Consider a system of two quasi-homogeneous equations

\[ xy \frac{dy}{dx} = z^2, \]
\[ z \frac{dz}{dx} = xy, \quad x, y, z > 0. \]

Find a change of variables which reduces it to a first order system.

**Solution:** We first determine the weights for which the system is quasi-homogeneous. There are a couple of (ultimately equivalent) ways of doing this.

**Method 1.** Let \( g_s(x, y, z) = (e^{s x}, e^{s y}, e^{s z}) \). Write the system of equations as

\[ \frac{dy}{xy} = z^2 dx, \]
\[ \frac{dz}{z} = xy dx. \]

If \( g_s \) is a symmetry of the system, then \((g_s)^* (dy) = (g_s)^* (z^2 dx)\) and \((g_s)^* (dz) = (g_s)^* (xy \, dx)\). Expanding these equations ultimately yields \( \beta = \gamma \) and \( 2\alpha + \beta = 2\gamma \), respectively, which together give \( 2\alpha = \beta = \gamma \).

Alternatively, one may prefer to use \( G_t(x, y, z) = (t^\alpha x, t^\beta y, t^\gamma z) \) instead of \( g_s \).

**Method 2.** Let \( g_s(x, y, z) = (e^{s x}, e^{s y}, e^{s z}) \). Let \( \lambda \) denote the line field in \( \mathbb{R}^3 \) given by

\[ \lambda|_{(x,y,z)} = \text{span} \left\{ (1, \frac{z^2}{xy}, \frac{xy}{z}) \right\}. \]

For the system to be quasi-homogeneous, we must have \((g_s)^* \lambda|_{(x,y,z)} = \lambda|_{g_s(x,y,z)}\). One can calculate that

\[ (g_s)^* \lambda|_{(x,y,z)} = \text{span} \left\{ (1, e^{(2\gamma-\alpha-\beta)s} \frac{z^2}{xy}, e^{(\alpha+\beta-\gamma)s} \frac{xy}{z}) \right\}, \]
\[ \lambda|_{g_s(x,y,z)} = \text{span} \left\{ (1, e^{(\beta-\alpha)s} \frac{z^2}{xy}, e^{(\gamma-\alpha)x} \frac{xy}{z}) \right\}, \]

which forces \( 2\gamma - \alpha - \beta = \beta - \alpha \) and \( \alpha + \beta - \gamma = \gamma - \alpha \). This implies \( 2\alpha = \beta = \gamma \), as above.

At any rate, we may take \( \alpha = 1 \) and \( \beta = \gamma = 2 \). Take the plane \( \Sigma = \{(x, y, z) : x = 1\} \subset \mathbb{R}^3 \) as the transverse surface. Then our coordinate change amounts to

\[ x = e^s \]
\[ y = e^{2s} u_1 \]
\[ z = e^{2s} u_2 \]

\[ \iff \]
\[ \quad \quad \iff \quad \quad \quad \quad s = \log(x) \]
\[ \quad \quad \quad \quad u_1 = yx^{-2} \]
\[ \quad \quad \quad \quad u_2 = zx^{-2}. \]

In these coordinates – that is, via pullback – our equations take the form

\[ \frac{du_1}{ds} = \frac{u_2 - 2u_1^2}{u_1}, \]
\[ \frac{du_2}{ds} = \frac{u_1 - 2u_2^2}{u_2}, \]

which together yield a first-order equation

\[ \frac{du_2}{du_1} = \frac{u_1 - 2u_2^2}{u_2 - 2u_1^2} \cdot \frac{u_1}{u_2}. \]
(3) Consider a pendulum $\ddot{x} = -\sin x$. Find the limit of its period when the amplitude goes to zero.

*Solution:* Defining $y = \dot{x}$, we can rewrite our second-order equation as a first-order system:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -\sin x.
\end{align*}
\]

Note that $(0, 0)$ is an equilibrium point of the system. We will assume without proof that the phase curves in a deleted neighborhood of $(0, 0)$ are all closed.

It is a theorem (*cf.* Arnold §2D: pages 13-14) that as the amplitude approaches zero, the period of oscillation in a neighborhood of $(0, 0)$ tends to the period of oscillation of the linearized system. So, we have to find the period of the linearized system.

Linearizing at the point $(0, 0)$ gives the linearized system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x.
\end{align*}
\]

In this linearized system, we have $\ddot{x} = -x$, whose general solution is $x(t) = A \cos t + B \sin t$, for any $A, B \in \mathbb{R}$. For $(A, B) \neq (0, 0)$, the period of $x(t)$ is $\boxed{2\pi}$. ☐
(4) Find the solutions, the criminant, and the discriminant of the equation

\[(y')^3 + 3xy' - 3y = 0.\]

**Solution:** Let \( F(x, y, p) = p^3 + 3xp - 3y, \) where \( p = y' \). Let us consider the surface \( M = \{(x, y, p): F(x, y, p) = 0\} \) in the space of 1-jets. Solutions of the differential equation correspond to curves in \( M \) whose tangent vectors lie on the contact planes \( dy - pdx = 0 \).

So, we need to find the integral curves of the following system

\[
\begin{align*}
F(x, y, p) &= 0 \quad \implies \quad p^3 + 3xp - 3y = 0 \quad \text{(belonging to } M) \\
dF &= 0 \quad \implies \quad 3(p^2 + x)dp + 3pdx - 3dy = 0 \quad \text{(tangent to } M) \\
dy - pdx &= 0 \quad \implies \quad dy - pdx = 0 \quad \text{(tangent to contact planes)}
\end{align*}
\]

Using \( dy = pdx \), the second equation becomes

\[(p^2 + x)dp = 0.\]

This gives two cases.

**Case One:** \( dp = 0 \). Then \( p = C \), a constant. Plugging this into \( p^3 + 3xp - 3y = 0 \) gives the solutions

\[y = Cx + \frac{1}{3}C^3.\]

**Case Two:** \( p^2 + x = 0 \). Then \( x = -p^2 \). Plugging this into \( p^3 + 3xp - 3y \) gives \( y = -\frac{2}{3}p^3 \), hence \( y = \pm \frac{2}{3}(p^2)^{3/2} \), and we obtain the solutions

\[y = \pm \frac{2}{3}(-x)^{3/2}.\]

The **criminant** is the set of points in the surface \( M = \{(x, y, p): F(x, y, p) = 0\} \) such that \( \frac{\partial F}{\partial p} = 0 \). That is, the set of points in the 1-jet space for which

\[
\begin{align*}
p^3 + 3xp - 3y &= 0 \\
3p^2 + 3x &= 0.
\end{align*}
\]

The second equation gives \( x = -p^2 \). Plugging this into the first equation gives \( y = -\frac{2}{3}p^3 \). Thus, the criminant can be described by the parametric curve in the 1-jet space given by

\[
\begin{align*}
x(t) &= -t^2 \\
y(t) &= -\frac{2}{3}t^3 \\
p(t) &= t.
\end{align*}
\]

The **discriminant** is the projection of the criminant onto the \( xy \)-plane via \((x, y, p) \mapsto (x, y)\). This results in the parametric curve in the \( xy \)-plane given by

\[
\begin{align*}
x(t) &= -t^2 \\
y(t) &= -\frac{2}{3}t^3.
\end{align*}
\]

One can also describe this curve as the graphs \( y = \pm \frac{2}{3}(-x)^{3/2} \). \( \diamondsuit \)
(5) Let \( f = \sum_{i}^{n} a_{ij} x_i x_j \) be a quadratic form on \( \mathbb{R}^n \). Show that its Legendre transform \( g(p) \) is again a quadratic form \( g(p) = \sum_{i}^{n} b_{ij} p_i p_j \), and the value of these forms and the points corresponding to each other under the Legendre map coincide.

**Solution:** Let \( A = (a_{ij}) \). Since \( f \) is a quadratic form, the matrix \( A \) is symmetric. We will assume for this problem that \( A \) is positive-definite, hence invertible.

We let \( \delta_{ij} \) denote the Kronecker delta symbol. That is, \( \delta_{ii} = 1 \) and \( \delta_{ij} = 0 \) if \( i \neq j \). Said another way, we can write the identity matrix as \( I = (\delta_{ij}) \).

By definition, the Legendre transform \( g(p) \) is given by

\[
g(p) = \sup_x \left( \sum_i p_i x_i - \sum_i \sum_j a_{ij} x_i x_j \right).
\]

Fix \( p \in \mathbb{R}^n \). For a point \( x \in \mathbb{R}^n \) at which the supremum is attained, we have

\[
0 = \frac{\partial}{\partial x_k} \left( \sum_i p_i x_i - \sum_i \sum_j a_{ij} x_i x_j \right)
= p_k - \sum_i \sum_j a_{ij} \frac{\partial}{\partial x_k} (x_i x_j)
= p_k - \sum_i \sum_j a_{ij} x_i \delta_{jk} - \sum_i \sum_j a_{ij} \delta_{ik} x_j
= p_k - \sum_i a_{ik} x_i - \sum_j a_{kj} x_j
= p_k - \sum_i 2a_{ik} x_i,
\]

where in the last line we used that \( A \) is a symmetric matrix. In coordinate-free notation, we have determined that \( p = 2A^{-1} x \), and so \( x = \frac{1}{2} A^{-1} p \).

Let \( B = (b_{ij}) \) denote the inverse matrix \( B = A^{-1} \). Then \( x_i = \sum \frac{1}{2} b_{ij} p_j \), so that (for this specific choice of \( x \))

\[
g(p) = \sum_i p_i x_i - \sum_i \sum_j a_{ij} x_i x_j
= \sum_i p_i x_i - \sum_j \frac{1}{2} p_j x_j
= \frac{1}{2} \sum_i p_i x_i
= \frac{1}{4} \sum_i \sum_j b_{ij} p_i p_j,
\]

which shows that \( g \) is a quadratic form.
Thus, our Legendre map is $L(x) = 2Ax$, and its inverse is $L^{-1}(p) = \frac{1}{2}Bp$. That our quadratic forms $f$ and $g$ coincide on corresponding points follows from

$$g(L(x)) = \frac{1}{4} \sum_i \sum_j b_{ij} \left( \sum_k 2a_{ki}x_k \right) \left( \sum_{\ell} 2a_{\ell j}x_{\ell} \right)$$

$$= \sum_i \sum_j \sum_k \sum_{\ell} b_{ij} a_{ki} a_{\ell j} x_k x_{\ell}$$

$$= \sum_j \sum_k \sum_{\ell} \delta_{jk} a_{\ell j} x_k x_{\ell}$$

$$= \sum_k \sum_{\ell} a_{\ell k} x_k x_{\ell}$$

$$= f(x),$$

where in the last line we used that $A$ is symmetric. ◊

Remark: It is, of course, possible to do all of the above calculations in a coordinate-free manner as well.