 13.8. *Let $W^n \subset V^n$ be compact manifolds with boundary and $\Delta \subset V \setminus W$ be an embedded k-disk transversely attached to W along its boundary. Let $\phi, \psi : V \to \mathbb{R}$ be two Morse functions with a unique index k critical point $p \in \Delta$ and regular level sets $\partial W = \phi^{-1}(a) = \psi^{-1}(a)$ and $\partial V = \phi^{-1}(b) = \psi^{-1}(b)$, $a < b$. Suppose that $\phi = \psi$ on $W \cup \Delta$ and their restrictions to Δ have a nondegenerate maximum at p. Then there exists a diffeomorphism $f : V \to V$ with $f|_{W \cup \Delta} = \mathbb{1}$, isotopic to $\mathbb{1}$ rel $W \cup \Delta$, such that $f^*\psi = \phi$.*

Proof. By the Morse lemma, there exists an orientation preserving diffeomorphism $g : U \to U'$ between neighborhoods of p such that $g^*\psi = \phi$. Moreover, we may assume the $g = \mathbb{1}$ on $U \cap \Delta$. (To see this, first find coordinates $x_1, \dots, x_k$ on Δ near p in which $\phi(x) = c - x_1^2 - \cdots - x_k^2$ and extend them to coordinates $x_1, \dots, x_n$ for V near p. Then apply the proof of the Morse lemma in [49] to find new coordinates $u_1, \dots, u_n$ near p in which $\phi(u) = c - u_1^2 - \cdots - u_k^2 + u_{k+1}^2 \cdots + u_n^2$. Inspection of the proof shows that $u_i = x_i$ on Δ. Pick corresponding coordinates v_i for ψ and define g by $u_i \to v_i$.) After shrinking U, U', we can extend g to a diffeomorphism $g : B \to B$ of a ball B containing U, U' such that $g = \mathbb{1}$ near ∂B and g is isotopic to $\mathbb{1}$ rel ∂B. Extend g to a diffeomorphism $g : W \cup N \to W \cup N$, where N is a neighborhood of $\Delta \cup B$ in V, such that $g = \mathbb{1}$ outside B. Using the flow of a gradient-like vector field for ψ on $N \setminus U'$, we can modify g by an isotopy fixed on $W \cup \Delta \cup U$ to a diffeomorphism $h : W \cup N \to W \cup N'$ satisfying $h^*\psi = \phi$. Now pick a gradient-like vector field X for ϕ on $N \setminus U$, tangent to ∂N, and set $X' := h_* X$ on $N' \setminus U'$. Extend X to a gradient-like vector field on $V \setminus (W \cup U)$ and normalize it such that $X \cdot \phi = 1$, similarly for X'. Denote the flows of X, X' by γ_t, γ'_t. For $x \in V \setminus (W \cup U)$, let $t(x) < 0$ be the unique time for which $\gamma_{t(x)}(x) \in \partial W$. Now define $f : V \to V$ by $f := h$ on $W \cup U$ and $f(x) := \gamma'_{-t(x)} \circ \gamma_{t(x)}(x)$ on $V \setminus (W \cup U)$. $\square$

13.2 The standard Weinstein handle

I have exchanged x and y.

We will be interested only in attaching handles of index $k \le n$ and view the handle $H = H^k_\varepsilon = D^k_{1+\varepsilon} \times D^{2n-k}_\varepsilon$ as canonically embedded in $\mathbb{C}^n$ as the bidisk

$$\{\sum_{j=1}^k x_j^2 + \sum_{j=k+1}^n |z_j|^2 \le \epsilon^2, \quad \sum_{j=1}^k y_j^2 \le (1+\varepsilon)^2, \}, \tag{13.1}$$

where $z_i = x_j + iy_j$, $j = 1, \dots, n$, are the complex coordinates in $\mathbb{C}^n$. In particular, the handle H carries the *standard complex structure* i, as well as the *standard symplectic structure* $\omega_{\rm st} = \sum dx_j \wedge dy_j$.

The symplectic form $\omega_{\rm st}$ on H admits a hyperbolic Liouville field

$$X_{\rm st} := \sum_{j=1}^k \left(2x_j \frac{\partial}{\partial x_j} - y_j \frac{\partial}{\partial y_j} \right) + \frac{1}{2} \sum_{j=k+1}^n \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right), \tag{13.2}$$

which is gradient-like for the function

$$\phi_{\rm st}(z) := 1 + \sum_{j=1}^k x_j^2 + \sum_{j=k+1}^n |z_j|^2 - \sum_{j=1}^k y_j^2.$$

More generally, $X_{\rm st}$ is gradient-like for a function on H of the form $\psi(u, v)$ with

$$u := \sum_{j=1}^k x_j^2 + \sum_{j=k+1}^n |z_j|^2, v := \sum_{j=1}^k y_j^2$$

provided that

$$\frac{\partial \psi}{\partial u}(u, v) > 0, \ \frac{\partial \psi}{\partial v}(u, v) < 0 \quad \text{for all } 0 \le u \le \varepsilon^2, \ 0 \le v \le (1+\varepsilon)^2. \tag{13.3}$$

Note that $\psi_{\rm st}(u, v) = 1 + u - v$ satisfies these conditions. The following lemma describes some more general functions satisfying these conditions, which will be needed in constructions below.

Lemma 13.9. *For any $\varepsilon > 0$ and $0 < \delta < 1 + \varepsilon^2$ there exists a smooth family of functions $\psi_t(u, v)$ on H^k_ε with the following properties:*

(a) ψ_t satisfies conditions (13.3) (hence is gradient-like for X_{st}) for all $t \in [0, 1]$;

(b) $\psi_0 = \psi_{\mathrm{st}}$ and $\psi_t = \psi_{\mathrm{st}}$ in a neighborhood of the set $\{\psi_{\mathrm{st}} \le 0\} \cup \{ u = \varepsilon^2\}$.

(c) $\psi_1 < \delta$ in a neighborhood of the core disk D.

Proof. This is pretty clear from Figure [fig:W-shape], but here is an explicit construction. Pick any smooth function $f : [1 - (1+\varepsilon)^2, 1 + \varepsilon^2] \to \mathbb{R}$ with the following properties:

- $f'(s) > 0$ and $f(s) \le s$ for all s;
- $f(s) = s$ for $s \le 0$;
- $f(s) < \delta$ for all s.

Pick $0 < a < b < \varepsilon^2$ and a smooth non-decreasing function $\rho : [0, \varepsilon^2] \to [0, 1]$ with $\rho = 0$ of $[0, 1]$ and $\rho = 1$ on $[b, \varepsilon^2]$. Let

$$g(u, s) := f(s) + \rho(u)\big(s - f(s)\big).$$

and define

$$\psi_1(u, v) := g\big(u, \psi_{\mathrm{st}}(u, v)\big), \qquad \psi_t := (1-t)\psi_{\mathrm{st}} + t\psi_1.$$

Let us verify the conditions in the lemma.

(a) The hypotheses on f and ρ imply $\frac{\partial g}{\partial u} = \rho'(u)\big(s - f(s)\big) \ge 0$ and $\frac{\partial g}{\partial s} = f'(s) + \rho(u)\big(1 - f'(s)\big) > 0$, and we find

$$\frac{\partial \psi_1}{\partial u} = \frac{\partial g}{\partial u} + \frac{\partial g}{\partial s}\frac{\partial \psi_{\mathrm{st}}}{\partial u} > 0, \qquad \frac{\partial \psi_1}{\partial v} = \frac{\partial g}{\partial s}\frac{\partial \psi_{\mathrm{st}}}{\partial v} < 0.$$

Hence ψ_1, and therefore also ψ_t, satisfies conditions (13.3).

(b) Clearly $\psi_0 = \psi_{\mathrm{st}}$. For $s \le 0$ we have $g(u, s) = s$, which shows $\psi_t(u, v) = \psi_{\mathrm{st}}(u, v)$ whenever $\psi_{\mathrm{st}}(u, v) \le 0$. For $u \ge b$ we have $g(u, s) = s$ and therefore $\psi_t(u, v) = \psi_{\mathrm{st}}(u, v)$.

(c) For $u \le a$ we have $g(u, s) = f(s)$ and therefore $\psi_1(u, v) = f\big(\psi_{\mathrm{st}}(u, v)\big) < \delta$ by the choice of f. □

Remark 13.10. Lemma 13.9 can be seen as a warm-up for the much more sophisticated study of shapes for i-convex functions on the handle in Chapter 4.

13.3 Weinstein handlebodies

Let us denote by $\xi^- := \ker(\lambda_{\mathrm{st}}|_{\partial^- H})$ the contact structure defined on $\partial^- H$ by To be rephrased.

the Liouville form $\lambda_{\rm st} = i(X_{\rm st})\omega_{\rm st}$. Note that the bundle $\xi^-|_S$ canonically splits as $(TS\otimes\mathbb{C})\oplus\varepsilon^{n-k}$, where ε^{n-k} is a trivial $(n-k)$-dimensional complex bundle. We will denote by σ_S the isomorphism

$$TS\otimes\mathbb{C}\oplus\varepsilon^{n-k}\to\xi_-|S.$$

We need some notation. Suppose we are given a real k-dimensional bundle E, a complex n-dimensional bundle F, $n\geq k$, and an injective totally real homomorphism $\phi:E\to F$. Then ϕ canonically extends to an injective complex homomorphism $\phi\otimes\mathbb{C}:E\otimes\mathbb{C}\to F$. If $n>k$ and $\phi\otimes\mathbb{C}$ extends to a fiberwise complex isomorphism $\Phi:(E\otimes\mathbb{C})\oplus\varepsilon^{n-k}\to F$ then Φ is called a *saturation* of ϕ. When $n=k$ the saturation is unique.

Now let (V,ω,X,ϕ) be a Weinstein manifold, p a critical point of index k of the function ϕ, and $a<b=\phi(p)$ a regular value of ϕ. Denote $W:=\{\phi\leq a\}$. Suppose that the stable manifold of p intersects $V\setminus{\rm Int}\,W$ along a disk D. By Proposition 12.6 the disk D is isotropic in (V,ω), and by Lemma 12.9 the attaching sphere $S=\partial D$ is isotropic in $(\partial W,\xi)$. Thus the inclusion $TS\hookrightarrow\xi$ extends canonically to an injective complex homomorphism $TS\otimes\mathbb{C}\to\xi$, while the inclusion $TD\hookrightarrow TV$ extends to an injective complex homomorphism $TD\otimes\mathbb{C}\to TV$. There exists a homotopically unique complex trivialization of the normal bundle to $TS\otimes\mathbb{C}$ in ξ which extends to D as a trivialization of the normal bundle to $TD\otimes\mathbb{C}$ in TV. This trivialization provides a canonical isomorphism $\Phi_D:TS\otimes\mathbb{C}\oplus\varepsilon^{n-k}\to\xi|_S$. We will call Φ_D *canonical saturation* of the inclusion $TS\hookrightarrow\xi$.

I'd rather speak of the "complex (or symplectic) normal framing"

The following result (at least the existence part) has been proved in [63].

Simplify formulation!

Proposition 13.11 (Weinstein [63])**.** *Let (W,ω,X,ϕ) be a $2n$-dimensional Weinstein domain with boundary ∂W and $\xi=\ker(\lambda|_{\partial W})$ the contact structure on ∂W defined by the Liouville for $\lambda=i_X\omega$. Let $h:S\to\partial W$ be an isotropic embedding of the $(k-1)$-sphere S covered by a saturation $\Phi:TS\otimes\mathbb{C}\oplus\varepsilon^{n-k}\to\xi$ of the differential $dh:TS\to\xi$. Then there exists a Weinstein domain $(\widetilde W,\widetilde\omega,\widetilde X,\widetilde\phi)$ such that $W\subset{\rm Int}\,\widetilde W$, and*

(i) $(\widetilde\omega,\widetilde X,\widetilde\phi)|_W=(\omega,X,\phi)$;

(ii) the function $\widetilde\phi|_{\widetilde W\setminus{\rm Int}\,W}$ has a unique critical point p of index k.

(iii) the stable disk D of the critical point p is attached to ∂W along the sphere $h(S)$, and the canonical saturation Φ_D coincides with Φ.

Given any two Weinstein extensions $(W_0,\omega_0,X_0,\phi_0)$ and $(W_1,\omega_1,X_1,\phi_1)$ of (W,ω,X,ϕ) satisfying properties (i)-(iii), there exists a diffeomorphism $g:W_0\to W_1$ fixed on W such that (ω_0,X_0,ϕ_0) and the pull-back structure $(g^*\omega_1,g^*X_1,g^*\phi_1)$ are homotopic in the class of Weinstein structures satisfying conditions (i)-(iii). In particular, the completions ${\rm Compl}(W_0,\omega_0,X_0,\phi_0)$ and ${\rm Compl}(W_1,\omega_1,X_1,\phi_1)$ are symplectomorphic via a symplectomorphism fixed on W.

We say that the Weinstein domain $(\widetilde{W}, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ is obtained from (W, ω, X, ϕ) by attaching a handle of index k along an isotropic sphere $h : S \to \partial W$ with a saturation homomorphism Φ.

Proof. Extend the Weinstein structure (ω, X, ϕ) to a slighly larger manifold $W' \supset W$. After adding a constant to ϕ we may assume that $\phi|_{\partial W} = -1$. Let $(H, \omega_{\rm st}, X_{\rm st}, \phi_{\rm st})$ be the standard Weinstein handle of index k. By Proposition 5.22 there exists an isomorphism of isotropic setups

$$F : (U, \omega_{\rm st}, X_{\rm st}, \phi_{\rm st}^{-1}(-1) \cap U, S) \to (U', \omega, X_1, \partial W \cap U', h(S))$$

between neighborhoods of S in H and $h(S)$ in W' inducing h and Φ. Thus (ω, X) and $(\omega_{\rm st}, X_{\rm st})$ fit together to a Liouville structure $(\tilde{\omega}, \tilde{X})$ on $W' \cup_F H$. Moreover, the level set $\{\psi_{\rm st} = -1\}$ corresponds via F to the level set $\{\psi = -1\}$. Thus after perturbing ψ on $W' \setminus W$, keeping it transverse to X, we may assume that the level sets $\{\psi = t\}$, $t \in [-1, -1 + \delta]$ correspond via F to level sets $\{\psi_{\rm st} = g(t)\}$ for some $\delta > 0$ and diffeomorphism $g : [0, \delta] \to [0, \delta']$. Now let $\tilde{\psi}$ be the function on $W' \cup_F H$ which equals $g^{-1} \circ \psi$ on W' and the function ψ_1 from Lemma 13.9 on H. Then $(\tilde{W} := \{\tilde{\psi} \leq \delta\}, \tilde{\omega}, \tilde{X}, \tilde{\phi})$ has the desired properties.

For uniqueness, pull back (ω_1, X_1, ϕ_1) by any diffeomorphism $g : W_0 \to W_1$ fixed on W and mapping the critical point and stable disk of ϕ_0 to those of ϕ_1. Compose $g^* \phi_1$ with a homotopy of functions $\mathbb{R} \to \mathbb{R}$ to arrange the same values at the critical point and apply Corollary **??**. □

Remark 13.12. Note that Proposition 13.11 implies that even in the case of infinitely many handles the handlebody description determines the symplectomorphism type of Weinstein manifold. Indeed, it follows that given 2 manifolds $(V_1, \omega_1, X_1, \phi_1)$ and $(V_1, \omega_2, X_2, \phi_2))$ with the same handlebody description, there is a symplectomorphism of a neighborhood U_1 of the core K_1 of the first manifold onto a neighborhood U_2 of the core K_2 of the second. Moreover, the neighborhoods can b chosen i such a way that their boundaries are transversal to the Liouville fields X_1 and X_2 respectively. On the other hand, $\bigcup_t X_1^t(U_1) = V_1$ and $\bigcup_t X_2^t(U_2) = V_2$, and hence the symplectomorphism $U_1 \to U_2$ can be extended to a symplectomorphism $V_1 \to V_2$ by matching the corresponding trajectories of the Liouville fields.

Rephrase, overlaps with Corollary **??**

Theorem 13.13. *Suppose that a $2n$-dimensional almost complex manifold (V, J) admits an exhausting Morse function ϕ with only critical points of index $\leq n$. Then there exists a Weinstein structure (ω, X, ϕ) (with the same ϕ!) on V such that J is homotopic to an almost complex structure compatible with ω.*

Proof. Let $\phi : V \to \mathbb{R}^+$ be an exhausting Morse function with critical points of index $\leq n$. The critical values of ϕ are discrete. Let us order them: $c_0 = 0 < c_1 < c_2 \dots$, introduce intermediate regular values $d_k = c_{k-1} + \frac{c_k - c_{k-1}}{2}$, $k = 1, \dots$, and set $W_k := \{\phi \leq d_k\}, \Sigma_k := \partial W_k, k = 1, \dots$. Note that there are

Still to be corrected.

only finitely many critical point on each critical level. We are going to construct the Weinstein structure inductively on W_k.

W_1 is a disjoint union of finitely many balls. We choose on each of them a Weinstein structure which consists of the standard symplectic structure of the unit ball in the standard symplectic $(\mathbb{R}^{2n}, \sum\limits_1^n dx_k \wedge dy_k)$, the radial Liouville field

$$X = \frac{1}{2}\left(\sum_1^n x_k \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial y_k}\right),$$

and the function ϕ, which can be assumed to be equal $d_1 \left(\sum\limits_1^k x_k^2 + y_k^2\right)$ on each of the balls. We can deform J on V so it becomes compatible with the chosen symplectic form on W_1.

Let us assume that we already constructed the required Weinstein structure (ω, X, ϕ) on W_l for some $l \geq 1$, so that $J|_{W_l}$ is compatible with the Weinstein structure on W_l. The standard Morse theory tells us that W_{l+1} can be obtained from W_l by a simultaneous attaching of several handles of index $\leq n$. Without a loss of generality we can assume that there is just one handle.

Let p be the corresponding critical point of the function ϕ, and Δ the intersection of its stable manifold (formed by the trajectories of X converging to p) with $V \setminus \operatorname{Int} W_l$. Then Δ is a disk of dimension $k = \operatorname{ind} p$, transversely attached to W_l in $W_{l+1} \subset V$. By Theorem 6.14 there exists an isotopy of Δ in $W_{l+1} \setminus \operatorname{Int} W_l$ into a totally real disk Δ' which is J-orthogonally attached to W_l along an isotropic submanifold of ∂W_l.

By Proposition 13.11 we can extend the Liuoville structure from W_l to a Lioville structure (ω', X', ϕ') on a domain $W'_{l+1} \subset \mathcal{O}p\,(W_l \cup \Delta' \subset W_{l+1}$, so that W'_{l+1} is obtained from W_l by attaching a handle of index k with the core disk Δ' using the canonical saturation of the attaching map provided by the totally real disk Δ'. In particular the almost complex structure on W'_{l+1} can be deformed to become compatible with ω''. Now observe that by construction there is an isotopy $\alpha_t : W'_{l+1} \to V$, $t \in [0,1]$, such that α_0 is the inclusion $W'_{l+1} \hookrightarrow V$, $\alpha_1(W'_{l+1}) = W_{l+1}$ and $\alpha_t|_{\mathcal{O}p\, W_l} = \mathrm{Id}$. Moreover, one can arrange that the function $\phi' \circ h$ differs from ϕ by a reparameterization of the image, i.e. $\phi = \beta \circ \phi'$ for a diffeomorphism $\beta : \mathbb{R}_+ \to \mathbb{R}_+$. The push-forward almost complex structure $(\alpha_1)_* J$ extends in the same homotopy class to V. Hence, $(\alpha_1)_*\omega', (\alpha_1)_* X', \phi)$ is the required extension of the Weinstein structure (ω, X, ϕ, ϕ) from W_l to W_{l+1}.

If the function ϕ has finitely many critical points then to complete the proof it remains to attach a cylindrical end to W_N where c_{N-1} is the last critical level. If there are infinitely many point that the resulted structure on $V = W_\infty = \bigcup\limits_1^\infty W_l$ is automatically Weinstein provided that the Liouville vector field X is complete. However, this can be easily achieved by an appropriate rescaling of ω and X in the neighborhood of all regular levels $\partial W_l = \{\phi = d_l\}$, $l = 1, \ldots, \infty$. □

13.4 Subcritical Weinstein manifolds

A $2n$-dimensional Weinstein manifold (V,ω) is called *subcritical* if it admits a Weinstein structure (X,ϕ) such that all critical points of the function ϕ have index $< n$. More precisely, it is called *k-subcritical*, $k \geq 1$ if all critical points of ϕ have index $\leq n-k$.

Theorem 13.14 (Cieliebak [13]). *Let (V,ω,X,ϕ) be a k-subcritical $2n$-dimesional Weinstein manifold. Then (V,ω) is symplectomorphic to the k-stabilization of a Weinstein manifold (V',ω',X',ϕ') of dimension $2(n-k)$.*

The proof requires some preparation. Note that if $k < n$, then the standard $2n$-dimensional handle $H(\mathbf{e})$ of index k contains the standard $(2n-2)$-dimensional handle $H'(\varepsilon) = H(\varepsilon) \cap \mathbb{C}^{n-1}$. The contact structure ξ_n^- on $\partial^- H$ canonically splits as $\xi_{n-1}^- \oplus \varepsilon^1$, where ξ_{n-1}^- is the canonical contact structure on $\partial^- H'$. In the next section we will need the following Still to be proofread.

Lemma 13.15. *Let (W,ω,X,ϕ) be a Weinstein domain of dimension $2n$, and W' a codimension 2 submanifold which is invariant with respect to X, and such that the restriction $(\omega|_{W'}, X|_{W'}, \phi|_{W'})$ defines on W' a Weinstein domain structure. Suppose that the normal bundle to W' in W is trivial. Let $h : S \to \partial W'$ be an isotropic embedding together with a saturation homomorphism $\Phi' : TS \otimes \mathbb{C} \oplus \mathbb{C}^{n-k-1} \to \xi'$. then one can simultaneously attach the handle H' to W' using h and Φ', and the handle H to W using h and $\Phi = \phi' \oplus \mathrm{Id} : TS \otimes \mathbb{C} \oplus \mathbb{C}^{n-k-1} \oplus \varepsilon^1 \to \xi|\partial W' = \xi' \oplus \varepsilon^1$ to get a pair a Weinstein domains $(\widetilde{W} = W \underset{h,\Phi}{\cup} H, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ and $(\widetilde{W'} = W' \underset{h,\Phi}{\cup} H', \widetilde{\omega}', \widetilde{X}, \widetilde{\phi}')$ such that*

- $(\widetilde{W'}, \partial\widetilde{W'}) \subset (\widetilde{W}, \widetilde{\partial}W)$*,* $\widetilde{X}'$ *is tangent to* $\widetilde{W'}$*;*
- $(\widetilde{\omega}', \widetilde{X}', \widetilde{\phi}') = (\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})|_{\widetilde{W'}}$*, and*
- *the normal bundle to* $\widetilde{W'}$ *in* $\widetilde{W}$ *is trivial.*

We will also need the following

Lemma 13.16. *Suppose that (M,ξ) be a $(2n+1)$-dimensional contact manifold and (N,ζ) its codimension 2 contact submanifold with a trivial normal bundle. Let S be a k-dimensional manifold, $k < n$, $f : S \to V$ an isotropic embedding and $\Phi : E = TS \otimes \mathbb{C} \oplus \varepsilon^{n-k} \to \xi$ a saturation of its differential $df : TS \to \xi$. Suppose that there exists a homotopy $f_t : S \to M, t \in [0,1]$, which begins with $f_1 = f$ and ends at a map $f_1 : S \to N$. Then there exists an isotropic isotopy $g_t : S \to M$ and a family Φ_t of saturation of $df_t, t \in [0,1]$, such that*

- $g_0 = f$*;*
- $g_1(S) \subset N$*;*
- g_t *is* C^0*-close to* f_t*.*

- *the restriction of* Φ_1 *to* $E_1 = TS \otimes \mathbb{C} \oplus \varepsilon^{n-k-1} \subset E = TS \otimes \mathbb{C} \oplus \varepsilon^{n-k}$ *is a saturation of the homomorphism* $dg_1 : S \to \zeta$.

Proof. By assumption, there is a splitting $\mu : \zeta \oplus \varepsilon^1 \to \xi|_N$, where ε^1 is a trivial complex bundle. Denote by ν the vector field $\mu(\mathbf{e})$ where $\mathbf{e}$ generates ε^1. Consider a homotopy $\Psi_t : TS \otimes \mathbb{C} \oplus \varepsilon^{n-k} \to \xi$ of complex homomorphisms which covers the homotopy f_t, $t \in [0,1]$, and begins with $\Psi_0 = \Phi$. We can assume that $\Psi_1(\mathbf{e}_{n-k}) = \nu$, where $\mathbf{e}_{n-k}$ is the generator of the second summand in the decomposition $\varepsilon^{n-k} = \varepsilon^{n-k-1} \oplus \varepsilon^1$. Indeed, the obstructions to do that lie in the groups $\pi_j(S^{2n-1})$, $j \le k < n$, which are trivial for any $n > 1$. Hence, we can further adjust Φ_t to ensure that $\Psi_1|_{E_1}$ is a saturation of a totally real homomorphism $\psi : TS \to \zeta$. Now we apply Gromov's h-principle for isotropic immersions 6.8 to C^0-approximate the map $f_1 : S \to N$ by an isotropic immersion $\widetilde{f}_1 : S \to N$, whose differential $d\widetilde{f}_1 : TS \to \zeta$ is homotopic to ψ through totally real homomorphisms $TS \to \zeta$. Note that the homotopy of complex homomorphisms Ψ_t can be modified into $\widetilde{\Psi}_t : E \to \xi$ so that it ends at a saturation $\widetilde{\Psi}_1 : TS \to \xi$ of the homomorphism $d\widetilde{f}_1$ such that $\widetilde{\Psi}_1(E_1) \subset \zeta$. Next, we apply again Theorem 6.8 and construct an isotropic regular homotopy g_t, $t \in [0,1]$, connecting $g_0 = f$ with $g_1 = \widetilde{f}_1$, together with a family $\Phi_t : E \to \xi$ of saturations of dg_t such that the paths $\widetilde{\Psi}_t$ and Φ_t, $t \in [0,1]$, are homotopic with fixed ends. It remains to note that by dimensional reasons (see Lemma 6.10) we can assume that g_t is an isotopy, rather than a regular homotopy. □

Proof of Theorem 13.14. It is sufficient to consider the case $k = 1$. As in the proof of Theorem 13.13 let $c_0 < c_1 \dots$ be the critical levels of the function ϕ, $d_1 < \dots$ intermediate regular values: $c_1 < d_1 < c_1 < d_2 < \dots$ and $W_l = \{\phi \le d_l\}$, $l = 1, \dots$. We will construct the required Weinstein manifold $V' \subset V$ inductively by successively adjusting the handlebody decomposition of V. On each step we will change the Weinstein domain structure on W_k by Weinstein homotopy, and change the attaching map by contact isotopies of ∂W_k. As it is explained above this will not affect the symplectomorphism type of the resulted Weinstein manifold.

Up to Weinstein homotopy we can assume that W_1 is a round ball in $\mathbb{C}^n = \mathbb{R}^{2n}$ with the standard symplectic structure and the radial Liouville field. We set $W_1' := W_1 \cup \mathbb{C}^{n-1}$. Suppose we already deformed a Weinstein domain structure on W_l, so that for the resulted Liouville structure $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi}$ there exists a codimension 2 submanifold with boundary $(W_l', \partial W_l') \subset (W_l, \partial W_l)$ such that

a) $\widetilde{X}$ is tangent to W_l';

b) the function $\widetilde{\phi}$ has no critical points outside W_l';

c) the normal bundle to W_l' in W_l is trivial.

We will consider the case when there is only 1 critical point on the level d_{l+1}. The general case differs only in the notation. Then the Weinstein domain W_{l+1}

can be obtained from W_l by attaching a handle H of index k with an isotropic embedding $h : S \to \partial W_l$ of the core $(k-1)$-dimensional sphere $S \subset H$ with a saturation homomorphism $\Phi : TS \otimes \mathbb{C} \oplus \mathbb{C}^{n-k} \to \xi$, where ξ denotes the contact structure on the boundary of the Weinstein domain $(W_l, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$. According to Lemma 13.16 we can adjust the attaching map via an isotropic isotopy (which is the same as via ambient contact isotopy) to ensure that $h(S) \subset \partial W'_l$ and that the saturation Φ restricted to $E_1 = TS \otimes \mathbb{C} \oplus \varepsilon^{n-k-1} \subset E = TS \otimes \mathbb{C} \oplus \varepsilon^{n-k}$ is a saturation of the homomorphism $dh : S \to \xi'$, where $\xi' = \xi \cap (\partial W'_l)$ is the induced contact structure on $\partial W'_l$. Then using Lemma 13.15 we can simultaneously attach index k handles to W_l and to W'_l. The resulted Weinstein structure on $(W_l \underset{h,\Phi}{\cup} H)$ coincides up to Weinstein homotopy with $(W_{l+1}, \omega, X_{l+1}, \phi)$ and we keep this notation for it. The Weinstein domain $W'_{l+1} = W'_l \underset{h,\Phi|_{E_1}}{\cup} H'$ is embedded in W_{l+1} in such a way that all the above properties a)-c) are satisfied. This gives a simultaneous handlebody description of Weinstein manifolds $(V'\omega')$ of dimension $2n-2$, and of $2n$-dimensional manifolds V, ω. Note that this handlebody decomposition of (V, ω) coincides with the decomposition of the stabilization $(V' \times \mathbb{R}^2, \omega' \oplus \omega_{\rm st})$, and hence, according to Propositions 13.11, 12.2 and Remark 13.12 the manifolds (V, ω) and $(V' \times \mathbb{R}^2, \omega' \oplus \omega_{\rm st})$ are symplectomorphic.

□

The following theorem is a slight modification of a result from [16].

Theorem 13.17. *Let $(V_1, \omega_1, X_1, \phi_1)$ and $(V_2, \omega_2, X_2, \phi_2)$ be two subcritical Weinsten manifolds. Suppose there exists a homotopy equivalence $h : V_1 \to V_2$ covered by a homomorphism $\Phi : TV_1 \to TV_2$ such that $\Phi^* \omega_2 = \omega_1$. Then h is homotopic to a symplectomorphism $f : (V_1, \omega_1) \to (V_2, \omega_2)$.*

Proof???

13.5 Morse-Smale theory for Weinstein structures

Lemma 13.18. *Let (V, ω, X, ϕ) be a Weinstein structure. Let a be a regular value of ϕ, and p a critical point with $\phi(p) = b > a$. Suppose that all the trajectories of the vector field $-X$ emanating from p hit the level set $\Sigma_a = \{\phi = a\}$, i.e the intersection of the stable manifold of p with $\{\phi \geq a\}$ is a disk D with boundary $S = \partial D \subset \Sigma_a$.*

(i) Then for any $c \in (a, b]$ there is another Lyapunov Morse function $\widetilde{\phi}$ for X such that $\widetilde{\phi}(p) = b$, while all other critical values of $\widetilde{\phi}$ and ϕ coincide.

(ii) *Given any contact isotopy* $h_t : \Sigma_a \to \Sigma_a, t \in [0,1]$, *there is family of Weinstein structures* (ω_t, X_t, ϕ) *such that the stable manifold of the point* p *for* X_t *intersects* Σ_a *along* $h_t(S)$, $t \in [0,1]$.

(iii) *Let* q *be another critical point of index* $\operatorname{ind} q = \operatorname{ind} p - 1$ *such that* $\phi(q) = c < a$ *and the intersection of the unstable manifold of* p *with* $\{\phi \leq a\}$ *is a disk* Δ *with boundary* $\Sigma = \partial\Delta \subset \Sigma_a$. *Suppose that* S *and* Σ *intersect transversely at 1 point. Then* (V, ω) *admits a Weinstein structure* $(\widetilde{X}, \widetilde{\phi})$ *such that* $\operatorname{Crit}(\widetilde{\phi}) = \operatorname{Crit}(\phi) \setminus \{p, q\}$, *where we denote by* $\operatorname{Crit}(\widetilde{\phi})$ *and* $\operatorname{Crit}(\phi)$ *the sets of critical points of the functions* $\widetilde{\phi}$ *and* ϕ.

To be continued... *Proof.* The first statements have been proved in [12]. □

Chapter 14

From Weinstein to Stein

14.1 Stein structures on Weinstein manifolds

Two Weinstein cobordisms or manifolds

$$\mathfrak{W} = (W, \omega, X, \phi) \quad \text{and} \quad \widetilde{\mathfrak{W}}' = (\widetilde{W}, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$$

are called *coarsely equivalent* if there exists a diffeomorphism $h : W \to \widetilde{W}$ such that

(i) $\widetilde{\phi} \circ h = g \circ \phi$ for a diffeomorphism $g : \mathbb{R} \to \mathbb{R}$;

(ii) $h^*\widetilde{\lambda} = g_1\lambda + g_2 d\phi$, where $\lambda, \widetilde{\lambda}$ are Liouville forms of $\mathfrak{W}$ and $\widetilde{\mathfrak{W}}$, g_1, g_2 are C^∞-functions on W such that $g_1 > 0$ and near critical points of ϕ, have $g_1 = 1$ and $g_2 = 0$. In other words h preserves the Liouville structure (i.e. ω and X) near critical points of ϕ and induces contactomorphism between the corresponding level sets of functions ϕ and $\widetilde{\phi}$.

The diffeomorphism h is called in this case a *coarse equivalence* between $\mathfrak{W}$ and $\widetilde{\mathfrak{W}}$. If both Weinstein structures are given on the same smooth manifold W then we will always require the equivalence h to be diffeotopic to the identity.

Lemma 14.1. *If $\mathfrak{W}$ and $\widetilde{\mathfrak{W}}$ are Weinstein manifolds then any coarse equivalence between $\mathfrak{W}$ and $\widetilde{\mathfrak{W}}$ is isotopic to a symplectomorphism. In the cobordism case the map h viewed as an embedding to the completion $\langle\widetilde{W}\rangle$ is isotopic to a symplectic embedding onto a domain with starshaped boundaries.*

By a *small adjustment* of a Weinstein cobordism $\mathfrak{W}$ we will mean a combination of the following operations:

(i) C^∞-small deformation;

(ii) C^1-small deformation near critical points of functions ϕ.

The following theorem is a ramification of Proposition 11.4.

Theorem 14.2. *Let $\mathfrak{W} = (W, \omega, X, \phi)$ be a Weinstein cobordism or manifold. Suppose the induced contact structure on $\partial_- W$ admits a compatible integrable CR-structure J. Then J extends to an integrable complex structure on W such that $(J, \alpha \circ \phi)$ is a Stein cobordism structure on W and $\mathfrak{W}(J, \phi)$ is coarsely equivalent to $\mathfrak{W}'$ which is obtained from $\mathfrak{W}$ by a small adjustment. Here α is a diffeomorphism $\mathbb{R} \to \mathbb{R}$.*

Remark 14.3. If $n = \dim W = 2$ then any sufficiently smooth CR-structure on $\partial_- W$ is integrable, see [**?**] . In the real analytic case the claim is straightforward.

check the reference

If $n > 2$ then according to a theorem of Grauert (??) any integrable CR-manifold symplectically fillable, while not every contact manifold is.

Research this

A Weinstein cobordism $\mathfrak{W} = (W, \omega, X, \phi)$ is called *elementary* if ϕ is a Morse function whose critical points are not connected by X-trajectories.

Possibly move this up somewhere in the Weinstein chapter

Proof of Proposition 14.2. Suppose first that the cobordism $\mathfrak{W}$ is elementary. We can assume that the stable discs $D_1, \dots, D_K$ of all critical points $q_1, \dots, q_K$ are real analytic. The complex structure J given on $U_1 = \mathcal{O}p\, \partial_- W \cup \bigcup\limits_1^K \mathcal{O}p\,(q_i)$ extends to $U_2 = \mathcal{O}p \left(\bigcup\limits_1^K D_i \right)$ in a unique way compatible with the real analytic structure of the discs. There exists a J-convex function $\widetilde{\phi} : U_1 \cup U_2 \to \mathbb{R}$ which coincides with ϕ on $U_1 \cup \bigcup\limits_1^K D_i$ and has $\nabla \widetilde{\phi} = X$ along $\bigcup\limits_1^K D_i$. According to **??** the Weinstein structure $\mathfrak{W}(J, \widetilde{\phi})$ given on $U_1 \cup U_2$ extends to W to a Weinstein structure $\widetilde{\mathfrak{W}}$ equivalent to $\mathfrak{W}$.

Next, we apply the Surrounding Lemma **??** and find a J-convex function $\widehat{\phi}$ on $U_1 \cup U_2$, which coincides with $h \circ \phi\widetilde{\phi}$ on $\mathcal{O}p\, \partial(U_1 \cup U_2)$, coincides with $\widetilde{\phi}$ on $\mathcal{O}p \bigcup\limits_1^K D_i$, and such that there exists $c > m = \widehat{\phi}|_{\partial_- W}$ for which

$$\bigcup_1^K \mathcal{O}p\,(q_i) \subset W_c = \{m \le \widehat{\phi} \le c\} \subset U_1 \cup U_2.$$

There exists a diffeomorphism $g : W \to W_c = \{m \le \widehat{\phi} \le c\}$ which is an equivalence between the Weinstein structure $\widetilde{\mathfrak{W}}$ on W and $\mathfrak{W}(J, \widehat{\phi})$ on W_c. Then the induced complex structure $J' = g^* J$ and the induced $g^* J$-convex function $\phi' = \widehat{\phi} \circ g$ on W define the required Stein cobordism structure on W such that $\mathfrak{W}(J', \phi')$ is equivalent to $\mathfrak{W}$.

When the cobordism $\mathfrak{W}$ is not necessarily elementary, let us take an admissible partition $m = c_0 < c_1 < \cdots < C_N = M$ such that the induced cobordism structures on $W_k = \{c_{k-1} \le \phi \le c_k\}$, $k = 1, \dots, N$, are elementary. Next, we consequently extend the Stein cobordism structures to $W_1, \dots, W_N$ in such a way that on each of W_k the Weinstein structure and Stein structure are equivalent. Though the equivalences h_k do not necessarily match into a global coarse equivalence, we can modify them on $U_i = \{c_i \le \phi \le c_i^+ c^i + \varepsilon\}$, $i = 1, \dots, N-1$, for any sufficiently small $\varepsilon > 0$ to get a global coarse equivalence $h : W \to W$.

□

Theorem 14.4. *Let $\mathfrak{W} = (V, \omega, X, \phi)$ be a Weinstein manifold structure which is Stein near critical points of ϕ. Then there exists a Stein complex structure J on V and a diffeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ such that the function $\widetilde{\phi} = \alpha \circ \phi$ is J-convex, and the Weinstein structures $\mathfrak{W}$ and $\mathfrak{W}(J, \widetilde{\phi})$ are coarsely equivalent.*

14.2 Constructing Stein homotopies

While the notion of homotopy of Weinstein or Stein *cobordism* structures is self-explanatory, the notion of homotopy of Stein or Weinstein *manifold* structures needs some clarification.

Slightly rephrasing a similar definition given in Section 12.2 in a more general context of convex Liouville manifolds, we call a family of Weinstein structures $(V, \omega_t, X_t, \phi_t)$, or Stein structures (V, J_t, ϕ_t), $t \in [0,1]$ a *simple homotopy* if there exists a family of functions $c_1 < c_2 < \dots$ on the interval $[0,1]$ such that for each $t \in [0,1]$, $c_i(t)$ is a regular value of the function ϕ_t and $\bigcup\limits_k \{\phi_t \le c_k(t)\} = V$.

A *homotopy* between two Weinstein structures is, by definition, a composition of finitely many simple homotopies. For any two exhausting J-convex functions $\phi_0, \phi_1 : V \to \mathbb{R}$ there exists a homotopy connecting (J, ϕ_0) and (J, ϕ_1) (see Section 12.4), and hence existence of a Stein homotopy connecting (J_0, ϕ_0) and (J_1, ϕ_1) depends only on the Stein complex structures J_0, J_1 and not on the functions ϕ_0, ϕ_1.

Two Weinstein homotopies $\mathfrak{W}_t = (W, \omega_t, X_t, \phi_t)$ and $\widetilde{\mathfrak{W}}_t = (\widetilde{W}_t, \widetilde{\omega}_t, \widetilde{X}_t, \widetilde{\phi}_t)$ are called *coarsely equivalent* if there exists a diffeotopy $h_t : W \to \widetilde{W}$ of coarse equivalences between the structures $\mathfrak{W}_t$ and $\widetilde{\mathfrak{W}}_t$.

By a *small adjustment* of a Weinstein homotopy $\mathfrak{W}_t$ we will mean a combination of the following operations:

(i) C^∞-small deformation;

(ii) C^1-small deformation near critical points of functions ϕ_t;

(iii) reparameterization $t \mapsto \alpha(t)$, where $\alpha : [0,1] \to [0,1]$ is a non-decreasing C^∞-function.

In the remaining part of this chapter we prove the following two theorems.

Theorem 14.5. *Let $\mathfrak{W}_t = (W, \omega_t, X_t, \phi_t)$ be a Weinstein homotopy such that $\mathfrak{W}_0 = \mathfrak{W}(J, \phi_0)$ for a Stein structure (J, ϕ_0) on W. Then after, possibly, a small adjustment of the homotopy $\mathfrak{W}_t$, there exists a homotopy of J-convex functions ψ_t such that $\psi_0 = \phi_0$ and the homotopies $\mathfrak{W}_t$ and $\mathfrak{W}(J, \psi_t)$ are coarsely equivalent.*

The above theorem applies both, to the manifold and cobordism cases.

Theorem 14.6. *Let (J_0, ϕ_0) and (J_1, ϕ_1) be two Stein manifold structures on the same manifold V, and $\mathfrak{W}_0 = \mathfrak{W}(J_0, \phi_0)$ and $\mathfrak{W}_1 = \mathfrak{W}(J_1, \phi_1)$ are the corresponding Weinstein structures. Suppose there exists a Weinstein homotopy $\mathfrak{W}_t$ connecting $\mathfrak{W}_0$ and $\mathfrak{W}_1$. Then after, possibly, a small adjustment of the homotopy $\mathfrak{W}_t$, there exists a homotopy of Stein structures (J_t, ϕ_t) such that the homotopies $\mathfrak{W}(J_t, \phi_t)$ and $\mathfrak{W}_t$ are coarsely equivalent.*

For the cobordism case Theorem 14.6 needs to be modified.

Theorem 14.7. *Let (J_0, ϕ_0) and (J_1, ϕ_1) be two Stein cobordism structures on the same manifold W, and $\mathfrak{W}_0 = \mathfrak{W}(J_0, \phi_0)$ and $\mathfrak{W}_1 = \mathfrak{W}(J_1, \phi_1)$ are the corresponding Weinstein structures. Suppose there exists a homotopy $\mathfrak{W}_t$ connecting $\mathfrak{W}_0$ and $\mathfrak{W}_1$, such that the induced homotopies ξ_t of contact structures on $\partial_- W$ can be covered by a homotopy of integrable CR-structures. Then there exists a homotopy of Stein structures (J_t, ϕ_t) such that the homotopies $\mathfrak{W}(J_t, \phi_t)$ and $\mathfrak{W}_t$ are coarsely equivalent.*

Clearly, Theorem 14.6 follows from Theorem 14.6 Indeed, it is clearly sufficient to prove 14.6 for simple homotopies, while the latter case follows from 14.6 inductively applied to cobordisms $\{c_{k-1} \le \phi_t \le c_k\}$, $k = 1, \dots$.

14.3 Special coarse equivalence of cobordisms and homotopies

For the purposes of this chapter let us first slightly expand the notion of an elementary Weinstein cobordism. We say, that a Weinstein cobordism $\mathfrak{W} = (W, \omega, X, \phi)$ is *elementary of type II* if either W contains a unique critical point which is of embryo type (see Section **??** above), or ϕ has exactly two critical points transversely (define!!) connected by a unique X-trajectory.

Elementary cobordisms introduced above in Section 14.1 will be called of type I if we need to distinguish them.

A coarse equivalence between two elementary cobordisms of type II is required by definition to be a Liouville symplectomorphism in a neighborhood of the unique X-trajectory connecting the critical points of ϕ.

A coarse equivalence $h : W \to \widetilde{W}$ between two elementary Weinstein cobordisms $\mathfrak{W} = (W, \omega, X, \phi)$ and $\widetilde{\mathfrak{W}} = (\widetilde{W}, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ is called *special* if h sends trajectories of X to trajectories of $\widetilde{X}$.

Lemma 14.8. *Any coarse equivalence $h : W \to \widetilde{W}$ between two elementary Weinstein cobordisms which near the critical points is a Liouvile diffeomorphism is isotopic to a special one through coarse equivalences.*

Proof. First, extend the diffeomorphism h from the neighborhoods of critical points in the case of type I, or from the neighborhood of the unique trajectory connecting critical points in the case of type II, to neighborhoods of stable and unstable manifolds of critical points of ϕ. In particular, this defines a new contactomorphism h_1 between neighborhoods of unstable spheres in $\partial_- W$ and $\partial_- \widetilde{W}$ in the case of type I, and between neighborhoods of unstable hemispheres in $\partial_- W$ and $\partial_- \widetilde{W}$ in the case of type II (see Section **??** above). This contactomorphism is contactly isotopic to h, and hence can be extended to a globally defined contactomorphism $h_1 : \partial_- W \to \partial_- \widetilde{W}$. Then h_1 uniquely extends trajectory-wise to the rest of W by a diffeomorphism preserving level sets of the functions ϕ and $\widetilde{\phi}$. Clearly, the constructed special coarse equivalence h_1 is isotopic to h through coarse equivalences. □

An *admissible partition* of a Weinstein cobordism $\mathfrak{W} = (W, \omega, X, \phi)$ is a finite sequence $m = c_0 < c_1 < \cdots < c_N = M$ of regular values of ϕ, where we denote $\phi|_{\partial_- W} = m, \phi|_{\partial_+ W} = M$, such that each subcobordism $W_k = \{c_{k-1} \leq \phi \leq c_k\}$, $k = 1, \ldots, N$, is elementary.

One similarly defines an admissible partition of a *Weinstein manifold*, with the only difference that c_i, $i = 0, 1, \ldots,$ form an increasing infinite sequence of regular values of ϕ converging to ∞.

Lemma 11.3 implies that

Lemma 14.9. *Any generic Weinstein cobordism admits an admissible partition into elementary cobordisms of type I.*

Let $m = c_0 < c_1 < \cdots < c_N = M$ be an admissible partition of a Weinstein cobordism $\mathfrak{W}$. Suppose that ε, $0 < \varepsilon < \min\limits_{1 \leq k \leq N} |c_{k+1} - c_k|$, is chosen in such a way that all values in the intervals $[c_k, c_k + \varepsilon]$, $k = 0, \ldots, N-1$, are regular. Let us denote $c_k^+ := c_k + \varepsilon$ for $k = 0, \ldots, N-1$. We further denote

$$W_k := \{c_{k-1} \leq \phi \leq c_k\}, \;\; W_k^\varepsilon := \{c_{k-1}^+ \leq \phi \leq c_k\}$$

for $k = 1, \ldots, N$ and $U_k := \{c_k \leq \phi \leq c_k^+\}$ for $k = 0, \ldots, N-1$.

Given two arbitrary Weinstein cobordisms

$$\mathfrak{W} = (W, \omega, X, \phi) \quad \text{and} \quad \widetilde{\mathfrak{W}} = (\widetilde{W}, \widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$$

and an admissible partition $\mathcal{P}: m = c_0 < c_1 < \cdots < c_N = M$ for $\mathfrak{W}$, we call a coarse equivalence $h : W \to \widetilde{W}$ *special* and compatible with the partition $\mathcal{P}$ if there exists an $\varepsilon > 0$ such that

(i) ϕ has no critical points in $U_k = \{c_k \leq \phi \leq c_k^+\}$, $k = 0 \dots, N-1$;

(ii) $h|_{W_k^\varepsilon}$ is a special equivalence between the elementary Weinstein cobordisms $\mathfrak{W}_k^\varepsilon$ and $\widetilde{\mathfrak{W}}_k^\varepsilon$, where we denote by $\mathfrak{W}_k^\varepsilon$ and $\widetilde{\mathfrak{W}}_k^\varepsilon$ the restrictions of the Weinstein structures $\mathfrak{W}$ to W_k^ε, and $\widetilde{\mathfrak{W}}$ to $\widetilde{W}_k^\varepsilon = h(W^\varepsilon)$, $k = 1, \dots, N$. respectively;

(iii) for each $k = 1, \dots, N$ the diffeomorphism h maps stable manifolds of critical points of ϕ in W_k to the stable manifolds of the corresponding critical points of $\widetilde{\phi}$ in $\widetilde{W}_k$.

maybe better to require preserving X-trajectories along stable manifolds

Remark 14.10. If a special coarse equivalence is compatible with some partition, then it is also compatible with any finer partition.

Lemma 14.11. *Let $h_0 : W \to \widetilde{W}$ be a coarse equivalence between two Weinstein structures $\mathfrak{W}$ and $\widetilde{\mathfrak{W}}$. Then given any partition $\mathcal{P}$ admissible for $\mathfrak{W}$, there exists a homotopy $\mathfrak{W}_t$, $t \in [0,1]$, $\mathfrak{W}_0 = \mathfrak{W}$, of Weinstein structures on W, and diffeotopy $h_t : W \to \widetilde{W}$ of coarse equivalences $\mathfrak{W}_t \to \widetilde{\mathfrak{W}}$, such that h_1 is a special coarse equivalence compatible with the partition $\mathcal{P}$. Moreover, if $\mathfrak{W}$ has the form $\mathfrak{W}(J,\phi)$ for a Stein cobordism structure (J,ϕ) on W and the Weinstein cobordism $\mathfrak{W}$ is Stein near critical points, then the homotopy $\mathfrak{W}_t$ can be chosen in the form $\mathfrak{W}_t = \mathfrak{W}(J,\phi_t)$.*

Proof. We first apply Lemma 14.8 and construct the isotopy $h_t : \bigcup\limits_1^N W_k^\varepsilon \to \bigcup\limits_1^N \widetilde{W}_k^\varepsilon$ to make it special on each elementary cobordism W_k^ε. This isotopy extends to $\bigcup\limits_0^{N-1} U_k$ as isotopy of coarse equivalences.

The only remaining thing to fix is the condition (iii) of the definition of special coarse equivalences. In the Weinstein case we can use Lemma 12.10 to deform the Liouville structure on each U_k, $k = 1, \dots, N-1$, so that the diffeomorphism $h_1|_{U_k}$ would preserve the trajectories of the Liouvile fields. In the Stein case, one can use Proposition 10.1 to deform the function $\phi|_{U_k}$ to make h_1 preserving the stable manifolds of critical points in W_{k+1}. □

The extension of the notion of special coarse equivalence to *Weinstein manifolds*, and an analog of Lemma 14.11 are straightforward.

A family $\mathfrak{W}_t = (W, \omega_t, X_t, \phi_t)$, $t \in [0,1]$, of Weinstein cobordisms is called an *elementary homotopy* of type I, IIb and IId, respectively, if

Type I. $\mathfrak{W}_t$ is an elementary cobordism for all $t \in [0,1]$;

Type IIb. (birth) there is $t_0 \in (0,1)$ such that for $t < t_0$ the function ϕ_t has no critical points, for $t < t_0$ has exactly two critical points of index j and $j+1$, $j = 0, \dots, n-1$, connected by exactly one X-trajectory, and for $t = t_0$ has a unique embryo critical point;

Type IId. (death) there is $t_0 \in (0,1)$ such that for $t > t_0$ the function ϕ_t has no critical points, for $t > t_0$ has exactly two critical points of index j and $j+1$, $j = 0, \dots, n-1$, connected by exactly one X-trajectory, and for $t = t_0$ has a unique embryo critical point.

Let $\mathfrak{W}_t = (W, \omega_t, X_t, \phi_t)$, $t \in [0,1]$, be an elementary homotopy of type I. Let us order critical points $c_1, \dots, c_K \in W$ of ϕ_0 and denote by $c_1(t), \dots, c_k(t)$ the corresponding critical points of ϕ_t, $t \in [0,1]$. Denote $\alpha_j(t) := \phi_t(c_j(t))$, $t \in [0,1]$, $j = 1, \dots, K$. The ordered set of fuctions $(\alpha_1, \dots, \alpha_K : [0,1] \to \mathbb{R}$ is called the *profile* of the elementary homotopy $\mathfrak{W}_t$. Two profiles $(\alpha_1, \dots, \alpha_K)$ and $(\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_K)$ are called *equivalent* if there is a diffeomorphism $[0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ which sends the graphs of functions $\alpha_1, \dots, \alpha_K$ to graphs of functions $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_K$.

Lemma 14.12. *Let $\mathfrak{W}_t$ and $\widetilde{\mathfrak{W}}_t$, $t \in [0,1]$, be two elementary homotopies of type I which have equivalent profiles. Denote by ξ_t and $\widetilde{\xi}_t$ the contact structures induced by $\mathfrak{W}$ and $\widetilde{\mathfrak{W}}$ on $\partial_- W$ and $\partial_- \widetilde{W}$. Let $h_0 : W \to W$ be a coarse equivalence between elementary cobordisms $\mathfrak{W}_0$ and $\widetilde{\mathfrak{W}}_0$, f_t be a Weinstein isomorphism between $\mathfrak{W}_t$ and $\widetilde{\mathfrak{W}}_t$ defined on a neighborhood of critical points of ϕ_t, and $g_t : \partial_- W \to \partial_- \widetilde{W}$ be a contact isotopy $(\partial_- W, \xi_t) \to (\partial_- \widetilde{W}, \widetilde{\xi}_t)$ such that $g_0 = h|_{\partial_- W}$. Then there exists a coarse equivalence $h_t : W \to W$ between the homotopies $\mathfrak{W}_t$ and $\widetilde{\mathfrak{W}}_t$, $t \in [0,1]$, such that $h_0 = h$, $h_t|_{\partial_- W} = g_t$ and $h_t = f_t$ near critical points of ϕ_t.*

Lemma 14.13. *Let $\mathfrak{W}_t$ and $\widetilde{\mathfrak{W}}_t$, $t \in [0,1]$, be two elementary homotopies of the same type IIb or IId. Denote by ξ_t and $\widetilde{\xi}_t$ the contact structures induced by $\mathfrak{W}$ and $\widetilde{\mathfrak{W}}$ on $\partial_- W$ and $\partial_- \widetilde{W}$. Suppose that both homotopies share the same death-birth moment $t_0 \in [0,1]$ and denote by σ the interval $(t_0, 1]$ in the case IIb, and $[0, t_0)$ in the case IId. Let us also consider a slightly bigger interval $\sigma' = [t_0 - \varepsilon, 1]$ or $[0, t_0 + \varepsilon]$ in the cases IIb and IId, respectively. For $t \in \sigma$ let us denote by γ_t the unique X_t-trajectory connecting critical point of the function ϕ_t, and let the notation $\widetilde{\gamma}_t$ have the same meaning for $\widetilde{\phi}_t$. We extend the notation γ_t $\widetilde{\gamma}_t$ to $\sigma' \supset \sigma$ choosing any continuous paths $t \mapsto W$ and $t \mapsto \widetilde{W}$ such that γ_{t_0} and $\widetilde{\gamma}_{t_0}$ are embryo points. Let $h_0 : W \to W$ be a coarse equivalence between elementary cobordisms $\mathfrak{W}_0$ and $\widetilde{\mathfrak{W}}_0$, $g_t : \partial_- W \to \partial_- \widetilde{W}$ be a contact isotopy $(\partial_- W, \xi_t) \to (\partial_- \widetilde{W}, \widetilde{\xi}_t)$ such that $g_0 = h|_{\partial_- W}$, and f_t be a family of Wenstein isomorphisms $\mathcal{O}p\,\Gamma_t \to \mathcal{O}p\,\widetilde{\Gamma}_t$, $t \in \sigma'$. Then there exists a coarse equivalence $h_t : W \to W$ between the homotopies $\mathfrak{W}_t$ and $\widetilde{\mathfrak{W}}_t$, $t \in [0,1]$, such that $h_0 = h$, $h_t|_{\partial_- W} = g_t$, and $h_t = f_t$ on $\mathcal{O}p\,\gamma_t$ for $t \in \sigma'$.*

An *admissible partition* of the homotopy $\mathfrak{W}_t$, $t \in [0,1]$, is a sequence $0 = t_0 < t_1 < \cdots < t_p = 1$ of parameter values, and for each $k = 1,\dots,p$ a finite sequence of functions $m(t) = c_0^k(t) < c_1^k(t) < \cdots < c_{N_k}^k(t) = M(t)$, $t \in [t_{k-1}, t_k]$, where we denote $m(t) := \phi_t|_{\partial_- W}, M(t) := \phi_t|_{\partial W_+}$, such that $c_j^k, j = 0,\dots,N_k(t)$ are regular values of ϕ_t and the restriction of the homotopy $\mathfrak{W}_t$ to each $W_k(t) = \{c_{k-1}(t) \le \phi \le c_k(t)\}$, $k = 1,\dots,N_k$, $t \in [t_{k-1}, t_k]$, is elementary.

A standard general position argument implies that

Lemma 14.14. *Any generic homotopy $\mathfrak{W}_t$, $t \in [0,1]$, of Weinstein cobordism structures on W admits an admissible partition.*

Proposition 14.15. *Let (J_0, ϕ_0) and (J_1, ϕ_1) be two elementary Stein cobordism structures on W. Suppose that there exist*

– *an isotopic to the identity equivalence $h : W \to W$ between the corresponding Weinstein cobordism structures $\mathfrak{W}(J_0, \phi_0)$ and $\mathfrak{W}(J_1, \phi_1)$ and*

– *a homotopy $\widetilde{J}_t$ of J-convex CR-structures on $\partial_- W$ connecting $J_0|_{\partial_- W}$ and $h^* J|_{\partial_- W}$.*

Then there exists

- *a homotopy of integrable complex structures (J_t), $t \in [0,1]$, on W which connects J_0 and J_1 and coincides with $\widetilde{J}_t$ on $\partial_- W$;*
- *a diffeotopy $h_t : W \to W$, $t \in [0,1]$, connecting the identity with h;*
- *a diffeotopy $g_t : \mathbb{R} \to \mathbb{R}$,*

such that for each $t \in [0,1]$, the function $\phi_t = g_t \circ \phi \circ h_t$ is J_t-convex and the diffeomorphism h_t is an equivalence between $\mathfrak{W}(J_0, \phi_0)$ and $\mathfrak{W}(J_t, \phi_t)$.

Remark 14.16. As it is explained above in ??? the second condition in the formulation of the lemma is automatically satisfied in the case $n = 2$.

Proof. To simplify the notation we consider the case when ϕ_0 and ϕ_1 have unique critical points. The general case is similar.

We can assume that J_0 and J_1 have the same underlying real analytic structures. Indeed, all real analytic structures compatible with a given smooth structure are isotopic. We can also assume that the homotopy $\widetilde{J}_t$ is real analytic and thus extends to a homotopy of integrable complex structures on a neighborhood $U \supset \partial_- W$, all compatible with the fixed real analytic structure. Let us define a family ϕ_t of $\widetilde{J}_t$ convex functions without critical points on $\mathcal{O}p\, \partial_- W \subset U$ which are constant on $\partial_- W$. Then there exist diffeotopies h_t, g_t such that $\phi_t = g_t \circ \phi \circ h_t$ on $\mathcal{O}p\, \partial_- W$. Let D_0 and D_1 denote stable discs of critical points p_0 and p_1 of the functions ϕ_0 and ϕ_1.

Let us consider first the case when D_0 and D_1 are real analytic and the index of the critical points is maximal, i.e. equal to $n = \dim_{\mathbb{C}} W$. By assumption, there exists an isotopy D_t, $t \in [0,1]$, between D_0 and D_1 such that D_t is totally real in U and $\widetilde{J}_t$ orthogonally attached to $\partial_- W$. According to **??** one can make this isotopy real analytic. We claim that the family of complex structures $\widetilde{J}_t$ on $\mathcal{O}p\, \partial_- W$ extends as a family of integrable complex structures to $\mathcal{O}p\, D_t$ which for $t = 0,1$ coincide with the given complex structures J_0 and J_1.

Indeed, let us parameterize discs D_t by real analytic embeddings $\gamma_t : D^n \to W$. We can assume that $p_t = \gamma_t(0)$ for $t = 0,1$. Let us denote $A_t := \gamma_t^{-1}(D_t \cap U)$. Consider the complex manifold $\widetilde{W}_t = (U, \widetilde{J}) \underset{x \in A_t^{\mathbb{C}} \sim \gamma_t^{\mathbb{C}}(x) \in U}{\cup} ((D^n)^{\mathbb{C}}, i)$, where we denote by $Y^{\mathbb{C}}$ a $\mathbb{C}^n$-neighborhood of a subset $Y \subset \mathbb{R}^n \subset \mathbb{C}^n$, and by $\gamma_t^{\mathbb{C}}$ the complexification of the real analytic embedding γ_t. We will keep the notation $\widetilde{J}_t$ for the complex structure on $\widetilde{W}_t$. For all t the inclusion $D \cup U \hookrightarrow W$ extends to a smooth embedding $\Gamma_t : \widetilde{W}_t \to W$ onto $U \cup \mathcal{O}p\, D_t$, and hence we can identify $\widetilde{W}_t$ with $\mathcal{O}p\, (\partial_- W \cup D_t)$. On the other hand, for $t = 0,1$ the real analytic totally real embeddings γ_0, γ_1 extend canonically to biholomorphic embeddings of $(D^n)^{\mathbb{C}} \to W$, and hence the embeddings Γ_t can be chosen biholomorphic for $t = 0,1$ and thus we can view $(\widetilde{W}_0, \widetilde{J}_0)$ and $(\widetilde{W}_1, \widetilde{J}_1)$ as holomorphic (codimension 0) submaniifolds of (W, J_0) and (W, J_1), respectively.

Now use Theorem 9.7 to find a family of $\widetilde{J}_t$-convex functions $\widetilde{\phi}_t$ on $\widetilde{W}_t$ such that for each $t \in [0,1]$ the function $\widetilde{\phi}_t$

- extends ϕ_t from $\mathcal{O}p\, \partial_- W$,
- has a critical point of index k with D_t as its stable disc and
- has one of its regular values surround $\partial_- W \cup D_t$.

We will denote by $\widehat{W}_t$ the domain in $\widetilde{W}_t$ bounded by that level. Moreover, using Proposition 4.20 we can arrange that for $t = 0,1$ the function $\widetilde{\phi}_t$ coincides with ϕ_t outside a bigger neighborhood of $\partial_- W \cup D_t$ which is compactly supported in $\widetilde{W}_t$. In the latter case we will keep the notation $\widetilde{\phi}_t$ for thus constructed function on the whole W.

This is a bad reference. One needs to separate an appropriate statement as a theorem in Section 5.

Consider a family of isotopies $g_{t,s} : W \to W$, $t, s \in [0,1]$, such that

- $g_{t,s}$ is the identity on $\mathcal{O}p\, (\partial_- W \cup D_t$ for all $t, s \in [0,1]$;
- $g_{t,0}$ is the identity map $W \to W$, and $g_{t,1}(W) = \widehat{W}_t$ for all $t \in [0,1]$;

Define a family of Stein cobordism structures (J_u, ϕ_u) on W, $u \in [0,1]$, as follows

$$(J_u, \phi_u) = \begin{cases} (g_{0,3u}^* J_0, \widetilde{\phi}_0 \circ g_{0,3u}), & u \in [0, \frac{1}{3}]; \\ (g_{3u-1,1}^* \widetilde{J}_{3u-1}, \widetilde{\phi}_{3u-1} \circ g_{3u-1,1}), & u \in (\frac{1}{3}, \frac{2}{3}]; \\ (g_{1,3-3u}^* J_1, \widetilde{\phi}_1 \circ g_{1,3-3u}), & u \in (\frac{2}{3}, 1]. \end{cases} \tag{14.1}$$

The constructed homotopy $\mathfrak{W}(J_u, \phi_u)$ is in the class of Weinstein structures coarsely equivalent to $\mathfrak{W}(J_0, \phi_0)$. This concludes the proof of Proposition 14.15 for the case $k = n$ and when the discs D_0 and D_1 are real analytic.

If the discs D_0 and D_1 are not really analytic, let us C^2-approximate the parameterizing maps γ_t, $t = 0, 1$, by real analytic totally real embeddings γ'_t which have the same 2-jet at 0 as γ_t. There exist J-convex functions ϕ'_t, $t \in [0,1]$, C^2-close to ϕ_t which have p_t as critical points, and $D'_t = \gamma'_t(D^n)$ as their stable discs. This can be done keeping the condition that the functions ϕ_0 and ϕ_1 are equivalent near critical points via a local biholomorphism. Applying the above construction to the functions ϕ'_0 and ϕ'_1 we construct a family of Stein cobordisms (J_u, ϕ'_u), $u \in [0,1]$. In particularly, the construction ensures existence of a family of local biholomorphisms h_u between a neighborhood G of the critical point p_0 and neighborhoods G_u of the critical points of the function ϕ'_u which sends the function ϕ'_0 to ϕ'_u. Consider a J_0-convex function ϕ''_0, C^2-close to ϕ'_0 on W (and hence to ϕ_0) which coincide with ϕ'_0 outside G and with ϕ_0 on a smaller neighborhood of p_0, and construct a modified family of functions ϕ''_u which are equal to ϕ'_u outside $h_u(G)$ and equal to $\phi''_0 \circ h_u$ on $h_U(G)$. Using criterion 14.12 the Weinstein cobordisms $\mathfrak{W}(J_u, \phi''_u)$ are all equivalent. On the other hand, the linear interpolation $\phi''_{0,t}$ between ϕ_0 and ϕ''_0 (resp. (resp. $\phi''_{1,t}$ between between ϕ''_1 and ϕ_1) consists of J-convex functions equal to ϕ_0 near p_0 (resp. ϕ_1 near p_1), and Hence, we can again apply the criterion 14.12 to conclude that the Weinstein cobordisms $\mathfrak{W}(J_0, \phi''_{0,u})$ as well as $\mathfrak{W}(J_1, \phi_{1,u})$ are all coarsely equivalent. Thus, concatenating the homotopies $(J_0, \phi"_{0,u}), (J_u, \phi''_u)$ and $(J_1, \phi_{1,u})$ we get the required homotopy between Stein cobordisms (J_0, ϕ_0) and (J_1, ϕ_1).

It would be good to have a reference for this in Section 2.

If $k < n$ we can first use the canonical framing of stable discs to extend D_0 and D_1 to totally real embeddings of $D^k \times D^{n-k}_\varepsilon$ for a sufficiently small $\varepsilon > 0$. Then the above proof works without any changes if replace D_0 and D_1 by these extended embeddings.

□

14.4 From Weinstein to Stein homotopies

Theorem 14.6 is a corollary of the following

Proposition 14.17. *Let $\mathfrak{W}_t = (\omega_t, X_t, \phi_t)$, $t \in [0,1]$, be a homotopy of Weinstein cobordism structures on W. Let (J, ψ), $t \in [0,1]$, be a Stein cobordism structure on W. Suppose there exists a course equivalence $h : W \to W$ between $\mathfrak{W}_0$ and $\mathfrak{W}(J, \psi)$. Let ξ_t and ζ be the contact structures induced, respectively, by $\mathfrak{W}_t$ and $\mathfrak{W}(J, \psi)$ on $\partial_- W$, and $g_t : \partial_- W \to \partial_- W$ be an isotopy such that $g_0 = h|_{\partial_- W}$ and $(g_t)_* \xi_t = \zeta$, $t \in [0,1]$. Then after, possibly, a small adjustment of the homotopy $\mathfrak{W}_t$, there exists a family of J-convex functions $\psi_t : W \to \mathbb{R}$ and a coarse equivalence $h_t : W \to W$ between the homotopies $\mathfrak{W}_t$ and $\mathfrak{W}(J, \psi_t)$ such that $h_0 = h$ and $h_t|_{\partial_- W} = g_t$, $t \in [0,1]$.*

Proof. First, we make a small adjustment of $\mathfrak{W}_t$ to ensure that it satisfies the genericity conditions needed for applications of Lemma 14.14. Consider an admissible partition

$$0 = t_0 < t_1 < \cdots < t_p = 1; \quad m(t) = c_0^k(t) < c_1^k(t) < \cdots < c_{N_k}^k(t) = M(t),$$

$t \in [t_{k-1}, t_k]$, $k = 1, \ldots, p$, of the homotopy $\mathfrak{W}_t$. We will assume that the partition points t_i are chosen sufficiently closely surrounding every point t for which the function ϕ_t has an embryo (death-birth) point so that every for elementary cobordism of type II the Weinstein structure is Stein near the the trajectory connecting critical points. Besides, without a loss of generality we can assume that all the hypersurfaces $\Sigma_j^k = \{\phi_t = c_j^k(t)\}$ are independent of t. Let us make the next small adjustment to make the homotopy $\mathfrak{W}_t$ constant near each t_i, $i = 0. \ldots, p$.

We will inductively extend the homotopy ψ_t and h_t to intervals $\Delta_i = [t_{i-1}, t_i], i = 1, \ldots, p$ and for each Δ_i we will do the construction inductively over elementary cobordisms W_j^i bounded by $\partial_- W = \Sigma_{j-1}^i$ and $\partial_+ W = \Sigma_j^i$. Suppose that ψ_t and h_t are already constructed for $t \leq t_{i-1}$. Using Lemma 14.11 we can arrange that $h_{t_{i-1}}$ is a special coarse equivalence compatile l with the partition of W into elementary cobordisms W_j^i, $j = 1, \ldots, N_j$. We will also assume that for $t \in \Delta_i$ the families h_t and ψ_t are already constructed on $\bigcup\limits_{j \leq k-1} W_j^i$. Denote $g_t := h_t|_{\Sigma_{k-1}^i}$. To simplify the notation we will assume that the interval Δ_i is $[0, 1]$, denote $h := h_{t_{i-1}}$, and write W instead of W_k^i.

We will consider separately cases when it is of type I,IIb and IId.

Type I. First make a small adjustment of $\mathfrak{W}_t$, so that near critical points of ϕ_t $\mathfrak{W}_t$ coincides with $\mathfrak{W}(J, \psi = \psi_0)$ for all $t \in \Delta$. Consider the profile $(\alpha_1, \ldots, \alpha_K)$ of the homotopy $\mathfrak{W}_t$. Using Proposition 10.6 we can construct a family of J-convex functions ψ_t on W which has an equivalent profile. Then, according to Lemma 14.12 the homotopies $\mathfrak{W}_t$ and $\mathfrak{W}(J, \psi_t)$ are coaresely equivalent, and there exists a coarse equivalence $h_t : W \to W$ such that $h_0 = h$ and $h_t|_{\partial_- W} = g_t$.

Type IIb. In this case the function $\psi = \psi_0$ has no critical points. The function ϕ_t has no critical points, for $t \in (t_0, 1]$ it has two critical points, p_t and q_t of index k and $k-1$, respectively, so that we have $\phi_t(p_t) > \phi_t(q_t)$, and for $t_0 \in (0, 1)$ the function ϕ_{t_0} has an embryo type singularity $p \in W$. As in Lemma 14.13 we denote by γ_t the unique X_t-trajectory connecting p_t and q_t, $t \in (t_0, 1]$ and for $t \in [t_0 - \varepsilon, t_0$ we choose any continuous path $t \mapsto \gamma_t \in \operatorname{Int} W$ such that γ_{t_0} is the embryo point. Let us use Proposition 10.8 to construct a creation family of J-convex functions $\psi_t : W \to \mathbb{R}$ with $\psi_0 = \psi$ such that the birth moment is t_0. Moreover, we can arrange that ψ_t has the same critical points p_t, q_t as the function ϕ_t, and that γ_t serves as the unique gradient trajectory of ψ_t connecting the critical points. Next, we use Lemma **??** to make a small adjustment of $\mathfrak{W}_t$ to make it isomorphic to $\mathfrak{W}(J, \psi_t)$ on $\mathcal{O}p\, \gamma_t$, $t \in [t_0 - \varepsilon, 1]$. It remains to apply

Lemma 14.13 to construct the required coarse equivalence $h_t : W \to W$between $\mathfrak{W}_t$ and $\mathfrak{W}(J, \psi_t)$, $t \in [0,1]$.

Type IId. The proof is similar to the case IIb using Proposition 10.9 instead of Proposition 10.8. □

Proposition 14.18. *Let (J_0, ϕ_0) and (J_1, ϕ_1) be two Stein cobordism structures on a manifold W. Suppose there exists a family $\mathfrak{W}_t$, $t \in [0,1]$, of Weinstein cobordisms connecting $\mathfrak{W}_0 = \mathfrak{W}(J_0, \phi_0)$ and $\mathfrak{W}_1 = \mathfrak{W}(J_1, \phi_1)$. Then (J_0, ϕ_0) and (J_1, ϕ_1) can be connected by a family (J_t, ϕ_t), $t \in [0,1]$ of Stein cobordisms such that, after a possible small adjustment of $\mathfrak{W}_t$, the homotopies $\mathfrak{W}_t$ and $\mathfrak{W}(J_t, \phi_t)$, $t \in [0,1]$ are coarsely equivalent.*

Proof. Let us first use Proposition 14.17 to construct a homotopy of Stein cobordisms (J_0, ϕ'_t), $t \in [0,1]$ such that $\phi'_0 = \phi_0$ and such that (after a possible small adjustment of $\mathfrak{W}_t$) the homotopies $\mathfrak{W}(J_0, \phi'_t)$ and $\mathfrak{W}_t$ are coarsely equivalent. Next, we will use Proposition 14.15 to construct a homotopy (J_t, ψ_t), $t \in [0,1]$, connecting (J_0, ϕ'_1) and (J_1, ϕ_1) in the class of cobordisms coarsely equivalent to (J_1, ϕ_1). To do that let us subdivide the cobordism (W, J_1, ϕ_1) into elementary cobordisms: $W = W_1 \cup \dots \cup W_N$. Using Proposition 14.15 we can construct the required family (J_t, ψ_t) on W_1, such that $\mathfrak{W}(W_1, J_t, \psi_t)$ is equivalent to $\mathfrak{W}(W_1, J_1, \phi_1)$. Next, we inductively extend the homotopy to $W_2, \dots W_N$. As it was pointed out above in Remark 14.10(iii) equivalent of elementary cobordisms do not necessarily can be glued together into coarsely equivalent cobordisms, because the condition (v) in the definition of coarse equivalence need not to be necessarily satisfied. However, as it is explained in this remark one can use Lemma 10.4 to change the family ψ_t in a small neighborhoods U_k of $\partial_- W_k$, $k = 1, \dots, N$ in order to satisfy this conditon as well. Thus for the constructed homotopy (J_t, ψ_t) the corresponding Weinstein homotopy $\mathfrak{W}(W, J_t, \psi_t)$ is in the class of Weinstein cobordisms coarsely equivalent to $\mathfrak{W}(W, J_1, \phi_1)$.

□

Proof of Theorem 14.7. According to the definition of homotopy of Weinstein structures there is a finite partition $0 = t_0 < t_1 < \dots < t_N = 1$ such that for each interval $\Delta_k = [t_{k-1}, t_k]$, $k = 1, \dots, N$, there exists a sequence of continuous functions $c^1_k(t) < \dots < c^2_k(t) < \dots$, $t \in \Delta_k$, such that all $c^k_j(t)$ are regular values for ϕ_t for all $t \in \Delta_k$ and $\bigcup\limits_j \{\phi_t \le c_j(t)\} = V$. Without a loss of generality we can assume that all the functions $c^j_i(t)$ are constant on Δ_i. Indeed, there exists an isotopy $g_t : V \to V$, $t \in \Delta_1$, such that $g^i_0 = \mathrm{Id}$ and $g^i_t(\{\phi_0 = c^j_1(0)\}) = \{\phi_t = c^j_1(t)\}$ for all $j = 1, \dots$. Pulling back the Weinstein structure $\mathfrak{W}_t$ by g_t makes the functions c^j_1 contant. Continuing this process for $j = 2, \dots, N$ we make all functions $c^j_i(t)$, $i = 1, \dots, N$, $j = 1, \dots$, constant.

First, apply consequently Proposition 14.17 to the homotopy $\mathfrak{W}_t$, $t \in \Delta_1$, restricted to $W_1 = \{\phi_0 \le c_1^1\}$, $W_2 = \{c_1^1 \le \phi_0 \le c_1^2\}, \dots$ to construct a family of J_0-convex functions $\psi_t : V \to \mathbb{R}$ such that the homotopies $\mathfrak{W}(V, J, \psi_t)$ and $\mathfrak{W}_t$, $t \in \Delta_1$ are coarsely equivalent. Next, repeat the construction for extending the family ψ_t to $\Delta_2, \dots, \Delta_N$.

□

Chapter 15

Subcritical Stein and Weinstein structures

15.1 Morse cobordisms

Let W be a smooth oriented cobordism between $\partial_- M$ and $\partial_+ M$, $\phi : W \to [m, M]$ be a Morse function such that $\phi|_{\partial_- W} = m, \phi|_{\partial_+ W} = M$, and X is a gradient vector field of ϕ for some Riemannian metric g on W. We will call the triple(W, X, ϕ) a *Morse cobordism.* Any Weinstein cobordism $\mathfrak{W}$ has an underlying Morse cobordism $\mathfrak{M}\mathfrak{W}$.

Mimicking the defined above notions of (coarse) equivalence of Weinstein cobordisms and Weinstein homotopy.

Proposition 15.1. *Let $\mathfrak{M}_t = (W, \underline{X}_t, \phi_t)$ be a homotopy of Morse cobordisms, such that $\mathfrak{M}_0 = \mathfrak{M}\mathfrak{W}$ for a a $2n$-dimensional Weinstein cobordism $\mathfrak{W}$. Suppose that for all $t \in [0, 1]$ the function ϕ_t has no critical points of index $\geq n$. Then there exists a Weinstein homotopy $\mathfrak{W}_t$ with $\mathfrak{W}_0 = \mathfrak{W}$ for which $\mathfrak{M}_t$ and $\mathfrak{W}_t$ are coarsely equivalent.*

Part IV

Additional topics

Chapter 16

Stein manifolds of complex dimension two

Chapter 17

Weinstein structures and Lefschetz fibrations

Chapter 18

Stein manifolds in symplectic topology

Appendix A

Immersions and embeddings

Some homotopy groups. Here we collect some results on homotopy groups that will be used in this book. For $1 \leq k \leq n$ denote by $V_{n,k}$ the *Stiefel manifold* of orthonormal k-frames in $\mathbb{R}^n$, and by $G_{n,k}$ the *Grassmannian* of k-dimensional subspaces in $\mathbb{R}^n$. The obvious projection $p : G_{n,k} \to V_{n,k}$ defines a fibration

$$O(k) \to V_{n,k} \to G_{n,k}$$

with fibre the orthogonal group $O(n)$. For $\ell < k \leq n$ the map $V_{n,k} \to V_{\ell,k}$ that forgets the last $k-\ell$ vectors defines a fibration

$$V_{n-\ell,k-\ell} \to V_{n,k} \to V_{n,\ell}.$$

Here an explicit inclusion $V_{n-\ell,k-\ell} \hookrightarrow V_{n,k}$ is given by adding to a $(k-\ell)$-frame in $\mathbb{R}^{n-\ell} \times \{0\} \subset \mathbb{R}^n$ the last ℓ standard basis vectors. Note that $V_{n,n} \cong O(n)$ and $V_{n,1} \cong S^{n-1}$. Thus the preceding fibration includes the following special cases:

$$V_{n-1,k-1} \to V_{n,k} \to S^{n-1}, \tag{A.1}$$

$$O(n-k) \to O(n) \to V_{n,k}, \tag{A.2}$$

$$O(n-1) \to O(n) \to S^{n-1}. \tag{A.3}$$

Of course, the preceding discussion carries over to the complex case: Just replace everywhere $V_{n,k}$ by the complex Siefel manifold $V^{\mathbb{C}}_{n,k}$, $G_{n,k}$ by the complex Grassmannian $G^{\mathbb{C}}_{n,k}$, $O(n)$ by the unitary group $U(n)$, and S^{n-1} by S^{2n-1}.

Lemma A.1. *(a) The map $\pi_i V_{n-1,k-1} \to \pi_i V_{n,k}$ induced by the inclusion is an isomorphism for $i < n-2$ and surjective for $i = n-2$. Similarly, the map $\pi_i V^{\mathbb{C}}_{n-1,k-1} \to \pi_i V^{\mathbb{C}}_{n,k}$ is an isomorphism for $i < 2n-2$ and surjective for $i = 2n-2$.*

(b) $V_{n,k}$ *is* $(n-k-1)$*-connected and* $V^{\mathbb{C}}_{n,k}$ *is* $(2n-2k)$*-connected.*

(c) For $n \geq k+2$*, the group* $\pi_k V_{n,n-k}$ *equals* $\mathbb{Z}$ *if* k *is even or* $k=1$*, and* $\mathbb{Z}_2$ *if* $k>1$ *is odd.*

Proof. Part (a) follows directly from the long exact sequence of the fibration (A.1) because S^{n-1} is $n-2$-connected. For Part (b), let $i<n-k$. Then it follows by induction from Part (a) that $\pi_i V_{n,k} = \pi_i V_{n-k+1,1} = \pi_i S^{n-k} = 0$. The complex cases are analogous.

For part (c), let $n \geq k+2$ and $k \geq 2$ (the case $k=1$ is trivial). Then it follows by induction from part (a) that $\pi_k V_{n,n-k} = \pi_k V_{k+2,2}$. Now observe that an element of $V_{k+2,2}$ is a unit vector in $\mathbb{R}^{k+2}$ and a second unit vector orthogonal to the first one. Thus $V_{k+2,2}$ equals the tangent sphere bundle of S^{k+1} and the fibration (A.1)

$$V_{k+1,1} \cong S^k \to V_{k+2,2} \to S^{k+1}$$

describes this bundle. Now for an oriented sphere bundle $S^k \to E \to B$, the boundary map $\pi_{k+1}B \to \pi_k S^k \cong \mathbb{Z}$ in the long exact sequence is given by evaluation of the Euler class $e(E) \in H^{k+1}(B)$ (this follows directly from the definition of the obstruction cocycle representing the Euler class in [60]). Thus the fibration above yields an exact sequence

$$\pi_{k+1}S^{k+1} \cong \mathbb{Z} \xrightarrow{\cdot\chi(S^{k+1})} \pi_k S^k \cong \mathbb{Z} \to \pi_k V_{k+2,2} \to 0,$$

where the first map is multiplication with the Euler characteristic of S^{k+1}. Since $\chi(S^{k+1})$ is 0 for k even and 2 for k odd, it follows that $\pi_k V_{n,n-k} = \pi_k V_{k+2,2}$ equals $\mathbb{Z}$ for k even and $\mathbb{Z}_2$ for k odd. □

In particular, setting $k=n$ in Lemma A.1 (a) we find

Corollary A.2. *The map* $\pi_i O(n-1) \to \pi_i O(n)$ *induced by the inclusion is an isomorphism for* $i<n-2$ *and surjective for* $i=n-2$*. Similarly, the map* $\pi_i U(n-1) \to \pi_i U(n)$ *is an isomorphism for* $i<2n-2$ *and surjective for* $i=2n-2$*.*

Define the *stable homotopy groups* $\pi_i O := \pi_i O(n)$ for $i<n-1$ and $\pi_i U := \pi_i U(n)$ for $i<2n$ (this is independent of n by the preceding corollary). These groups are determined by the celebrated

Theorem A.3 (Bott Periodicity Theorem [7])**.** *(a) The stable homotopy group* $\pi_i U$ *equals* 0 *if* i *is even and* $\mathbb{Z}$ *if* i *is odd.*

(b) The stable homotopy group $\pi_i O$ *equals* $\mathbb{Z}_2$ *if* $i \equiv 0$ *or* 1 *(mod 8),* $\mathbb{Z}$ *if* $i \equiv 3$ *or* 7 *(mod 8), and* 0 *otherwise.*

The h-principle for immersions. Fix integers $1 \leq k < n$. Let $f: D^k \to \mathbb{R}^n$ be an immersion of the closed k-disk into $\mathbb{R}^n$ with $f(x)=(x,0)$ near ∂D^k. Its differential yields a fibrewise injective bundle homomorphism $df: T(D^k) =$

$D^k \times \mathbb{R}^k \to \mathbb{R}^n$. Taking the images of the standard basis vectors $e_1, \dots, e_k \in \mathbb{R}^k$, this can be viewed as a map $df : D^k \to V_{n,k}$ to the Stiefel manifold of k-frames in $\mathbb{R}^n$ satisfying $df(x) = (e_1, \dots, e_k) =: v_0$ for $x \in \partial D^k$. Thus $df : (D^k, \partial D^k) \to (V_{n,k}, v_0)$ represents an element

$$\Omega(f) := [df] \in \pi_k V_{n,k} = \pi_k(V_{n,k}, v_0)$$

which we call the *Smale invariant* of f. Clearly $\Omega(f)$ is invariant under regular homotopies of f fixed near ∂D^k. The following is the simplest version of the h-principle for immersions, proved by Smale first for $k = 2$ [58] and then in general.

Theorem A.4 (Smale [59])**.** *For $k < n$, Ω defines a bijection between regular homotopy classes of immersions $f : D^k \to \mathbb{R}^n$ with $f(x) = (x, 0)$ near ∂D^k and $\pi_k V_{n,k}$.*

Although we will only use this version, let us mention some generalizations. Smale [59] extended the result to immersions of spheres as follows. Fix base points $x_0 \in V_k(S^k)$ in the frame bundle of S^k and $y_0 \in \mathbb{R}^n \times V_{n,k}$ and call an immersion $f : S^k \to \mathbb{R}^n$ *based* if $df(x_0) = y_0$. Consider two based immersions $f, g : S^k \to \mathbb{R}^n$. After a small perturbation, we may assume that dg agrees with df in a neighborhood of x_0. Cutting out this neighborhood, we obtain maps $df, dg : D^k \to V_{n,k}$ that agree on ∂D^k. The continuous map $S^k \to V_{n,k}$ that equals df on the upper and dg on the lower hemisphere represents a homotopy class $\Omega(f, g) \in \pi_k V_{n,k}$. Clearly $\Omega(f, g)$ depends only on the based regular homotopy classes of f and g. We call $\Omega(f, g)$ the *relative Smale invariant* of f and g.

Theorem A.5 (Smale [59])**.** *Fix a based immersion $f : S^k \to \mathbb{R}^n$, $k < n$. Then $g \mapsto \Omega(f, g)$ defines a bijection between based regular homotopy classes of immersions $g : S^k \to \mathbb{R}^n$ and $\pi_k V_{n,k}$.*

Remark A.6. The theorem holds for non-based immersions provided that $\pi_1 V_{n,k}$ acts trivially on $\pi_k V_{n,k}$. E.g., this is the case if $n > k + 1$ (because then $\pi_1 V_{n,k} = 0$) or if $k = 2$ and $n = 3$ (because $\pi_2 V_{3,2} = 0$). The latter case gives the famous "sphere eversion" [58]: The standard sphere $S^2 \subset \mathbb{R}^3$ can be turned inside out by a regular homotopy.

Immersions of half dimension. Observe that $\pi_n V_{2n,n}$ equals $\mathbb{Z}$ for n even and $\mathbb{Z}_2$ for $n > 1$ odd, which suggests that for immersions of half dimension the isomorphism of Theorem A.5 may correspond to the self-intersection index. This is indeed the case:

Theorem A.7 (Smale [59])**.** *The self-intersection index defines a bijection between regular homotopy classes of immersions $S^n \to \mathbb{R}^{2n}$ (or $D^n \to \mathbb{R}^{2n}$ standard near ∂D) to $\mathbb{Z}$ (for n even) resp. $\mathbb{Z}_2$ (for $n > 1$ odd).*

We will reproduce below the short proof of this theorem from [59]. It is based on some results by Lashof and Smale [44]. Let M^k be a closed oriented connected

k-manifold, $k \geq 1$. To an immersion $f : M^k \to \mathbb{R}^{2k}$ we can assign, besides its self-intersection index, several other invariants. Denote by ν_f the oriented normal bundle of f and by $\chi(\nu_f) \in \mathbb{Z}$ its Euler characteristic (i.e., its Euler class evaluated on the fundamental class of M). We have a map $S(\nu_f) \to S^{2k-1}$, $(x, v) \mapsto v$ from the normal sphere bundle of f to the unit sphere in $\mathbb{R}^{2k}$. Its mapping degree $d_\nu(f) \in \mathbb{Z}$ is called the *normal degree of* f. Similarly, the map $SM \to S^{2k-1}$, $(x, v) \mapsto d_x f \cdot v$ from the tangent sphere bundle of M gives rise to the *tangential degree* $d_\tau(f) \in \mathbb{Z}$. Of course, the Euler characteristic of the tangent bundle TM is just the usual Euler characteristic $\chi(M)$ of M. Finally, f induces the *tangential map* $T_f : M \to G_{2k,k}$, $x \mapsto df(T_x M)$ to the Grassmannian of oriented k-planes in $\mathbb{R}^{2k}$. Clearly, the numbers $\chi(\nu_f)$, $d_\nu(f)$, $d_\tau(f)$ and the homotopy class of T_f are invariant under regular homotopies.

Theorem A.8 (Lashof and Smale [44]). *For an immersion $f : M^k \to \mathbb{R}^{2k}$, $k \geq 1$, of a closed oriented connected manifold the following holds:*

(a) $d_\tau(f) = -\chi(\nu_f)$.

(b) $d_\nu(f) = \chi(M)$.

(c) If k is even then $d_\tau(f) = 2I_f$.

(d) Let $f, g : M^k \to \mathbb{R}^{2k}$ be two immersions, $k \geq 2$, satisfying $\chi(\nu_f) = \chi(\nu_g)$ if k is even. Then the tangential maps induce the same map $T_f^* = T_g^* : H^*(G_{2k,k}) \to H^*(M)$ on integral cohomology.

Remark A.9. Statements (b) and (d) have generalizations to immersions $M^k \to \mathbb{R}^n$ not of half dimension in terms of integral Stiefel-Whitney classes, see [44].

Proof of Theorem A.7. By Theorem 6.2, I_f attains every possible value. So it remains to show that $I_f = I_g$ for two immersions $f, g : S^k \to \mathbb{R}^{2k}$ implies that f and g are regularly homotopic.

If $k > 1$ is odd, by Theorem A.5 regular homotopy classes are in one-to-one correspondence to $\pi_k V_{2k,k} = \mathbb{Z}_2$. Since I is surjective onto $\mathbb{Z}_2$, it must be bijective.

For k even consider the commutative diagram

$$\begin{array}{ccccc} \pi_k(S^k) & \xrightarrow{T_{f\#}, T_{g\#}} & \pi_k(G_{2k,k}) & \xleftarrow{p_\#} & \pi_k(V_{2k,k}) \\ \downarrow h_0 & & \downarrow h_1 & & \downarrow h_2 \\ H_k(S^k) & \xrightarrow{T_{f*}, T_{g*}} & H_k(G_{2k,k}) & \xleftarrow{p_*} & H_k(V_{2k,k}), \end{array}$$

where $p : V_{2k,k} \to G_{2k,k}$ is the projection, $T_f, T_g : S^k \to G_{2k,k}$ are the tangential maps, and h_0, h_1, h_2 are the Hurewicz maps. By definition of Ω and the tangential map we have $p_\# \Omega(f, g) = (T_{f\#} - T_{g\#})[S^k]$. Now suppose $I_f = I_g$. Then $\chi(\nu_f) = \chi(\nu_g)$ by Theorem A.8 (a) and (c), hence $T_f^* = T_g^*$ by Theorem A.8 (d), and therefore $T_{f*} = T_{g*}$ on homology. By the diagram, this implies

$$0 = (T_{f*} - T_{g*})h_0[S^k] = h_1(T_{f\#} - T_{g\#})[S^k] = h_1 p_\# \Omega(f, g) = p_* h_2 \Omega(f, g).$$

Now $V_{2k,k}$ is $(k-1)$-connected (Lemma A.1), so h_2 is an isomorphism by the Hurewicz theorem. As p_* is also injective ([44]), it follows that $\Omega(f,g)=0$. But then f and g are regularly homotopic by Theorem A.5. □

Now consider a family of immersions $f_t : D^{n-1} \to \mathbb{R}^{2n-1}$, $t \in [0,1]$, with $f_t(x) = (x,0)$ for (x,t) near $\partial(D^{n-1} \times [0,1])$. It induces an immersion

$$F : D^{n-1} \times [0,1] \to \mathbb{R}^{2n}, \qquad (x,t) \mapsto \big(f_t(x), t\big)$$

with $F(x,t) = (x,t,0)$ for (x,t) near $\partial(D^{n-1} \times [0,1])$. Let I_F be its self-intersection index.

On the other hand, let $\beta_t : D^{n-1} \times \mathbb{R}^n \to \mathbb{R}^{2n-1}$ be a family of normal framings for f_t, i.e. $df_t \oplus \beta_t : D^{n-1} \times \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ is a fibrewise orthogonal isomorphism, with $\beta_t(x)v = (0,v)$ for (x,t) near $\partial D^{n-1} \times [0,1] \cup D^{n-1} \cup \{0\}$. The restriction $\beta_1 : D^{n-1} \times \mathbb{R}^n \to \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{2n-1}$ equals fibrewise the identity for x near ∂D^{n-1} and thus represents an element $[\beta_1] \in \pi_{n-1}O(n)$. Its image in $\pi_{n-1}O(2n-1)$ is represented by $df_1 \oplus \beta_1$, which is homotopic via $df_t \oplus \beta_t$ to the constant map $df_0 \oplus \beta_0$. So

$$[\beta_1] \in K := \ker[\pi_{n-1}O(n) \to \pi_{n-1}O,$$

where we have used that $\pi_{n-1}O(2n-1)$ equals the stable group $\pi_{n-1}O$.

Proposition A.10. *The element $[\beta_1] \in K$ depends only on the self-intersection index I_F and the map $I_F \mapsto [\beta_1]$ is surjective onto K.*

Proof. In view of Theorem A.4, Theorem A.7 and Remark 6.5, we may replace I_F by the element $\Omega(F) \in \pi_n V_{2n,n}$ represented by F. The fibrewise injective differentials $df_t : D^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}^{2n-1}$ define a map $df : D^{n-1} \times [0,1] \to V_{2n-1,n-1}$ which represents an element $\Omega(f) \in \pi_n V_{2n-1,n-1}$. Since

$$dF = \begin{pmatrix} df_t & \frac{\partial f_t}{\partial t} \\ 0 & 1 \end{pmatrix}$$

is homotopic to $df_t \oplus 1$ through fibrewise injective maps, $\Omega(f)$ maps to $\pm\Omega(F)$ under the natural map $\pi_n V_{2n-1,n-1} \to \pi_n V_{2n,n}$. Since this map is an isomorphism by Lemma A.1, we may replace $\Omega(F)$ by $\Omega(f)$.

Now consider the fibre bundle

$$O(n) \cong V_{n,n} \to O(2n-1) \cong V_{2n-1,2n-1} \to V_{2n-1,n-1}$$

(a special case of (A.1)). We are given a map $df : D^{n-1} \times [0,1] \to V_{2n-1,n-1}$ which equals the basepoint $v_0 := (e_1, \dots, e_{n-1})$ near $\partial(D^{n-1} \times [0,1])$, and a lift $df \oplus \beta : D^{n-1} \times [0,1] \to V_{2n-1,2n-1}$ which equals the basepoint $w_0 := (e_1, \dots, e_{2n-1})$ near $\partial D^{n-1} \times [0,1] \cup D^{n-1} \times \{0\}$. Hence $[\beta_1] = \partial\Omega(f)$ by definition of the boundary map in the homotopy exact sequence

$$\pi_n V_{2n-1,n-1} \xrightarrow{\partial} \pi_{n-1}O(n) \xrightarrow{i_\#} \pi_{n-1}O(2n-1).$$

This proves that $[\beta_1]$ depends only on $\Omega(f)$, and the map $\Omega(f) \mapsto [\beta_1]$ is surjective because ∂ is surjective onto $K = \ker(i_\#)$. □

Remark A.11. In Proposition A.10, the self-intersection index I_F takes values in $\mathbb{Z}$ if n is even or $n = 1$ and $\mathbb{Z}_2$ if $n > 1$ is odd. The kernel K is isomorphic to $\mathbb{Z}$ for n even, 0 for $n = 1, 3, 7$, and $\mathbb{Z}_2$ for $n \neq 1, 3, 7$ odd (this is essentially proved in [60], see Appendix 5 of [42]).

Isotopies of embeddings. Finally, we briefly discuss *isotopies*, i.e. homotopies through embeddings.

Theorem A.12 (Haefliger [36]). *Let M^k, N^n be manifolds of dimensions k, n, M closed, such that M is q-connected and N is $(q-1)$-connected for some $q \geq 0$. Suppose that $n \geq 2k + 2 - q$ and $2q < k + 1$. Then any two embeddings $M \hookrightarrow N$ are isotopic.*

The case $q = 0$ is due to Whitney [64]. In the case $q = 1$ we obtain

Corollary A.13. *For $k > 1$ any two embeddings $M^k \hookrightarrow N^{2k+1}$ of a closed connected k-manifold into a simply connected $(2k+1)$-manifold are isotopic. In particular, this holds for embeddings $S^k \hookrightarrow \mathbb{R}^{2k+1}$ with $k > 1$.*

Remark A.14. (1) The relative Smale invariant gives no obstruction to regular homotopies of maps $S^k \to \mathbb{R}^{2k+1}$ because it takes values in $\pi_k V_{2k+1,k} = 0$ (Lemma A.1).

(2) Corollary A.13 fails for $k = 1$: There are many non-isotopic knots $S^1 \hookrightarrow \mathbb{R}^3$.

Appendix B

The Thurston-Bennequin invariant

The rotation invariant. Let (M, ξ) be a contact manifold and choose a compatible almost complex structure J on ξ. Given an isotropic immersion $f : \Lambda \to (M, \xi)$, the space $df(T_p\Lambda)$, $p \in \Lambda$, is isotropic in $(\xi_{f(p)}, d\alpha)$ and thus totally real in $(\xi_{f(p)}, J)$. Hence $d_pf : T_p\Lambda \to \xi_{f(p)}$ extends to a complex monomorphism $d_pf \otimes \mathbb{C} : T_p\Lambda \otimes \mathbb{C} \to \xi_{f(p)}$ defined by

$$d_pf \otimes \mathbb{C}(X + iY) := d_pf \cdot X + Jd_pf \cdot Y.$$

The homotopy class of $df \otimes \mathbb{C}$ in the space of complex monomorphisms $T\Lambda \otimes \mathbb{C} \to \xi$ is invariant under isotropic regular homotopies of f. We call it the *rotation invariant* of f and denote it by $r(f)$.

The following h-principle states that the rotation invariant is the only invariant of isotropic immersions. It was proved by Gromov in 1971 ([31], see also [32], [18]).

Theorem B.1 (h-principle for isotropic immersions and embeddings)**.** *Let (M^{2n+1}, ξ) be a contact manifold and Λ^k a manifold, $k \leq n$.*

(a) The rotation invariant defines a homotopy equivalence from isotropic immersions $\Lambda \to M$ to complex monomorphisms $T\Lambda \otimes \mathbb{C} \to \xi$.

(b) For every continuous map $f : \Lambda \to M$ there exists a C^0-small homotopy f_t to an isotropic embedding f_1. If f is an embedding f_t can be chosen to be a smooth isotopy. If f is an isotropic immersion f_t can be chosen to be a C^∞-small isotropic regular homotopy.

(c) In the subcritical case *$k < n$, the rotation invariant defines a homotopy equivalence from isotropic* embeddings *$\Lambda \to M$ to complex monomorphisms $T\Lambda \otimes \mathbb{C} \to \xi$ covering embeddings. Moreover, an isotropic regular homotopy $f_t : \Lambda \to M$ between isotropic embeddings f_0, f_1 can be deformed, through C^∞-close isotropic regular homotopies with fixed ends, to an isotropic isotopy.*

The rotation invariant for Legendrian spheres in $\mathbb{R}^{2n+1}$. Given a Legendrian immersion $f : S^n \to (\mathbb{R}^{2n+1}, \xi_0)$, its Lagrangian projection $P_{\rm Lag} \circ f : S^n \to \mathbb{R}^{2n}$ is a Lagrangian immersion with respect to the standard symplectic form $dp \wedge dq$ on $\mathbb{R}^{2n}$. Since the Lagrangian projection maps each contact hyperplane isomorphically onto $\mathbb{R}^{2n}$, the standard complex structure i on $\mathbb{R}^{2n} = \mathbb{C}^n$ induces via this projection a compatible almost complex structure J on ξ. The class of $df \otimes \mathbb{C}$ in the space of complex isomorphisms $TS^n \otimes \mathbb{C} \to (\xi, J)$ can thus be identified with the class of $d(P_{Lag} \circ f) \otimes \mathbb{C}$ in the space of complex isomorphisms $TS^n \otimes \mathbb{C} \to \mathbb{C}^n$. Picking Hermitian metrics on $TS^n \otimes \mathbb{C}$ and $\mathbb{C}^n$, we can reduce the space of complex isomorphisms to the space of unitary isomorphisms $U(TS^n \otimes \mathbb{C}, \mathbb{C}^n)$. On this space the group of continuous maps $S^n \to U(n)$ acts freely and transitively by pointwise composition, so the complex isomorphisms associated to two Legendrian immersions $f, g : S^n \to \mathbb{R}^{2n+1}$ differ by a map $S^n \to U(n)$. We call the homotopy class of this map the *relative rotation invariant*

$$r(f, g) \in \pi_n U(n) = \begin{cases} \mathbb{Z} & n \text{ odd }, \\ 0 & n \text{ even }. \end{cases}$$

Remark B.2. The rotation invariant $r(f, g)$ of two Legendrian immersions $f, g : S^n \to \mathbb{R}^{2n+1}$ is related to their Smale invariant $\Omega(f, g)$ by

$$\Omega(f, g) = i_{n\#} r(f, g),$$

where $i_n : U(n) \hookrightarrow V_{2n,n}$ is the natural inclusion (viewing the columns of a unitary matrix as a real n-frame).

Remark B.3. For $n = 1$ the rotation number $r(f)$ is just the *winding number* (i.e., the degree of the Gauss map) of the immersion $P_{Lag} \circ f : S^1 \to \mathbb{R}^2$. It can also easily be computed from the front projection $P_{front} \circ f$ which is generically the oriented graph of a multivalued function with transverse self-intersections and a finite number of standard cusps (see Figure [fig:???]). The cusps correspond to vertical points of the Lagrangian projection, i.e., points where $P_{Lag} \circ f$ is parallel to the p-axis. Thus the winding number of $P_{Lag} \circ f$ is given by

$$r(f) = \frac{1}{2}\Big(\#(\text{up} - \text{cusps}) - \#(\text{down} - \text{cusps})\Big)$$

(see Figure [fig:???]).

The Thurston-Bennequin invariant. Legendrian embeddings possess an additional invariant, the Thurston-Bennequin invariant. It was defined by Bennequin [5] in dimension 3 and generalized to higher dimensions by Tabachnikov [62].

Let $\Lambda^n \subset (M^{2n+1}, \xi = \ker \alpha)$ be a closed orientable Legendrian submanifold. Suppose first that the homology class $[\Lambda] \in H_n(M)$ is trivial. Push Λ slightly in the direction of the Reeb vector field to a submanifold Λ' disjoint from Λ and define the *Thurston-Bennequin invariant* as the linking number of Λ and Λ',

$$\text{tb}(\Lambda) := \text{lk}(\Lambda, \Lambda').$$

Here the *linking number* is defined as the algebraic intersection number $Z \cdot \Lambda'$ of Λ' with an $(n+1)$-chain Z satisfying $\partial Z = \Lambda$. (For the independence of the choice of Z, suppose that $\tilde{Z}$ is another $(n+1)$-chain with $\partial \tilde{Z} = \Lambda$. Then the difference between the intersection numbers equals the intersection number of Λ' with the $(n+1)$-cycle $Z - \tilde{Z}$. But the intersection number $\Lambda' \cdot (Z - \tilde{Z})$ depends only on the homology classes $[Z - \tilde{Z}] \in H_{n+1}(M)$ and $[\Lambda'] \in H_n(M)$, so it vanishes because $[\Lambda'] = [\Lambda] = 0$.)

Remark B.4. If the orientation of Λ is reversed, then the orientation of the $(n+1)$-chain Z is also reversed. So the Thurston-Bennequin invariant is independent of the orientation of Λ. If Λ is not orientable we can still define the Thurston-Bennequin invariant as an integer mod 2.

If Λ_0, Λ_1 are two disjoint (not necessarily homologically trivial) Legendrian submanifolds in (M, ξ) with $[\Lambda_0] = [\Lambda_1] \in H_n(M)$ we can define the *relative Thurston-Bennequin invariant*

$$\mathrm{tb}(\Lambda_0, \Lambda_1) := \mathrm{lk}(\Lambda_0 - \Lambda_1, \Lambda_0' - \Lambda_1'),$$

where Λ_i' is obtained by pushing Λ_i in the direction of the Reeb vector field.

The Thurston-Bennequin invariant for Legendrian embeddings in $\mathbb{R}^{2n+1}$. Consider a closed orientable Legendrian submanifold $\Lambda \subset \mathbb{R}^{2n+1}$ such that the Lagrangian projection $P_{\mathrm{Lag}}(\Lambda)$ has only transverse self-intersections (this can always be arranged by a generic perturbation).

Pick an orientation of Λ. To each self-intersection point c of $P_{\mathrm{Lag}}(\Lambda)$ we assign a number $I(c) = \pm 1$ as follows. Let a, b be the points on Λ with $P_{\mathrm{Lag}}(a) = P_{\mathrm{Lag}}(b) = c$ and z-coordinates $z(a) > z(b)$. Set $I(c) := +1$ if the orientation of $P_{\mathrm{Lag}}(T_a\Lambda) \oplus P_{\mathrm{Lag}}(T_b\Lambda)$ (in this order!) agrees with the complex orientation of $\mathbb{C}^n$, and $I(c) := -1$ if not (Note that this definition does not depend on the chosen orientation of Λ). Then

$$\mathrm{tb}(\Lambda) = \sum_c I(c), \tag{B.1}$$

where the sum is taken over all self-intersection points of the Lagrangian projection of Λ. To prove this formula, pick the $(n+1)$-chain $Z \subset \mathbb{R}^{2n+1}$ to be the cone over Λ through a point with very large negative z-coordinate and push Λ slightly upwards in z-direction to an embedded submanifold Λ'. Then the intersections of Z with Λ' are in 1-1 correspondence with the 'undercrossings' of Λ, i.e. with the double points of $P_{\mathrm{Lag}}(\Lambda)$, and the sign of an intersection equals the number $I(c)$ of the corresponding double point c.

Lemma B.5. *Let $\Lambda \subset \mathbb{R}^{2n+1}$ be a closed orientable Legendrian submanifold. Then*

(a) The parity of $\mathrm{tb}(\Lambda)$ *equals the self-intersection index of* $P_{\mathrm{Lag}}(\Lambda)$ *mod 2 (and is therefore determined by the rotation invariant $r(\Lambda)$).*

(b) If n is even,

$$\mathrm{tb}(\Lambda) = \frac{(-1)^{n/2}}{2}\chi(\Lambda).$$

Proof. Part (a) follows immediately from formula (B.1). For (b) note that if n is even the index $I(c)$ of a double point of $P_{\text{Lag}}(\Lambda)$ does not depend on the order of the preimages a, b and their sum $\text{tb}(\Lambda) = \sum_c I(c)$ equals the self-intersection index I_f of the immersion $f = P_{\text{Lag}} : \Lambda \to \mathbb{C}^n$. Now by Theorem A.8, $I_f = -\frac{1}{2}\chi(\nu_f)$, where $\chi(\nu_f)$ is the Euler characteristic of the normal bundle of f. Since f is Lagrangian, its normal bundle is isomorphic to its tangent bundle $T\Lambda$. However, the orientation on the normal bundle induced by the complex orientation of $\mathbb{C}^n$ differes from the orientation of TL by the sign $(-1)^{n(n-1)/2} = (-1)^{n/2}$ (recall that n is even). Thus $\chi(\nu_f) = (-1)^{n/2}\chi(\Lambda)$ and (b) follows. □

Remark B.6. For $n = 1$ the Thurston-Bennequin invariant can also be computed from the front projection as

$$\text{tb}(\Lambda) = \#(\text{left} - \text{over} - \text{crossings}) - \#(\text{right} - \text{over} - \text{crossings}) - \frac{1}{2}\#(\text{cusps}).$$

This formula is easily derived by computing the rotation number of ∂_z with respect to the normal framing coming from a Seifert surface (see e.g. [19]).

Stabilization. The following refinement of Proposition 5.25 describes how the Thurston-Bennequin invariant can be changed by suitable Legendrian regular homotopies.

Proposition B.7. *Let $\Lambda_0 \subset (M^{2n+1}, \xi = \ker\alpha)$ be a closed orientable Legendrian submanifold and k an integer. Suppose $n > 1$. Then the Legendrian submanifold $\Lambda_1 \subset M$ and Legendrian regular homotopy Λ_t constructed in Proposition 5.25 have the following additional properties:*

(a) Λ_1 coincides with Λ_0 outside a small ball and is smoothly isotopic to Λ_0.

(b) The relative Thurston-Bennequin invariant equals

$$\text{tb}(\Lambda_1, \Lambda_0) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -2k & \text{if } n > 1 \text{ is odd.} \end{cases}$$

Note that Part (b) is consistent with Lemma B.5.

We will later need the following consequence of Proposition B.7 for Legendrian spheres. Let the Legendrian immersions Λ_t be parametrized by $f_t : S^n \to M$. A compatible almost complex structure J on ξ and the Reeb vector field R_α induce normal framings $Jdf_t \oplus R_\alpha : S^n \times \mathbb{R}^{n+1} \cong TS^n \oplus \mathbb{R} \to TM$ along f_t. Let $\tilde{f}_t : \Lambda \hookrightarrow M$ be the smooth isotopy provided by (a). Let $\beta_0 : S^n \times \mathbb{R}^{n+1} \to TM$ be any normal framing and extend it to normal framings β_t along $\tilde{f}_t$. We can write $Jdf_0 \oplus R_\alpha = \beta_0 g_0$ and $Jdf_1 \oplus R_\alpha = \beta_1 g_1$ for unique elements $g_0, g_1 : S^n \to O(n+1)$.

Corollary B.8. *We have $[g_1] - [g_0] \in K_n = \ker[\pi_n O(n+1) \to \pi_n O]$. For $n > 1$, using the construction in Proposition B.7 we can arrange for $[g_1] - [g_0]$ to be any given element in K_n.*

Proof. Consider the loop of immersions following f_t from and then $\tilde{f}_t$ backwards. Define a path of normal framings along this loop as follows: Follow $Jdf_t \oplus R_\alpha$ from $Jdf_0 \oplus R_\alpha = \beta_0 g_0$ to $Jdf_1 \oplus R_\alpha = \beta_1 g_1$ and then $\beta_t g_1$ backwards to $\beta_0 g_1 = (\beta_0 g_0)(g_0^{-1} g_1)$. Note that we can arrange for all the data to be fixed outside a ball. So we can apply Proposition A.10 to this situation. It follows that $[g_0^{-1} g_1] = [g_1] - [g_0] \in K_n$ can be made equal to any given element in K_n by choosing the self-intersection index I_L appropriately. But by Proposition B.7, I_L can be arbitrarily prescribed for $n > 1$ (as an integer if n is odd and mod 2 if n is even). □

Arguing as in the proof of Proposition 5.25, Proposition B.7 follows from the following refinement of Lemma 5.26.

Lemma B.9. *The family of Legendrian immersions $\Lambda_t \subset \mathbb{R}^{2n+1}$, $t \in [0,1]$, constructed in Lemma 5.26 has the following additional properties:*

(a) Λ_1 is smoothly isotopic to Λ_0 by an isotopy that is fixed outside the branch $\{z = 0\}$ of Λ_0.

(b) The relative Thurston-Bennequin invariant equals

$$\mathrm{tb}(\Lambda_1, \Lambda_0) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -2(-1)^{n(n-1)/2} \chi(\{f \geq 1\}) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Part (a) follows from Corollary A.13.

For (b), perturb f as in the proof of Lemma 5.26. Then the Thurston-Bennequin invariant $\mathrm{tb}(t) := \mathrm{tb}(\Lambda_t, \Lambda_0)$ changes precisely when $t_0 f$ has a critical point q_0 on level 1 for some $t_0 \in (0,1)$. Let $I(q_0)$ be the oriented intersection number of the branches $\{z = 1\}$ and $\{z = t_0 f(q)\}$ at q_0 (in this order). $I(q_0)$ is the contribution of q_0 to $\mathrm{tb}(t)$ for $t < t_0$ because then the branch $\{z = 1\}$ passes *over* $\{z = tf(q)\}$, compare formula (B.1). For $t > t_0$ the branch $\{z = 1\}$ passes *under* $\{z = tf(q)\}$, so the contribution of q_0 to $\mathrm{tb}(t)$ equals the intersection number in the opposite order, which is $(-1)^n I(q_0)$. Hence the change of $\mathrm{tb}(t)$ at t_0 equals

$$\Delta \mathrm{tb}(t_0) = \begin{cases} 0 & n \text{ even.} \\ -2I(q_0) & n \text{ odd.} \end{cases}$$

This proves $\mathrm{tb}(\Lambda_1, \Lambda_0) = 0$ if n is even. To compute $I(q_0)$ for n odd, pick coordinates near q_0 as in (c). Then the tangent spaces in $\mathbb{R}^{2n}$ of the Lagrangian projections of the branches $\{z = 1\}$ and $\{z = t_0 f(q)\}$ are given by

$$\tau_1 = \{p_1 = \cdots = p_n = 0\},$$
$$\tau_2 = \{p_i = -t_0 q_i \text{ for } i \leq k, p_i = +t_0 q_i \text{ for } i \geq k+1\}.$$

Again suppose that the basis $(\partial_{q_1}, \dots, \partial_{q_n})$ represents the orientation of τ_1. Since the two branches of Λ_0 are oppositely oriented, the orientation of τ_2 is then represented by the basis

$$-(\partial_{q_1} - t_0\partial_{p_1}, \dots, \partial_{q_n} + t_0\partial_{p_n}).$$

Hence the orientation of (τ_1, τ_2) is represented by

$$-(\partial_{q_1}, \dots, \partial_{q_n}, -\partial_{p_1}, \dots, +\partial_{p_n}),$$

which equals $(-1)^{k+1+n(n-1)/2}$ times the complex orientation $(\partial_{q_1}, \partial_{p_1}, \dots, \partial_{q_n}, \partial_{p_n})$ of $\mathbb{R}^{2n} = \mathbb{C}^n$. So

$$I(q_0) = (-1)^{\mathrm{ind}_f(q_0)+1+n(n-1)/2} = I_L(q_0)$$

if n is odd, where $I_L(q_0)$ is the local intersection index of L from (c). Summing over all critical points above level 1 and using (c), we find for n odd:

$$\mathrm{tb}(\Lambda_1, \Lambda_0) = -2I_L = -2(-1)^{n(n-1)/2}\chi(\{f \geq 1\}).$$

□

The 3-dimensional case. The preceding proof fails for $n = 1$ because 1-dimensional manifold with boundary always has Euler characteristic $\xi \geq 0$. Therefore for $n = 1$ the local construction in Lemma B.9. allows us only to *decrease* the Thurston-Bennequin invariant by multiples of 2. This failure to increase the Thurston-Bennequin invariant is unavoidable in view of

Theorem B.10 (Bennequin's inequality [5]). *Every embedded Legendrian curve $\Lambda \subset \mathbb{R}^3$ satisfies*

$$tb(\Lambda) + |r(\Lambda)| \leq \chi(\Sigma),$$

where Σ is an embedded surface (Seifert surface) bounded by Λ in $\mathbb{R}^3$.

However, no analog of Bennequin's inequality exists in overtwisted contact manifolds, and one can change the invariant *tb* arbitrarily.

[To be continued...]

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