Math 177: Homework N3

Due on Wednesday, May 27

1. A particle of mass $m$ is moving in $\mathbb{R}^3$ in a central field with potential energy $U(r)$. Write its Hamiltonian function and the equation of motion in the canonical coordinates $(r, \phi, \theta, p_r, p_{\phi}, p_{\theta})$ associated with the spherical coordinates coordinates $(r, \phi, \theta)$.

2. Find an area preserving transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(P, Q) = f(p, q)$, if its graph is given by the generating function $F(q, P) = (q + q^3)P$. In other words, the graph of the area preserving map $f$ in $(\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2, dp \wedge dq - dP \wedge dQ)$ given by the generating function $F$ with respect to the polarization of $\mathbb{R}^4$ by the coordinate plane $(q, P)$ and $(p, Q)$.

3. Poisson bracket $\{f, g\}$ of two functions $f, g : M \to \mathbb{R}$ on a symplectic manifold $(M, \omega)$ is a function $M \to \mathbb{R}$ defined by the formula $\{f, g\} = dg(X_f)$, where $X_f$ is the Hamiltonian vector field of the function $f$, i.e. $X_f \omega = -df$. Verify the following properties of the Poisson bracket:

   **Skew-symmetricity:** $\{f, g\} = -\{g, f\}$;

   **Jacobi identity:** $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$;

   **Leibniz rule:** $\{f, gh\} = \{f, g\}h + \{f, h\}g$.

4. Suppose that $\mathbb{R}^2$ is endowed with an area form $\omega = dp \wedge dq$. Let $H_t : \mathbb{R}^2 \to \mathbb{R}, t \in [0, 1]$, be a family of smooth functions equal to 0 outside of the unit disc $D$. Let $X_t := X_{H_t}$ be the
Hamiltonian vector field generated by $H_t$, i.e. $X_t \omega = -dH_t$. Let $f_t : \mathbb{R}^2 \to \mathbb{R}^2$ be the flow of area preserving transformations generated by $X_t$, i.e.

$$\frac{df_t}{dt}(x) = X_t(f_t(x)).$$

Let $z_0 \in \text{Int}D$ be a fixed point of $f_1$, i.e. $f_1(z_0) = z_0$. Denote by $\gamma$ the loop $\gamma : [0, 1] \to \mathbb{R}^2$ defined by the formula $\gamma(t) = f_t(z_0)$, $t \in [0, 1]$. Then the integral $S(z_0) := \int_{\gamma} pdq - H_t dt$ is called action of the fixed point $z_0$.

Prove that for any path $\delta : [0, 1] \to \mathbb{R}^2$ such $\delta(0) \in \mathbb{R}^2 \setminus D$ and $\delta(1) = z_0$ one has

$$\int_{\delta} pdq - \int_{f_1(\delta)} pdq = S(z_0).$$

In particular, the integral in the left hand side of the equation is independent of the choice of the path $\delta$, so that the action depends only on $f_1$ and not on a choice of the Hamiltonian $H_t$ which generates it.

Each problem is 10 points.