Math 116 Complex Analysis

Additional chapters

Yakov Eliashberg

October 5, 2018
# Contents

1 Complex Analysis Basics 5

1 Linear algebra 7

1.1 Complex numbers ............................................. 7
1.2 Complex linear function from the real perspective .......... 9

2 Holomorphic functions 13

2.1 Differentiability and the differential .......................... 13
2.2 Holomorphic functions, Cauchy-Riemann equations .......... 14

3 Differential 1-forms and their integration 17

3.1 Complex-valued differential 1-forms .......................... 17
3.2 Holomorphic 1-forms ............................................ 20
3.3 Integration of differential 1-forms along curves ................ 20
3.4 Integrals of closed and exact differential 1-forms .......... 21

4 Cauchy integral formula 25

4.1 Stokes/Green theorem .......................................... 25
4.2 Cauchy theorem and Cauchy integral formula ................. 26

5 Convergent power series and holomorphic functions 29

5.1 Recollection of basic facts about series ....................... 29
5.2 Power series ...................................................... 30
5.3 Analytic vs holomorphic ....................................... 32
6 Properties of holomorphic functions

6.1 Exponential function and its relatives ........................................... 35
6.2 Entire functions ........................................................................... 36
6.3 Analytic continuation ................................................................. 37
6.4 Complex logarithm .................................................................... 39
Part I

Complex Analysis Basics
Chapter 1

Linear algebra

1.1 Complex numbers

The space $\mathbb{R}^2$ can be endowed with an associative and commutative multiplication operation. This operation is uniquely determined by three properties:

- it is a bilinear operation;
- the vector $(1, 0)$ is the unit;
- the vector $(0, 1)$ satisfies $(0, 1)^2 = -(1, 0)$.

The vector $(0, 1)$ is usually denoted by $i$, and we will simply write $1$ instead of the vector $(1, 0)$. Hence, any point $(a, b) \in \mathbb{R}^2$ can be written as $a + bi$, where $a, b \in \mathbb{R}$, and the product of $a + bi$ and $c + di$ is given by the formula

$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$ 

The plane $\mathbb{R}^2$ endowed with this multiplication is denoted by $\mathbb{C}$ and called the set of complex numbers. The real line generated by $1$ is called the real axis, the line generated by $i$ is called the imaginary axis. The set of real numbers $\mathbb{R}$ can be viewed as embedded into $\mathbb{C}$ as the real axis. Given a complex number $z = x + iy$, the numbers $x$ and $y$ are called its real and imaginary parts, respectively, and denoted by $\text{Re}z$ and $\text{Im}z$, so that $z = \text{Re}z + i\text{Im}z$. 


For any non-zero complex number $z = a + bi$ there exists an inverse $z^{-1}$ such that $z^{-1}z = 1$. Indeed, we can set

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$ 

The commutativity, associativity and existence of the inverse is easy to check, but it should not be taken for granted: it is impossible to define a similar operation any $\mathbb{R}^n$ for $n > 2$.

Given $z = a + bi \in \mathbb{C}$ its conjugate is defined as $\bar{z} = a - bi$. The conjugation operation $z \mapsto \bar{z}$ is the reflection of $\mathbb{C}$ with respect to the real axis $\mathbb{R} \subset \mathbb{C}$. Note that $\text{Re}z = \frac{1}{2}(z + \bar{z})$, $\text{Im}z = \frac{1}{2i}(z - \bar{z})$.

Let us introduce the polar coordinates $(r, \phi)$ in $\mathbb{R}^2 = \mathbb{C}$. Then a complex number $z = x + yi$ can be written as $r \cos \phi + ir \sin \phi = r(\cos \phi + i \sin \phi)$. This form of writing a complex number is called, sometimes, trigonometric. The number $r = \sqrt{x^2 + y^2}$ is called the modulus of $z$ and denoted by $|z|$ and $\phi$ is called the argument of $\phi$ and denoted by $\text{arg}z$. Note that the argument is defined only mod $2\pi$. The value of the argument in $[0, 2\pi)$ is sometimes called the principal value of the argument. When $z$ is real than its modulus $|z|$ is just the absolute value. We also not that $|z| = \sqrt{zz^*}$.

An important role plays the triangle inequality

$$| |z_1| - |z_2| | \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

**Exponential function of a complex variable**

Recall that the exponential function $e^x$ has a Taylor expansion

$$e^x = \sum_{0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots.$$ 

We then define for a complex $z$ the exponential function by the same formula

$$e^z := 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \ldots.$$ 

One can check that this power series absolutely converging for all $z$ and satisfies the formula

$$e^{z_1 + z_2} = e^{z_1}e^{z_2}.$$
In particular, we have

\[ e^{iy} = 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \cdots + \ldots \]  
\[ = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{2k!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}, \]

(1.1.1)

But \( \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{2k!} = \cos y \) and \( \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} = \sin y \), and hence we get Euler’s formula

\[ e^{iy} = \cos y + i \sin y, \]

and furthermore,

\[ e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y), \]

i.e. \( |e^{x+iy}| = e^x \), \( \arg(e^x) = y \). In particular, any complex number \( z = r(\cos \phi + i \sin \phi) \) can be rewritten in the form \( z = re^{i\phi} \). This is called the exponential form of the complex number \( z \).

Note that

\[ (e^{i\phi})^n = e^{in\phi}, \]

and hence if \( z = re^{i\phi} \) then \( z^n = r^n e^{in\phi} = r^n (\cos n\phi + i \sin n\phi) \).

Note that the operation \( z \mapsto iz \) is the rotation of \( \mathbb{C} \) counterclockwise by the angle \( \frac{\pi}{2} \). More generally a multiplication operation \( z \mapsto zw \), where \( w = \rho e^{i\theta} \) is the composition of a rotation by the angle \( \theta \) and a radial dilatation (homothety) in \( \rho \) times.

Exercise 1.1.  
1. Compute \( \sum_{k=0}^{n} \cos k\theta \) and \( \sum_{k=1}^{n} \sin k\theta \).  
2. Compute \( 1 + \binom{n}{4} + \binom{n}{8} + \binom{n}{12} + \ldots \).

1.2 Complex linear function from the real perspective

A linear map \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) is by definition is required to satisfy two conditions:

\[ F(z_1 + z_2) = F(z_1) + F(z_2); \]
\[ F(\lambda z) = \lambda F(z), \]

\(^1\)Convergence of power series will be discussed later.
where $z_1, z_2, z$ are any vectors from $\mathbb{R}^2$ and $\lambda \in \mathbb{R}$ is a real number. Any such map is a multiplication by a $2 \times 2$-matrix:

$$F(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

A linear function of one complex variable is a linear map $F : \mathbb{C} \to \mathbb{C}$ which satisfies in addition, the condition

$$F(\lambda z) = \lambda F(z), \text{ for any complex number } \lambda. \quad (1.2.1)$$

Any such function has to satisfy $F(z) = F(1)z = cz$, where $c = a + ib = F(1)$. Equivalently,

$$F(x + iy) = (a + ib)(x + iy) = ax - by + i(ay + bx).$$

Thus, viewing a complex number $z = x + iy$ as a vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ we get

$$F(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

In other words, we proved the following

**Lemma 1.2.** A real linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ with a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is complex linear map $\mathbb{C} \to \mathbb{C}$ if and only if $a = d$ and $b = -c$.

In particular, the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the matrix of multiplication by $i$.

Note that

$$\det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2 + b^2 = |c|^2, \text{ where } c = a + ib. \quad (1.2.2)$$

In other words, the (real) determinant of the matrix of the multiplication by a complex number $c$ is equal to $|c|^2$.

We can also view a real linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ as a map $\mathbb{R}^2 \to \mathbb{C}$, i.e. as a complex-valued linear (in a real sense) function $F(x, y) = f_1(x, y) + if_2(x, y)$. If $F$ was given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $f_1(x, y) = ax + by, f_2(x, y) = cx + dy$. We also have

$$F(x, y) = f_1(x, y) + if_2(x, y) = ax + by + i(cx + dy) = (a + ic)x + (b + id)y = Ax + By. \quad (1.2.3)$$
Note that $x = \frac{1}{2}(z + \bar{z}), y = -\frac{1}{2}(z - \bar{z})$, and hence
\[
F(x, y) = Ax + By = \frac{A}{2}(z + \bar{z}) - \frac{B}{2}(z - \bar{z}) = \frac{1}{2}(A - iB)z + \frac{1}{2}(A + iB)\bar{z} = \alpha z + \beta \bar{z}, \quad (1.2.4)
\]
where we denoted $\alpha := \frac{1}{2}(A - iB), \beta := \frac{1}{2}(A + iB)$. Note that the function $l_1(z) = \alpha z$ is complex linear, while the function $l_2(z) = \beta \bar{z}$ is complex anti-linear, which means that it is linear in the real sense, but satisfied the condition $l_2(\lambda z) = \bar{\lambda} l_2(z)$.

If $F$ is a complex linear map, then $F$ is anti-linear and vice versa. In particular, every complex anti-linear map $F$ has the form $F(z) = az$ for a complex number $a$.

The following lemma summarizes the above discussion.

**Lemma 1.3.** Any linear in the real sense map $F : \mathbb{C} \to \mathbb{C}$ can be uniquely written as a sum $F = F_1 + F_2$, where $F_1$ is complex linear and $F_2$ is complex anti-linear. If $F$ is given by a matrix
\[
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\]
then $F_1(z) = \alpha z, F_2(z) = \beta \bar{z}$, where $\alpha := \frac{1}{2}(A - iB), \beta := \frac{1}{2}(A + iB), A = a + ic, B = b + id$.

\footnote{Complex-valued linear (in the real sense) functions on $\mathbb{R}^2$ form a 2-dimensional complex vector space. Formulas (1.2.3) and (1.2.4) say that the pairs of functions $(x,y)$ as well as the pair $(z, \bar{z})$ form a basis of this space.}
Chapter 2

Holomorphic functions

2.1 Differentiability and the differential

For any point \( z = (x, y) \in \mathbb{R}^2 \) we denote by \( \mathbb{R}^2_z \) the space \( \mathbb{R}^2 \) with the origin, shifted to the point \( z \).

Though the parallel transport allows one to identify spaces \( \mathbb{R}^2 \) and \( \mathbb{R}^2_z \) it will be important for us to think about them as different spaces.

Let \( U \) be a domain in \( \mathbb{R}^2 \) and a map \( f : U \rightarrow \mathbb{R}^2 \) a function on on it. A vector-valued function \( f : U \rightarrow \mathbb{R}^2 \), where \( U \subset \mathbb{R}^2 \) a domain in \( \mathbb{R}^2 \), is called differentiable at a point \( a \in U \) if near the point \( a \) in can be well approximated by a linear function. More precisely, if there exists a linear map \( A : \mathbb{R}^2_a \rightarrow \mathbb{R}^2_{f(a)} \) such that

\[
f(a + h) - f(a) = A(h) + o(||h||)
\]

for any sufficiently small vector \( h = (h_1, h_2) \in \mathbb{R}^2 \), where the notation \( o(t) \) stands for any vector-valued function such that \( \frac{o(t)}{t} \rightarrow 0 \). The linear function \( A \) is called the differential of the function \( f \) at the point \( a \) and is denoted by \( d_a f \). In other words, \( f \) is differentiable at \( a \in U \) if for any \( h \in \mathbb{R}^2_a \) there exists a limit

\[
d_a f(h) = \lim_{t \rightarrow 0} \frac{f(a + th) - f(a)}{t},
\]

and the limit \( A(h) \) linearly depends on \( h \). By identifying \( \mathbb{R}^2_a \) and \( \mathbb{R}^2_{f(a)} \) with \( \mathbb{R}^2 \) via the parallel transport we can associate with the linear map \( d_a f \) its matrix \( J_a(f) \), called the Jacobi matrix or derivative of the map \( f \). If we denote by \( u(x, y) \) and \( v(x, y) \) the coordinate functions of the map \( f \),
then

\[ J(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}. \]

The function \( f \) is called differentiable on the whole domain \( U \) if it is differentiable at each point of \( U \).

### 2.2 Holomorphic functions, Cauchy-Riemann equations

Let us now view the map \( f : U \to \mathbb{R}^2 \) as a complex valued function \( f(z) = u(z) + iv(z) \), \( z = x + iy \).

The function \( f \) is called differentiable at the point \( a \) in a complex sense, or holomorphic at \( a \) if the differential \( d_a f \) is a complex linear map. The following theorem lists equivalent definitions of holomorphicity.

**Theorem 2.1.** The function \( f = u + iv : U \to \mathbb{C} \) is holomorphic at a point \( a \in U \) if one of the following equivalent conditions is satisfied:

1. \( f(a + h) - f(a) = ch + o(|h|) \), for a complex number \( c \);

2. There exists a limit \( \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \); the limit is denoted by \( f'(a) \) and called a complex derivative at the point \( a \);

3. \( f \) is differentiable in the real sense and the following Cauchy–Riemann equations are satisfied at the point \( a \):

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\
\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.
\end{align*}
\]  

(2.2.1)

4. \( f \) is differentiable in the real sense and \( \frac{df}{dx}(a) := \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right) = 0. \)

**Proof.** Statement (1) just a reformulation of the fact that the differential \( d_a f \) is a complex linear map. Equivalence (1) and (2) is straightforward. According to Lemma 1.2 condition (3) just means that the Jacobi matrix is a matrix of a complex linear map.
To deal with condition (4) let us recall that according to Lemma 1.3 for any differentiable at \( a \) function \( f \) we can decompose the linear map \( d_a f \) into a complex linear and anti-linear:

\[
d_a f = \partial_a f + \bar{\partial}_a f,
\]

so that we have \( \partial_a f(h) = \alpha h, \bar{\partial}_a f(h) = \beta \bar{h} \). Rephrasing Lemma 1.3 we have

\[
\alpha = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) - i \frac{\partial f}{\partial y}(a) \right) h + \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right) \bar{h}.
\]

If we introduce the notation

\[
\frac{\partial f}{\partial z}(a) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) - i \frac{\partial f}{\partial y}(a) \right),
\]

\[
\frac{\partial f}{\partial \bar{z}}(a) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right),
\]

then we can write

\[
d_a f(h) = \frac{\partial f}{\partial z}(a) h + \frac{\partial f}{\partial \bar{z}}(a) \bar{h}.
\]

Hence, \( f \) is holomorphic at a point \( a \) if and only if

\[
\frac{\partial f}{\partial \bar{z}}(a) = 0,
\]

and this condition is just another form of the Cauchy-Riemann equations 2.2.1.

It is important to note that \( f \) is holomorphic at \( a \), i.e. \( \frac{\partial f}{\partial \bar{z}}(a) = 0 \) then \( f'(a) = \frac{\partial f}{\partial z}(a) \).

Note that the determinant \( \det J(f)(a) \) for a holomorphic at \( a \) map is equal to \( |f'(a)|^2 \), see formula 1.2.2.

The function \( f : U \to \mathbb{C} \) is called holomorphic in \( U \) if it is holomorphic at every point of \( U \).

This is equivalent to the condition \( \frac{\partial f}{\partial \bar{z}} = 0 \) in \( U \).

The following proposition summarizes property of complex differentiation which are analogous to the corresponding facts in the real case.

**Proposition 2.2.**  (1) If \( f, g \) are holomorphic at \( a \in \mathbb{C} \) the \( f \pm g \) and \( fg \) is holomorphic at \( a \) at \( (f \pm g)'(a) = f'(a) \pm g'(a), \ (fg)'(a) = f'(a)g(a) + f(a)g'(a) \); if \( g(a) \neq 0 \) then \( \frac{f}{g} \) is holomorphic at \( a \) and \( \left( \frac{f}{g} \right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)} \);

(2) If \( f \) is holomorphic at \( a \) and \( g \) is holomorphic at \( f(a) \) then the composition \( g \circ f \) is holomorphic at \( a \) and \( (g \circ f)'(a) = g'(f(a))f'(a) \).
The proof of (1) repeats the corresponding proofs in the real case, while (2) is the chain rule with an additional observation that a composition of complex linear maps is itself complex linear.

According to our definition of a holomorphic function it is not even clear whether this function is $C^1$-smooth, i.e. whether its derivative continuously depends on a point of the domain. It turns out that this is automatically true, which is the subject of the following theorem.

**Theorem 2.3** (H. Looman, D. Menchoff). *Every holomorphic function in a domain $U$ is of class $C^1$, i.e. its derivative continuously depends on the point of $U$.*

The proof of this theorem goes beyond this course. Interested students can read the proof in the book “Complex Analysis on One Variable” by R. Narasimhan and Y. Nievergelt, Birkhäuser, 2001. See Section 1.6 in this book.

We will assume in what follows the conclusion of this theorem, i.e. that a holomorphic function is of class $C^1$. Those who feel uncomfortable using unproven fact can assume in what follows that the $C^1$-condition is a part of the definition of a holomorphic function. As we will prove below that this condition is in turn implies that a holomorphic function is infinitely differentiable, and moreover analytic, i.e it is the sum of its Taylor series expansion in a neighborhood of each point point of $U$. 

Chapter 3

Differential 1-forms and their integration

3.1 Complex-valued differential 1-forms

Let us first recall some basics of the theory of real differential forms. For our purposes we will need only 1-forms on domains in $\mathbb{R}^2$. By definition a differential 1-form $\lambda$ on a domain $U \subset \mathbb{R}^2$ is a field of linear functions $\lambda_z : \mathbb{R}^2 \to \mathbb{R}$. Thus a differential 1-form is function of arguments of 2 kind: of a point $z \in U$ and a vector $h \in \mathbb{R}^2_z$. It depends linearly on $h$ and arbitrarily (but usually continuously and even differentiably) on $z$, i.e. we have $\lambda_z(h) = a_1(z)h_1 + a_2(z)h_2$, where $h_1, h_2$ are Cartesian coordinates of $h \in \mathbb{R}^2_z$. Any differential 1-form can be multiplied by a function ("a field of scalars"): $(f\lambda)_z(h) = f(z)\lambda_z(h)$.

Given a real-valued function $f : U \to \mathbb{R}^2$ on $U$ its differential $df$ is an example of a differential form: $d_z(f)(h) = \frac{\partial f}{\partial x}h_1 + \frac{\partial f}{\partial y}h_2$. In particular differentials $dx$ and $dy$ of the coordinate functions $x, y$ are differential 1-forms, and any other differential form can be written as a linear combination of $dx$ and $dy$:

$$\lambda = Pdx + Qdy,$$

where $P, Q : U \to \mathbb{R}$ are functions on the domain $U$. In particular,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$
A differential 1-form is called \textit{exact} if \( \lambda = df \). The function \( f \) is called the \textit{primitive} of the 1-form \( \lambda \). The primitive is defined uniquely up to adding a constant.

Not every closed differential 1-form \( \lambda = Pdx + Qdy \) is exact. The necessary condition for exactness is that \( \lambda \) is \textit{closed} which by definition means \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \). The necessity of this condition for exactness follows from the mixed derivatives equality (assuming that the coefficients \( P, Q \) are \( C^1 \)-smooth). Indeed, if \( P = \frac{\partial f}{\partial x} \) and \( Q = \frac{\partial f}{\partial y} \). Then

\[
\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.
\]

On the other hand the closedness of \( \lambda \) is not sufficient for its exactness, as it is demonstrated by an example of a 1-form \( d\phi \) on \( \mathbb{R}^2 \setminus 0 \) (written in polar coordinates). We will discuss a bit later the precise argument for that, but intuitively clear that the primitive of this form is not a univalent function on \( \mathbb{R}^2 \setminus 0 \). On the other hand, as we will see below any closed 1-form on \( \mathbb{R}^2 \), or more generally on any \textit{simply connected domain} in \( \mathbb{R}^2 \) is exact.

We will also consider complex-valued differential 1-forms. A \( \mathbb{C} \)-valued differential 1-form is a field of \( \mathbb{C} \)-valued linear in the real sense functions, or simply it is an expression \( \alpha + i\beta \), where \( \alpha, \beta \) are usual real-valued differential 1-forms. All usual operations on complex valued 1-forms are defined in the same way as for real-valued forms, and in addition such forms can be multiplied by complex-valued functions.

Note that a complex-valued function (or 0-form) on a domain \( U \subset \mathbb{C} \) is just a map \( f = u + iv : U \to \mathbb{C} \). Its differential \( df \) is the same as the differential of this map, but it also can be viewed as a \( \mathbb{C} \)-valued differential 1-form \( df = du + idv \).

\textbf{Example 3.1.}

\[
dz = dx + idy, d\bar{z} = dx - idy, zdz = (x + iy)(dx + idy) = xdx - ydy + i(xdy + ydx),
\]

We have

\[
dx = \frac{1}{2}(dz + d\bar{z}), \quad dy = -\frac{i}{2}(dz - d\bar{z}).
\]

Hence, any complex valued 1-form \( \lambda \) can be written as a linear combination of forms \( dz \) and \( d\bar{z} \):

\[
\lambda = f dz + g d\bar{z},
\]

which is a decomposition of \( \lambda \) into a sum of complex linear and complex anti-linear parts.
Lemma 3.2. The form \( \lambda = f dz + gd\bar{z} \) is closed if and only if
\[
\frac{\partial f}{\partial z} = \frac{\partial g}{\partial \bar{z}}.
\]

Proof.

\[ \lambda = f dz + gd\bar{z} = f(dx + idy) + g(dx - idy) = (f + g)dx + i(f - g)dy. \]

The closedness of \( \lambda \) means by definition that
\[
\frac{\partial(f + g)}{\partial y} = \frac{\partial(i(f - g))}{\partial x},
\]
which is equivalent to
\[
\frac{\partial f}{\partial y} - i\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y} - i\frac{\partial g}{\partial x}.
\]

Dividing both parts by \((-i)\) we get
\[
\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y},
\]
and hence
\[
\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y} \right) = \frac{\partial g}{\partial z}.
\]

Let us express the differential \( df \) of a complex valued function \( f \) as a combination of differential forms \( dz = dx + idy \) and \( d\bar{z} = dx - idy \) parts.

Lemma 3.3. For any complex valued function \( f = u + iv \) we have
\[
df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.
\]

In particular, when \( f \) is holomorphic we have
\[
df = \frac{\partial f}{\partial z} dz = f'(z)dz.
\]

Proof. We have
\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{1}{2} \left( \frac{\partial f}{\partial x} dx + d\bar{z} \right) - \frac{i}{2} \left( \frac{\partial f}{\partial y} dx - d\bar{z} \right)
\]
\[
= \frac{1}{2} \left( \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} \right) dz + \frac{1}{2} \left( \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \right) d\bar{z} = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.
\]

\[\Box\]
3.2 Holomorphic 1-forms

A complex-valued 1-form $\lambda$ is called holomorphic if it is equal to $f dz$ for a holomorphic function $f$.

**Lemma 3.4.** The form $f dz$ is closed in a domain $U$ if and only if the function $f$ is holomorphic in $U$.

**Proof.** According to Lemma 3.2, closedness of $f dz$ is equivalent to $\frac{\partial f}{\partial \bar{z}} = 0$, which, in turn, is equivalent to the holomorphicity of $f$. $\blacksquare$

**Example 3.5.** The holomorphic form $\frac{dz}{z^n}$, $n \geq 1$, on $\mathbb{C} \setminus 0$ is always closed. It is exact if and only if $n > 1$.

Indeed, if $n > 1$ then

$$\frac{dz}{z^n} = d\left(\frac{dz}{(1-n)z^{n-1}}\right).$$

If $n = 1$ we have in polar coordinates

$$\frac{dz}{z} = \frac{d(re^{i\phi})}{re^{i\phi}} = \frac{e^{i\phi} dr + ire^{i\phi} d\phi}{r} = \frac{dr}{r} + id\phi = d(lnr) + id\phi,$$

but we already discussed above that the form $d\phi$ is not exact.

3.3 Integration of differential 1-forms along curves

**Curves as paths**

A path, or parametrically given curve in a domain $U \subset \mathbb{R}^2 \gamma : [a, b] \rightarrow U$. We will assume in what follows that all considered paths are differentiable. Given a differential 1-form $\alpha = Pdx + Qdy$ in $U$ we define the integral of $\alpha$ over $\gamma$ by the formula

$$\int_{\gamma} \alpha = \int_{a}^{b} \gamma^* \alpha.$$  

Denoting the coordinate functions of $\gamma(t)$ by $x(t)$ and $y(t)$ (i.e. $\gamma(t) = (x(t), y(t))$) the pull-back differential form $\gamma^* \alpha$ is by definition equal to

$$\gamma^* \alpha = P(x(t), y(t))dx(t) + Q(x(t), y(t))dy(t) = (P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t))dt,$$
so that
\[
\int_{\gamma} \alpha = \int_{a}^{b} \gamma^* \alpha = \int_{a}^{b} (P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t))dt.
\]

An important property of the integral of a differential 1-form is that it does not depend on the parameterization of the curve.

**Proposition 3.6.** Let a path \( \tilde{\gamma} \) be obtained from \( \gamma : [a,b] \to U \) by a reparameterization, i.e. \( \tilde{\gamma} = \gamma \circ \phi \), where \( \phi : [c,d] \to [a,b] \) is an orientation preserving diffeomorphism. Then \( \int_{\tilde{\gamma}} \alpha = \int_{\gamma} \alpha \).

Thus the integral \( \int_{\gamma} \alpha \) depends only on the curve \( \gamma \) as an oriented submanifold and not on a particular parameterization which is compatible with the orientation. For instance, given a unit circle \( S^1 = \{ |z| = 1 \} \) oriented counter-clockwise we can compute \( \int_{S^1} d\phi = 2\pi \). Indeed, the circle can be parameterized by the angular coordinate \( \phi \in [0,2\pi] \), and this parameterization is compatible with the counter-clockwise orientation. Hence, \( \int_{S^1} d\phi = \int_{0}^{2\pi} d\phi = 2\pi \).

**Exercise 3.7.** Compute \( \int_{S} \frac{dz}{z} \)

**Solution.** Let us parameterize the circle by polar coordinates: \( z = e^{i\phi}, \phi \in [0,2\pi] \). Then \( \frac{dz}{z} = id\phi \) and \( \int_{S} \frac{dz}{z} = i \int_{S^1} d\phi = 2\pi i \).

### 3.4 Integrals of closed and exact differential 1-forms

**Theorem 3.8.** Let \( \alpha = df \) be an exact 1-form in a domain \( U \subset \mathbb{C} \). Then for any path \( \gamma : [a,b] \to U \) which connects points \( A = \gamma(a) \) and \( B = \gamma(b) \) we have
\[
\int_{\gamma} \alpha = f(B) - f(A).
\]

In particular, if \( \gamma \) is a loop then \( \oint_{\gamma} \alpha = 0 \).

Similarly for an oriented curve \( \Gamma \subset U \) with boundary \( \partial \Gamma = B - A \) we have
\[
\int_{\Gamma} \alpha = f(B) - f(A).
\]
Proof. We have \( \int_{\gamma} df = \int_{a}^{b} \gamma^* df = \int_{a}^{b} d(f \circ \gamma) = f(\gamma(b)) - f(\gamma(a)) = f(B) - f(A). \) ■

It turns out that closed forms are \textit{locally exact}. A domain \( U \subset V \) is called \textit{star-shaped} with respect to a point \( a \in V \) if with any point \( x \in U \) it contains the whole interval \( I_{a,x} \) connecting \( a \) and \( x \), i.e. \( I_{a,x} = \{ a + t(x - a); t \in [0,1] \} \). In particular, any convex domain is star-shaped.

\textbf{Proposition 3.9.} Let \( \alpha \) be a closed 1-form in a star-shaped domain \( U \subset V \). Then it is exact.

\textbf{Proof.} Define a function \( F : U \to \mathbb{R} \) by the formula

\[
F(x) = \int_{I_{a,x}} \alpha, \ x \in U,
\]

where the intervals \( I_{a,x} \) are oriented from 0 to \( x \).

We claim that \( dF = \alpha \). Let us identify \( V \) with the \( \mathbb{R}^n \) choosing \( a \) as the origin \( a = 0 \). Then \( \alpha \) can be written as \( \alpha = \sum_{k=1}^{n} P_k(x) dx_k \), and \( I_{0,x} \) can be parameterized by

\[
t \mapsto tx, \ t \in [0,1].
\]

Hence,

\[
F(x) = \int_{I_{0,x}} \alpha = \int_{0}^{1} \sum_{k=1}^{n} P_k(tx) x_k dt.
\] (3.4.1)

Differentiating the integral over \( x_j \) as parameters, we get

\[
\frac{\partial F}{\partial x_j} = \int_{0}^{1} \sum_{k=1}^{n} tx_k \frac{\partial P_k}{\partial x_j}(tx) dt + \int_{0}^{1} P_j(tx) dt.
\]

But \( d\alpha = 0 \) implies that \( \frac{\partial P_k}{\partial x_j} = \frac{\partial P_j}{\partial x_k} \), and using this we can further write

\[
\frac{\partial F}{\partial x_j} = \int_{0}^{1} \sum_{k=1}^{n} tx_k \frac{\partial P_j}{\partial x_k}(tx) dt + \int_{0}^{1} P_j(tx) dt = \int_{0}^{1} t \frac{dP_j(tx)}{dt} dt + \int_{0}^{1} P_j(tx) dt
\]

\[
= (tP_j(tx))|_0^1 - \int_{0}^{1} P_j(tx) dt + \int_{0}^{1} P_j(tx) dt = P_j(tx)
\]
Thus

\[ dF = \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} dx_j = \sum_{j=1}^{n} P_j(x) dx = \alpha \]

Given a differential 1-form \( \alpha = P \, dx + Q \, dy \) we will define \( \int_{\Gamma} |\alpha| \) as

\[ \int_{\Gamma} |\alpha| := \int_{a}^{b} \left| P(x(t), y(t)) x'(t) + Q(x(t), y(t)) y'(t) \right| \, dt, \]

where \((x(t), y(t), t \in [a, b])\), is a parameterization of \( \Gamma \). Unlike \( \int_{\Gamma} \alpha \) the integral \( \int_{\Gamma} |\alpha| \) is non-negative and does not depend on the orientation of \( \Gamma \). Clearly, we have

\[ \left| \int_{\Gamma} \alpha \right| \leq \int_{\Gamma} |\alpha|, \]

and

\[ \int_{\Gamma} |\alpha + \beta| = \int_{\Gamma} |\alpha| + \int_{\Gamma} |\beta|. \]
Chapter 4

Cauchy integral formula

4.1 Stokes/Green theorem

Given a bounded domain $U \subset \mathbb{C}$ with a smooth (or piece-wise smooth boundary) we will always orient it boundary $\partial U$ as follows. For each point $p \in \partial U$ take an outward normal vector $\nu$. Then $i\nu$ is tangent to $\partial U$ and defines its orientation. For instance, suppose $A$ is the annulus $1 \leq |z| \leq 2$. Its boundary is the union of two circles: $S_1 = \{|z| = 1\}$ and $S_2 = \{|z| = 2\}$. Then $A$ induces on the outer circle $S_2$ the counter-clockwise orientation, and on the inner circle $S_1$ the clockwise orientation.

The fundamentally important fact about integration of 1-forms is the following theorem which belongs to George Green and it is a special case of a more general result, called Stokes’ theorem (which was not actually proved by George Stokes!)

**Theorem 4.1.** Let $U \subset \mathbb{C}$ be a bounded domain with a piecewise smooth boundary $\partial U$, and $\alpha = Pdx + Qdy$ a differential 1-form on $U$ with coefficients which are $C^1$-smooth in $U$ and continuous in the closure $\overline{U}$. Let us orient the curve $\partial U$ as the boundary of $U$. Then

$$\int_{\partial U} Pdx + Qdy = \iint_U \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$ 

**Corollary 4.2.** Suppose a 1-form $\alpha$ is closed in a domain $U$. Then

$$\int_{\partial U} \alpha = 0.$$
Indeed, closedness of \( \alpha = Pdx + Qdy \) just means that \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \).

### 4.2 Cauchy theorem and Cauchy integral formula

**Corollary 4.3 (Cauchy theorem).** Let \( f \) be a function, holomorphic in the domain \( U \) and continuous up to the boundary. Then

\[
\oint_{\partial U} f(z) \, dz = 0.
\]

**Proof.** According to Lemma 3.4 the holomorphic differential 1-form \( f(z) \, dz \) is closed in \( U \). \( \blacksquare \)

**Example 4.4.** Let \( U \subset \mathbb{C} \) be any domain such that \( 0 \in U \). Then

\[
\oint_{\partial U} \frac{dz}{z^n} = \begin{cases} 2\pi i, & n = 1, \\ 0, & \text{otherwise.} \end{cases}
\]

Indeed, for \( n > 1 \) the 1-form \( \frac{dz}{z^n} \) is exact in \( U \setminus 0 \), and the integral of an exact form over any closed loop is equal to 0.

If \( n = 1 \) consider a disc \( D_\epsilon = \{ |z| < \epsilon \} \subset U \). Then according to Corollary 4.2

\[
0 = \oint_{\partial (U \setminus \{D_\epsilon\})} \frac{dz}{z} = \oint_{\partial U} \frac{dz}{z} - \oint_{\partial D_\epsilon} \frac{dz}{z}.
\]

But we already computed that \( \oint_{\{|z|=\epsilon\}} \frac{dz}{z} = 2\pi i \), and therefore

\[
\oint_{\partial U} \frac{dz}{z} = 2\pi i.
\]

**Theorem 4.5 (Cauchy integral formula).** Suppose that \( f : \overline{U} \to \mathbb{C} \) is a continuous function which is holomorphic in \( U \). Then for any \( u \in U \) we have

\[
\frac{1}{2\pi i} \oint_{\partial U} \frac{f(z) \, dz}{z - u} = f(u).
\]

**Proof.** Let \( D_\delta(u) \) denote the disc \( \{ |z - u| < \epsilon \} \) centered at \( u \), where \( 0 < \delta < | - |u| \). The function \( \frac{f(z)}{z - u} \) is holomorphic in \( U \setminus u \), and therefore according to the Cauchy theorem 4.3 we have

\[
\oint_{\partial (U \setminus D_\delta(u))} \frac{f(z) \, dz}{z - u} = 0.
\]
Hence,
\[ \int_{\partial U} \frac{f(z)dz}{z-u} = \int_{\partial D(u)} \frac{f(z)dz}{z-u} = \int_{|w| = \delta} \frac{f(u+w)dw}{w}. \]

The function \( f \) is continuous at the point \( u \). Hence, for any \( \epsilon \) there exists \( \delta > 0 \) such that if \( |w| \leq \delta \) then
\[ |f(u + w) - f(u)| < \epsilon. \]

Hence,
\[ \left| \int_{|w| = \delta} \frac{f(u+w)dw}{w} - \int_{|w| = \delta} \frac{f(u)dw}{w} \right| \leq \frac{2\pi}{\delta} \int_{0}^{\delta} |f(u + w) - f(u)|d\phi \leq 2\pi \epsilon. \]

Note that according to Example 4.4
\[ \int_{|w| = \delta} \frac{f(u)dw}{w} = \frac{2\pi i f(u)}{w}. \]
Thus
\[ \left| \int_{\partial U} \frac{f(z)dz}{z-u} - 2\pi i f(u) \right| \leq 2\pi \epsilon \]
for any \( \epsilon > 0 \). But the left-hand side is independent of \( \epsilon \), and therefore
\[ \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)dz}{z-u} = f(u). \]

As the first application of the Cauchy integral formula we prove the infinite differentiability of a holomorphic function.

**Corollary 4.6.** Any holomorphic in a domain \( U \) function is infinitely differentiable at every point. Its derivatives can be computed by the formula
\[ f^{(k)}(u) = \frac{k!}{2\pi i} \int_{\partial U} \frac{f(z)dz}{(z-u)^{k+1}}. \]

**Proof.** The variable \( u \) enters the integral \( \int_{\partial D} \frac{f(z)dz}{z-u} \) as a parameter. The integrand \( \frac{f(z)dz}{z-u} \) is differentiable with respect to the parameter, and hence the integral itself is differentiable with respect to
and we can compute the derivative \( f'(u) \) by the differentiating the integral with respect to the parameter, i.e.

\[
f'(u) = \frac{d}{du} \left( \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)dz}{z-u} \right) = \frac{1}{2\pi i} \int_{\partial U} \left( \frac{f(z)dz}{z-u} \right) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)dz}{(z-u)^2}.
\]

Applying the same argument to the integral \( \int_{\partial U} \frac{f(z)dz}{(z-u)^2} \) we compute \( f''(u) \), etc. 

**Corollary 4.7** (Cauchy inequality). Let \( f : U \to \mathbb{C} \) be a holomorphic function. Suppose that for a point \( z_0 \) the closed disc \( \overline{D}_r(z_0) = \{ |z - z_0| \leq r \} \) is contained in \( U \). Then

\[
|f^n(z_0)| \leq \frac{M n!}{r^n},
\]

where \( M := \max_{|z-z_0|=r} |f(z)| \).

**Proof.** By the Cauchy integral formula we have

\[
f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|\zeta|=r} \frac{f(z_0 + \zeta)d\zeta}{\zeta^{n+1}}.
\]

Therefore,

\[
|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{Mrd\theta}{r^{n+1}} = \frac{M n!}{r^n}.
\]
Chapter 5

Convergent power series and holomorphic functions

5.1 Recollection of basic facts about series

Let us recall that a series \( \sum_{0}^{\infty} b_k \), where \( b_j \) are complex numbers, is called \emph{converging} if there exists a finite limit of \emph{partial sums}

\[
S := \lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{k=0}^{N} b_k.
\]

In this case we write \( \sum_{0}^{\infty} b_k = S \). A necessary and sufficient condition for convergence is given by \emph{Cauchy criterion}:

For any \( \epsilon > 0 \) there exists \( N \) such that for any \( n \geq N \) and \( m > 0 \) we have

\[
\left| \sum_{k=n}^{n+m} b_k \right| < \epsilon.
\]

A series \( \sum_{0}^{\infty} b_k \) is called \emph{absolutely converging} if the series \( \sum_{0}^{\infty} |b_k| \) is converging. Absolute convergence implies convergence, as it immediately follows from the Cauchy criterion and the inequality

\[
\left| \sum_{k=n}^{n+m} b_k \right| \leq \sum_{k=n}^{n+m} |b_k|.
\]

An important tool for establishing an absolute convergence (or divergence) is the following \emph{comparison criterion}: 

\[
\sum_{k=n}^{n+m} b_k \leq \sum_{k=n}^{n+m} |b_k|.
\]
Lemma 5.1. Let $\sum a_n$ and $\sum b_n$ be two series such that $a_n, b_n \geq 0$. Suppose that there exists $N$ such that for $n \geq N$ we have $a_n \leq b_n$. Then if $\sum b_n$ is converging so so does $\sum a_n$, and if $\sum a_n$ is diverging so so does $\sum b_n$.

5.2 Power series

A power series is a series of the form $\sum_{0}^{\infty} a_n z^n$, $a_n z \in \mathbb{C}$.

A remarkable fact about power series is existence of a radius of convergence.

Proposition 5.2. For any power series $\sum_{0}^{\infty} a_n z^n$ there exists $R$ (which could be $0$ or $\infty$) such that for $|z| < R$ the power series is absolutely converging and for $|z| > R$ it is diverging. The radius of convergence $R$ can be computed by the following formula (due to Jacques Hadamard):

$$\frac{1}{R} = \lim \sup |a_n|^\frac{1}{n}.$$

Proof. The proof follows from the comparison with a geometric series $\sum r^n$ which converges when $r < 1$ and diverges when $r \geq 1$. Indeed, for any $r < R$ we have $|a_n| < \frac{1}{r^n}$ for a sufficiently large $n$. Hence, if $|z| < r$ then $|a_n||z|^n < \left(\frac{|z|}{r}\right)^n$, and therefore the power series $\sum_{0}^{\infty} a_n z^n$ is absolutely converging due to the comparison with the geometric series $\sum_{0}^{\infty} \left(\frac{|z|}{r}\right)^n$. But $r$ is any number $< R$, and hence $\sum_{0}^{\infty} a_n z^n$ is absolutely converging for all $Z < R$.

If $|z| > R$ then there exists infinitely many $n$ such that $|a_n|^n > \frac{1}{|z|}$, and hence for these values of $n$ we have $|a_n||z|^n > 1$. This implies that $\sum_{0}^{\infty} a_n z^n$ is diverging because the common term of a converging series must converge to 0.

The disc $\{ |z| < R \}$ is called the disc of convergence.

Exercise 5.3. Verify the following statements.

1. The radius of convergence of the geometric series geometric sequence $\sum_{0}^{\infty} z^n$ is equal to 1. On the boundary of $\{ |z| = 1 \}$ of the disc of convergence the series diverges at every point.

2. The radius of convergence of the geometric series geometric sequence $\sum_{1}^{\infty} z^n$ is also equal to 1. However, the behavior n the boundary of the disc of convergence is different: the series is convergent at every point except $z = 1.$
3. The radius of convergence of the exponential series \( \sum_{n=0}^{\infty} \frac{z^n}{n!} \) is equal to \( \infty \), i.e. the series is absolutely converging on the whole \( \mathbb{C} \).

4. The radius of convergence of the series \( \sum_{n=0}^{\infty} n!z^n \) is equal to 0, i.e. the series id divergent for any \( z \neq 0 \).

**Exercise 5.4.** [Operations on converging power series] Suppose power series \( \sum_{n=0}^{\infty} a_n z^n \) and \( \sum_{n=0}^{\infty} b_n z^n \) are converging for \( |z| < R \). Denote \( A(z) := \sum_{n=0}^{\infty} a_n z^n \), \( B(z) := \sum_{n=0}^{\infty} b_n z^n \) and \( \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) z^n \) are also converging for \( |z| < R \) and

\[
A(z) + B(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n,
\]
\[
A(z)B(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) z^n.
\]

**Proposition 5.5.** Suppose the series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is converging in the disc \( D_R := \{ |z| < R \} \). Then

a) The series \( \sum_{n=1}^{\infty} n a_n z^{n-1} \) is also converging in \( D_R \) and \( \sum_{n=1}^{\infty} n a_n z^{n-1} = f'(z) \). Thus, the sum of a power series is holomorphic in the disc of its convergence.

b) The series \( \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} \) is converging in \( D_R \) to a function \( F(z) \) such that \( F'(z) = f(z) \).

**Proof.** First observe that according to the Hadamard criterion the power series \( \sum_{n=0}^{\infty} a_n z^n \), \( \sum_{1}^{\infty} n a_n z^{n-1} \) and \( \sum_{0}^{\infty} \frac{a_n}{n+1} z^{n+1} \) have the same radius of convergence. Indeed,

\[
\limsup(n|a_n|)^{\frac{1}{n}} = \limsup \left( \frac{|a_n|}{n+1} \right)^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}}.
\]

I refer the reader to Stein-Shakarchi’s book, page 17, for a proof of statement a). To prove b) we apply a) to the series \( F(z) = \sum_{0}^{\infty} \frac{a_n}{n+1} z^{n+1} \).
5.3 Analytic vs holomorphic

A function \( f : U \to \mathbb{C} \) is called analytic if in a neighborhood of any point \( z_0 \in U \) it can be presented as a sum of a converging power series.

**Lemma 5.6.** Given an analytic function \( f : U \to \mathbb{C} \), then the coefficients of its power expansion are equal to its Taylor coefficients, i.e. for any point \( z_0 \) and a sufficiently small \( \epsilon > 0 \) we have

\[
    f(z_0 + u) = f(z_0) + f'(z_0)u + \frac{f''(z_0)}{2}u^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}u^n,
\]

for \( |u| < \epsilon \).

Indeed, according to Proposition 5.5, a converging power series can be differentiated term-wise in the disc of its convergence. Hence, if \( f(z_0 + u) = \sum_{n=0}^{\infty} a_n u^n \), then \( f(z_0) = a_0 \), \( f'(z_0 + u) = \sum_{n=1}^{\infty} na_n u^{n-1} \), and hence \( f'(z_0) = a_1 \). Continuing this process we get the required formula \( a_n = \frac{f^{(n)}(z_0)}{n!} \).

**Theorem 5.7.** The notions of a holomorphicity and analyticity are equivalent.

**Proof.** According to Proposition 5.5, any analytic function is holomorphic. To see the converse, take any point \( z_0 \in U \) and choose \( r > 0 \) such that the closed disc \( \overline{D} = \{|z - z_0| \leq r\} \) of radius \( r \) centered at \( z_0 \) is contained in \( U \). Changing the variable \( u := z - z_0 \) we can express the function \( f(u) \) in the open disc \( D = \{|u| < r\} \) by Cauchy formula

\[
    f(u) = \frac{1}{2\pi i} \int_{|\zeta| = r} \frac{f(\zeta)d\zeta}{\zeta - u}.
\]

We have

\[
    \frac{f(\zeta)}{\zeta - u} = \frac{f(\zeta)}{\zeta} \frac{1}{1 - \frac{u}{\zeta}} = \frac{f(\zeta)}{\zeta} \sum_{n=0}^{\infty} \frac{u^n}{\zeta^n}.
\]

Let us prove that the power series in the right hand side can be integrated term-wise and thus we get

\[
    \int_{|\zeta| = r} \frac{f(\zeta)d\zeta}{\zeta - u} = \sum_{n=0}^{\infty} \left( \int_{|\zeta| = r} \frac{f(\zeta)d\zeta}{\zeta^{n+1}} \right) u^n. \tag{5.3.1}
\]

Then for any \( u \in D, |u| \leq \rho \) we have

\[
    \left| \frac{f(\zeta)}{\zeta^{n+1}} \right| \leq \frac{M_r}{\rho^{n+1}},
\]

where \( M_r \) is the maximum of \( f(\zeta) \) on the circle \( |\zeta| = r \).
where we denoted $M_r := 2\pi \max_{|\zeta|=r} |f(\zeta)|$, and hence the power series in the right-hand side absolutely converges in $D$. Let us choose any $\rho < r$. Then for any $u \in D$, $|u| \leq \rho$ we have

\[
\left| \int_{|\zeta|=r} \frac{f(\zeta)d\zeta}{\zeta - u} - \sum_{n=0}^{N} \left( \int_{|\zeta|=r} \frac{f(\zeta)d\zeta}{\zeta^{n+1}} \right) u^n \right| \leq \sum_{N+1}^{\infty} \left( \int_{|\zeta|=r} \frac{f(\zeta)d\zeta}{\zeta^{n+1}} \right) |u|^n \\
\leq \frac{M_r}{r} \sum_{N+1}^{\infty} \left( \frac{\rho}{r} \right)^n = \frac{M_r \rho^{N+1}}{r^{N+2}(1 - \frac{\rho}{r})} = \frac{M_r \rho^{N+1}}{r^{N+1}(r - \rho)} \xrightarrow{N \to \infty} 0.
\]

This proves formula (5.3.1) for any $u \in D$ because $\rho$ is an arbitrary number $< r$. ■

**Remark 5.8.** The above argument also shows that if $f : U \to \mathbb{C}$ is a holomorphic function and for $a \in U$ the disc $D_r(a) = \{|z - a| < r\}$ is contained in $U$ then the radius $R$ of convergence of the Taylor expansion of $f$ at the point $a$ satisfies the inequality $R \geq r$. 

33
Chapter 6

Properties of holomorphic functions

6.1 Exponential function and its relatives

So far the only examples of holomorphic functions we had were polynomials and rational functions $P(z)/Q(z)$, where $P, Q$ are polynomials, in the domain where $Q(z) \neq 0$. The theorem equating holomorphic and analytic functions allows us to greatly extend the set of examples. We begin in this section with exponential function and its close relatives.

As we already pointed out the exponential function is defined by the formula

$$f(z) = \sum_{0}^{\infty} \frac{z^n}{n!}.$$ 

We also define

$$\sin z = \sum_{0}^{\infty} (-1)^{\infty} \frac{z^{2n+1}}{(2n+1)!!},$$

$$\cos z = \sum_{0}^{\infty} (-1)^{\infty} \frac{z^{2n}}{(2n)!!},$$

$$\sinh z = \frac{e^z - e^{-z}}{2} = \sum_{0}^{\infty} \frac{z^{2n+1}}{(2n+1)!!},$$

$$\cosh z = \frac{e^z + e^{-z}}{2} = \sum_{0}^{\infty} \frac{z^{2n}}{(2n)!!}.$$ 

The radius of convergence of all these series is $\infty$, and hence the above formulas define holomorphic function on the whole $\mathbb{C}$. 

35
Lemma 6.1. 1) $e^{z_1+z_2} = e^{z_1}e^{z_2};$

2) $(e^z)' = e^z;

3) $e^{iz} = \cos z + i \sin z;

4) \cos z = \cosh iz, \sin z = -i \sinh iz.$

First two properties follows from the formulas of multiplication and differentiation of power of series, see Exercise 5.4 and Proposition 5.5a). Formula 3) follows the comparison of series in the left and right hand sides. Formula 4) follows from 3).

It is also interesting to observe that the exponential function is periodic with the imaginary period $2\pi i$.

6.2 Entire functions

Functions which are holomorphic on the whole $\mathbb{C}$ are called entire. The functions $e^z, \sin z, \cos z, \sinh z, \cosh z$ considered in Section 6.1 are examples of entire functions. The sum of any power series with the infinite radius of convergence is an entire holomorphic function. Remark 5.8 implies that the converse is also true:

If $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function, then its Taylor expansion at any point has a infinite radius of convergence.

Theorem 6.2 (Liouville’s theorem). If an entire function is bounded it is a constant.

Proof. This is a corollary of the Cauchy inequality 4.7. Indeed, suppose $|f(z)| \leq M$, then the inequality 4.7 implies that for any $k$ we have

$$|f^k(0)| \leq \frac{Mk!}{R^k} \text{ for any } R > 0.$$ 

Hence $f^k(0) = 0$ for $k > 0$. Therefore,

$$f(z) = f(0) + f'(0)z + \frac{1}{2}f''(0)z^2 + \cdots = f(0).$$
**Theorem 6.3** (Fundamental theorem of algebra). Any polynomial \( P(z) = a_0 + a_1 z + \cdots + a_n z^n, a_n \neq 0 \), of degree \( n > 0 \) has a root.

**Proof.** Suppose \( P(z) \neq 0 \) for all \( z \in \mathbb{C} \). Then \( g(z) := \frac{1}{P(z)} \) is an entire holomorphic function. On the other hand, \(|P(z)| \geq |z^n| \left( |a_n| - \frac{|a_{n-1}|}{|z|} - \frac{|a_{n-2}|}{|z|^2} - \cdots - \frac{|a_0|}{|z|^n} \right)\). But \( \frac{|a_{n-2}|}{|z|^2} + \cdots + \frac{|a_0|}{|z|^n} \to 0 \), and therefore \( \frac{|a_{n-2}|}{|z|^2} + \cdots + \frac{|a_0|}{|z|^n} \leq \frac{|a_n|}{2} \) for \(|z|\) sufficiently large. But then \( |P(z)| \geq |z^n| \left( |a_n| - \frac{|a_{n-1}|}{|z|} - \frac{|a_{n-2}|}{|z|^2} - \cdots - \frac{|a_0|}{|z|^n} \right) \geq \frac{|a_n||z|^n}{2} \), and therefore \(|g(z)| \leq \frac{2}{|a_n||z|^n} |z| \to 0\). This implies that the function \( g \) is bounded, and therefore by Liouville’s theorem it is constant. But this contradicts to the assumption that the degree \( n \) is positive. \( \blacksquare \)

Theorem 6.3 implies that the polynomial \( P(z) \) of degree \( n \) with complex coefficients has \( n \) roots, counted with multiplicities. Indeed, by Theorem 6.3 there exists at least one root \( z_1 \). Then \( P(z) \) can be divided by \( (z - z_1) \):

\[ P(z) = (z - z_1)P_1(z), \]

where \( \deg P_1 = \deg P - 1 \). If degree of \( P_1(z) \) is still positive one can continue the process and get \( P(z) = (z - z_1)(z - z_2)P_2(z) \). Continuing the process we decompose \( P \) into a product of linear terms:

\[ P(z) = a(z - z_1) \cdots (z - z_n). \]

**6.3 Analytic continuation**

Let us recall that a domain \( U \) is called *connected* if one cannot present it as the union \( U = U_1 \cup U_2 \) of disjoint non-empty open sets. Equivalently, a *disconnected* domain is a domain which admits a continuous function on \( U \) which takes exactly two values: 0 and 1.
There is a related notion of path-connectedness. A domain $U \subset \mathbb{C}$ is called path-connected if for any two points $A, B \in U$ there exists a continuous path $\phi : [0, 1] \to U$ such that $\phi(0) = A, \phi(1)$. The notions of connectedness and path connectedness coincide for open sets (for more general sets path connectedness is a stronger notion).

**Lemma 6.4.** Let $f : U \to \mathbb{C}$ be a holomorphic function on a connected domain $U$. Suppose that there exists a sequence of distinct points $z_n \in \mathbb{C}$, $n = 1, \ldots, \infty$, such that $f(z_n) = 0$, and $\lim_{n \to \infty} z_n = a \in U$. Then $f \equiv 0$ in $U$.

**Proof.** Denote by $A$ the set of all points $a \in U$ which satisfy the conditions of the lemma, i.e. that there exists a sequence of distinct points $z_n \in \mathbb{C}$, $n = 1, \ldots, \infty$, such that $f(z_n) = 0$, and $\lim_{n \to \infty} z_n = a$. Then by continuity we have $f(a) = 0$ for every $a \in U$. Let us prove that the set $A$ is open. For every $a \in A$ the holomorphic function $f$ can be expanded to a converging power series in a sufficiently small disc centered at $a$:

$$f(a + u) = c_1 u + c_2 u^2 + \ldots.$$  

If $f$ is not identically 0 in a neighborhood of $a$ then there is $k > 0$ such that $c_k \neq 0$ and $c_j = 0$ for all $j < k$. Then

$$f(a + u) = c_k u^k (1 + g(u)), \text{ where } g(u) = \frac{c_{k+1}}{c_k} u + \frac{c_{k+2}}{c_k} u^2 + \ldots.$$  

The function $g$ is holomorphic in a neighborhood of $u = 0$ and we have $g(0) = 0$. Hence, there exists $r > 0$ such that $|g(u)| < \frac{1}{2}$ for $|u| < r$. Therefore,

$$|f(a + u)| \geq \frac{1}{2} |c_k||u|^k, \text{ for } |u| < r.$$  

But this implies that $f(z) \neq 0$ provided that $z \neq a$ and $|z - a| < r$. But this contradicts the assumption of existence of a sequence $z_n \to a$ such that $f(z_n) = 0$. Hence, $f$ is identically equal to 0 in a neighborhood of $a$, i.e. $A$ is open.

Suppose that $U \setminus A \neq \emptyset$. Then for any $b \in U \setminus A$ the point $b$ there is a neighborhood $U_b \subset U$ where there is no zeroes of $f$ with a possible exception of $b$. But then $U_b \subset U \setminus A$, and hence $U \setminus A$ is open. But this contradicts the connectedness of $U$, and hence $A = U$, i.e. the function $f$ is equal to 0 identically on $U$.  

■
Given domains $U \subset V \subset \mathbb{C}$ we say that a holomorphic function $f : V \to \mathbb{C}$ is a holomorphic extension of a holomorphic function $g : U \to \mathbb{C}$ if $f|_U = g$. Lemma 6.4 implies that any two holomorphic extensions of $f$ to a bigger domain coincide.

**Example 6.5.** 1) The radius of convergence of the series $\sum_0^\infty z^n$ is 1. However the function $f(z) = \frac{1}{1-z}$ provides a holomorphic extension of $\sum_0^\infty z^n$ from the unit disc $\{|z| < 1\}$ to $\mathbb{C} \setminus 0$.

### 6.4 Complex logarithm

Consider a closed differential 1-form $\frac{dz}{z}$ on $\mathbb{C} \setminus 0$. While this form is not exact on $\mathbb{C} \setminus 0$ it becomes exact when restricted to any simply connected subdomain $U \subset \mathbb{C} \setminus 0$. For instance, take $U := \mathbb{C} \setminus R$, where $R$ is the ray $R = \{z \in \mathbb{C}; \Re z \leq 0, 3z = 0\}$. The primitive of $\frac{dz}{z}$, which is called logarithm and denoted by $\log z$ (or $\ln z$), can be computed by the formula

$$\log z = \int_{\Gamma_z} \frac{dz}{z},$$

where $\Gamma_z$ is any path connecting 1 with the point $z$.

As we already computed above $\frac{dz}{z} = d \ln r + id\phi$, and therefore

$$\log z = \int_{\Gamma_z} \frac{dz}{z} = \int_{\Gamma_z} d(\ln r) + i \int_{\Gamma_z} d\phi$$

$$= \ln r + i\phi.$$ 

Thus the real part of the complex logarithm $\log z$ is equal to $\log |z|$, while the imaginary part is equal to $\arg z$.

**Lemma 6.6** (Properties of the logarithm). 1) $e^{\log z} = z$

2) $\log(1 + z) = \sum_1^\infty (-1)^n \frac{z^n}{n}$ for $|z| < 1$.

Indeed, $e^{\log z} = e^{\log r + i\phi} e^{i\phi} = z$. To prove 2) we observe that $d(\log(1 + z)) = \frac{dz}{1+z}$. But $\frac{1}{1+z} = \sum_1^\infty (-1)^n z^n$ for $|z| < 1$. Hence, by integrating both parts of this equality we get $\log(1 + z) = \sum_1^\infty (-1)^n \frac{z^n}{n}$ for $|z| < 1$. 

39
When trying to extend $\log z$ to the whole punctured plane $\mathbb{C} \setminus 0$ we get a multivalued function defined up to a multiple of $2\pi i$.

In particular, the equality $\log z_1 z_2 = \log z_1 + \log z_2$ holds only up to a multiple of $2\pi i$. 