Fixed points of symplectic tranformations

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1 Arnold's fixed point conjecture

Let (M, ω) be a closed symplectic manifold. Given a function $H : M \to \mathbb{R}$ the *Hamiltonian vector field* X_H determined by the *Hamiltonian* H is defined by the formula

$$X_H \lrcorner\, \omega = -dH.$$

Then $L_{X_H}\omega = 0$, and hence the flow generated by X_H preserves the symplectic form ω.

If one has a family of functions $H_t : M \to \mathbb{R}$, $t \in [0, 1]$, one gets a family of Hamiltonian vector fields X_{H_t} which generate an isotopy $f_t : M \to M$ which starts at $f_0 = \mathrm{Id}$ and defined by the differential equation

$$\frac{df_t}{dt}(x) = X_{H_t}(f_t(x)).$$

The isotopy of this type is called *Hamiltonian (flow)*. It consists of symplectomorphisms, $f_t^*\omega = \omega$. A diffeomorphism $f : M \to M$ is called *Hamiltonian* if there exists a Hamiltonian isotopy f_t with $f_0 = \mathrm{Id}$ and $f_1 = f$.

Conjecture 1.1 (Arnold's fixed points conjecture)**.** *Let (M, ω) be a closed symplectic manifold and $f : M \to M$ a Hamiltonian diffeomorphism. Then f has at least as many fixed points, as the minimal number of critical points of a smooth function $\varphi : M \to \mathbb{R}$.*

Remark 1.2. In this generality the conjecture is still open. For the case of 2-torus and other surfaces it was first proven by myself in [3]. For the case of an n-dimensional torus it was proven by C. Conley and E. Zehnder in [**?**]. It was generalized to other manifolds by the work of many people: M. Gromov ([6]), A. Floer ([5]), K. Fukaya and K Ono ([7]) and others. It is now known for all symplectic manifolds, but the lower bound for the manifold is not quite as good as predicted by Conjecture 1.1.

We note that this minimal number is at least 2 because any function on a closed manifold has at least two critical points, the minimum and the maximum. In fact, it is usually larger. For instance, for the 2-dimensional torus this number is 3, if one allows fixed points to be degenerate, and 4 in the non-degenerate case.

2 Proof of Arnold's conjecture for the 2-torus

Lemma 2.1. *Given a loop $\gamma : S^1 \to M$ consider a map $F : S^1 \times [0,1] \to M$ given by the formula $\gamma(u,t) = f_t(\gamma(u)),\ u \in S^1, t \in [0,1]$. Then $\int\limits_{S^1 \times [0,1]} F^*\omega = 0$.*

Proof. The tangent space to $S^1 \times [0,1]$ is generated by the vector fields $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$. We have

$$\frac{\partial}{\partial t} \lrcorner F^*\omega = \frac{\partial F}{\partial t} \lrcorner \omega = X_{H_t} \lrcorner \omega = -dH_t.$$

Hence,

$$\int\limits_{S^1 \times [0,1]} F^*\omega = \int\limits_0^1 \left(\frac{\partial}{\partial t} \lrcorner F^*\omega \right) dt = -\int\limits_0^1 \left(\int\limits_{f_t(\gamma)} dH_t \right) dt = 0,$$

because the integral of the exact 1-form dH_t over a closed curve $f_t(\gamma)$ is equal to 0. $\square$

We prove below Arnold's fixed point conjecture for the 2-torus, but we will only prove existence of 1 fixed point. A slightly more precise argument allows to prove existence of at least 3 fixed points. The current proof was first given in [4].

What is remarkable about this proof that it could be given by H. Poincaré. In fact, the first half of the proof almost precisely follows the first page of Poincaré's paper [2].

Theorem 2.2 (C. Conley and E. Zehnder,[1])**.** *Any Hamiltonian diffeomorphism f of the 2-torus $(T^2, \omega) = (\mathbb{R}^2/\mathbb{Z}^2, dp \wedge dq)$ must have at least 1 fixed point.*

Proof. We view the torus T^2 as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, i.e. the set of points $(p,q) \in \mathbb{R}^2$ up to addition of a vector with integer coordinates. Let us denote by π the projection $\mathbb{R}^2 \to T^2$. The area form $\Omega = dp \wedge dq$ on $\mathbb{R}^2$ descends to the an area form ω on $\mathbb{R}^2$, i.e. $\pi^*\omega = \Omega$.

The Hamiltonian isotopy $f_t : T^2 \to T^2$ lifts to a Hamiltonian isotopy $F_t : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F_0 = \mathrm{Id}$ and $\varphi \circ F_t = f_t$ for all $t \in [0,1]$.

We have $F(p,q) = (P(p,q), Q(p,q))$ and $dP \wedge dQ = dp \wedge dq$. Let us first assume that F is C^1-close to the identity. Then its graph

$$\Gamma_F = \{(p,q,P,Q) |\ P = P(p,q), Q = Q(p,q)\} \subset \mathbb{R}^4$$

is graphical with respect to the splitting of $\mathbb{R}^{4n}$ into the (q,P)- and (p,Q)-coordinate subspaces, i.e.

$$\Gamma_F = \{p = p(q,P), Q = Q(q,P)\},$$

and hence the equation $dp \wedge dq = dP \wedge dQ$ is equivalent to the existence of a function $G(q,P)$ such that $pdq + QdP = dG$. Fixed points $p = P, Q = q$ of F are zeroes of the 1-form $(p-P)dq + (Q-q)dP = d(G-qP)$. In other words, fixed points are exactly the critical points of the function $\widetilde{G}(q,P) := G(q,P) - qP$.

Lemma 2.3. *The function* $\widetilde{G}$ (called a generating function of the canonical transformation$F$) *is* 1*-periodic in variables* q*,* P*, i.e.* $\widetilde{G}(q+1,P) = \widetilde{G}(q,P+1) = \widetilde{G}(q,P)$.

Proof. Take a path γ in the coordinate plane (p,q) connecting points $\gamma(0) = (q_0,p_0)$ and $\gamma(1) = (q_0+1,p_0)$. Note that the projection $\widetilde{\gamma} := \pi \circ \gamma$ of this path to the torus T^2 is a loop. Consider a family of paths $\delta_s : [0,1] \to \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, $s \in [0,1]$, defined by the formula $(p,q) = \delta(t), (P,Q) = F_s(\delta(t))$, so that the path δ_s lies on the graph Γ_{F_s}. Denote $(P_s,Q_s) := F_s(p_0,q_0)$. Then $F_s(p_0,q_0+1) = (P_s,Q_s+1)$. Thus, $\delta_s(0) = (p_0,q_0,P_s,Q_s)$, $\delta_s(1) = (p_0,q_0+1,P_s,Q_s+1)$. Then by Stokes' formula

$$\widetilde{G}(q_0+1,P_1) - \widetilde{G}(q_0,P_1) = \int\limits_{\delta_1} d\widetilde{G} = \int\limits_{\delta_1} (p-P)dq + (Q-q)dP.$$

But $(p-P)dq+(Q-q)dP = pdq - PdQ + d(P(Q-q))$. Hence,

$$\widetilde{G}(q_0+1,P_0)-\widetilde{G}(q_0,P_0) = \int\limits_{\delta_1} pdq - PdQ + d(P(Q-q))$$

$$= \int\limits_{\gamma} pdq - \int\limits_{F\circ\gamma} pdq + \int\limits_{\delta_1} d(P(Q-q)),$$

But the latter integral is equal to 0 because the function $P(Q-q)$ is equal to 0 at the end points of the path δ_1. On the other hand, $\int\limits_{\gamma} pdq = \int\limits_{F\circ\gamma} pdq$. Indeed, denote $\beta(s) := (P_s, Q_s)$ $\overline{\beta}(s) := (P_s, Q_s+1)$, Then, $\int\limits_{\beta} pdq - \int\limits_{\overline{\beta}} pdq$. Hence,

$$\int\limits_{\gamma} pdq - \int\limits_{F\circ\gamma} pdq = \int\limits_{\gamma} pdq + \int\limits_{\overline{\beta}} pdq - \int\limits_{F\circ\gamma} pdq - \int\limits_{\beta} pdq.$$

Consider a square $A = [0,1]\times[0,1]$ and a map $\Phi : A \to \mathbb{R}^2$ defined by the formula $\Phi(t,s) = F_s(\gamma(t))$. Then

$$\int\limits_{\gamma} pdq + \int\limits_{\overline{\beta}} pdq - \int\limits_{F\circ\gamma} pdq - \int\limits_{\beta} pdq = \int\limits_{\partial A} \Phi^*(pdq) = \int\limits_{A} \Phi^*(dp\wedge dq).$$

Denote $\overline{\Phi} := \pi\circ\Phi : A \to T^2$. Recall that the projection $\pi : \mathbb{R}^2 \to T^2$ satisfies $\pi(p,q) = \pi(p,q)+1)$. Therefore, $\overline{\Phi}(0,s) = \overline{\Phi}(1,s)$ for $s \in [0,1]$. We have $dp\wedge dq = \pi^*\omega$, and hence $\Phi^*(dp\wedge dq) = \overline{\Phi}^*\omega$. But by Lemma 2.1 we have $\int\limits_{A} \overline{\Phi}^*\omega = 0$. Hence,

$$\widetilde{G}(q_0+1,P_0)-\widetilde{G}(q_0,P_0) = \int\limits_{A} \Phi^*(dp\wedge dq) = \int\limits_{A} \overline{\Phi}^*\omega = 0.$$

We similarly check that $\widetilde{G}(q_0,P_0+1) = wtG(q_0,P_0)$. □

Thus the function $\widetilde{G}$ hence descends to the torus T^2, and hence must have at least 2 critical points, the maximum and the minimum. In fact, one can show that it has to have at least 3 critical points. But Its critical points are in 1-1 correspondence with the fixed points of f, and therefore, f has as many fixed points. This concludes the proof of Arnold's conjecture for the 2-torus for the case when f (and hence F) is C^1-small

Consider now the of the general F. Recall that the Hamiltonian isotopy F_t connects $F_0 = \mathrm{Id}$ with $F_1 = F$. For any integer $N > 0$ we can present F as a composition $F = \widetilde{F}_N \circ \ldots \widetilde{F}_1$, where we denote

$$\widetilde{F}_k = F_{\frac{k}{N}} F_{\frac{k-1}{N}},\ k = 1, \ldots, N.$$

By taking N sufficiently large we can make all the diffeomorphisms $\widetilde{F}_k$ arbitrarily C^1-small.

We consider below the case $N = 2$, the general case differs only in the notation.

As above, we can conclude, that the product $\Gamma := \Gamma_{\widetilde{F}_1} \times \Gamma_{\widetilde{F}_2} \subset \mathbb{R}^8$ of the graphs of $\widetilde{F}_1$ and $\widetilde{F}_2$ is given by the equations

$$p_1 = p_1(q_1, P_1), Q_1 = Q_1(q_1, P_1), p_2 = p_2(q_2, P_2), Q_2 = Q_2(q_2, P_2).$$

Furthermore, we have $p_i dq_i + Q_i dP_i = dG_i$ and the functions $\widetilde{G}_i = G_i - q_i P_i$ are $\mathbb{Z}^2$-periodic, $i = 1, 2$. Set $\widetilde{G}(q_1, P_1, q_2, P_2) := G_1(q_1, P_1) + G_2(q_2, P_2)$. Fixed points of F are in 1-1 correspondence with the intersection $\Gamma \cap \{p_2 = P_1, Q_1 = q_2, p_1 = P_2, Q_2 = q_1\}$, i.e. with the *zeroes of the 1-form*

$$\begin{aligned}\alpha :=&(p_1 - P_2)dq_1 + (Q_1 - q_2)dP_1 + (p_2 - P_1)dq_2 + (Q_2 - q_1)dP_2 \\ =&dG(q_1, q_2, P_1, P_2) + d\Big((P_1 - P_2)(q_1 - q_2)\Big).\end{aligned}$$

Changing the variables $(q_1, q_2, P_1, P_2) \mapsto (q_1, u_1 := q_2 - q_1, P_1, U_1 := P_2 - P_1)$ we get

$$\alpha = d(\widehat{G} + u_1 U_1),\ \text{where}\ \widehat{G}(q_1, u_1, P_1, U_1) := \widetilde{G}(q_1, q_1 + u_1, P_1, P_1 + U_1).$$

Similarly to the proof of Lemma 2.3, one can check that the function $\widehat{G}$ is periodic with respect to all variables, and in particular, in variables (q_1, P_1), and hence it descends to a function

$$T^2 \times \mathbb{R}^2 = \mathbb{R}^2/\{q_1 \sim q_1 + 1, P_1 \sim P_1 + 1\} \to \mathbb{R}.$$

Note also that this function and its derivatives are bounded. Then the following lemma implies that the function $\widetilde{G}(q_1, P_1, u_1, U_1)$ must have some critical points, which, as we showed above, corresponds to fixed points of F.

Lemma 2.4. *Let M be a closed manifold, $C : \mathbb{R}^n \to \mathbb{R}$ a non-degenerate quadratic form, and $\varphi : M \times \mathbb{R}^n \to \mathbb{R}$ a smooth function which is bounded and has bounded 1st derivatives. Then the function $\psi(x, y) = \varphi(x, y) + C(y),\ x \in M, y \in \mathbb{R}^n$ has at least 1 critical point.*

□

Sketch of the proof. We can assume that $C(y) = \sum_1^k y_j^2 - \sum_{k+1}^n y_j^2$. Suppose that $k \neq 0$ (if $k = 0$ we can change the sign of the function ψ). Consider a map $h : \mathbb{R}^k \to M \times \mathbb{R}^n$ such that $h(y_1, \dots, y_k) = (x_0, y_1, \dots, y_k, 0, \dots, 0)$ when $||y||^2 = \sum_1^k y_j^2$ is large enough. Let us denote by $\mathcal{H}$ the space of all maps h with this property. For any $h \in \mathcal{H}$ the function $\psi \circ h : \mathbb{R}^k \to \mathbb{R}$ is bounded below and achieves its minimal value at a point $a_h \in \mathbb{R}^k$. Indeed, $\lim_{||y|| \to \infty} \psi \circ h = +\infty$. Denote $b_h = h(a_h) \in M \times \mathbb{R}^n$. There exists a point $b \in M \times \mathbb{R}^n$ such that $\psi(b) \geq \psi(b_h)$ for all $h \in \mathcal{H}$ (why?). Then b is a critical point of ψ (why?).

□

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