Math 116 Complex Analysis

Additional chapters

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1 Complex Analysis Basics

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Chapter 1

Complex Analysis Basics

1.1 Complex numbers

The space \( \mathbb{R}^2 \) can be endowed with an associative and commutative multiplication operation. This operation is uniquely determined by three properties:

- it is a bilinear operation;
- the vector \((1, 0)\) is the unit;
- the vector \((0, 1)\) satisfies \((0, 1)^2 = -(1, 0)\).

The vector \((0, 1)\) is usually denoted by \(i\), and we will simply write 1 instead of the vector \((1, 0)\). Hence, any point \((a, b) \in \mathbb{R}^2\) can be written as \(a + bi\), where \(a, b \in \mathbb{R}\), and the product of \(a + bi\) and \(c + di\) is given by the formula

\[(a + bi)(c + di) = ac - bd + (ad + bc)i.\]

The plane \( \mathbb{R}^2 \) endowed with this multiplication is denoted by \(\mathbb{C}\) and called the set of complex numbers. The real line generated by 1 is called the real axis, the line generated by \(i\) is called the imaginary axis. The set of real numbers \(\mathbb{R}\) can be viewed as embedded into \(\mathbb{C}\) as the real axis. Given a complex number \(z = x + iy\), the numbers \(x\) and \(y\) are called its real and imaginary parts, respectively, and denoted by Re\(z\) and Im\(z\), so that \(z = \text{Re}z + i\text{Im}z\).
For any non-zero complex number $z = a + bi$ there exists an inverse $z^{-1}$ such that $z^{-1}z = 1$.

Indeed, we can set

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

The commutativity, associativity and existence of the inverse is easy to check, but it should not be taken for granted: it is impossible to define a similar operation any $\mathbb{R}^n$ for $n > 2$.

Given $z = a + bi \in \mathbb{C}$ its conjugate is defined as $\bar{z} = a - bi$. The conjugation operation $z \mapsto \bar{z}$ is the reflection of $\mathbb{C}$ with respect to the real axis $\mathbb{R} \subset \mathbb{C}$. Note that

$$\operatorname{Re}z = \frac{1}{2}(z + \bar{z}), \quad \operatorname{Im}z = \frac{1}{2i}(z - \bar{z}).$$

Let us introduce the polar coordinates $(r, \phi)$ in $\mathbb{R}^2 = \mathbb{C}$. Then a complex number $z = x + yi$ can be written as $r \cos \phi + ir \sin \phi = r(\cos \phi + i \sin \phi)$. This form of writing a complex number is called, sometimes, trigonometric. The number $r = \sqrt{x^2 + y^2}$ is called the modulus of $z$ and denoted by $|z|$ and $\phi$ is called the argument of $\phi$ and denoted by $\arg z$. Note that the argument is defined only mod $2\pi$. The value of the argument in $[0, 2\pi)$ is sometimes called the principal value of the argument. When $z$ is real than its modulus $|z|$ is just the absolute value. We also note that $|z| = \sqrt{z \bar{z}}$.

An important role plays the triangle inequality

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$  

**Exponential function of a complex variable**

Recall that the exponential function $e^x$ has a Taylor expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots.$$  

We then define for a complex $z$ the exponential function by the same formula

$$e^z := 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \ldots.$$  

One can check that this power series absolutely converging for all $z$ and satisfies the formula

$$e^{z_1 + z_2} = e^{z_1}e^{z_2}.$$  

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In particular, we have

\[ e^{iy} = 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + \cdots + \ldots \]  

\[ = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{2k!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}. \]  

But \( \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{2k!} = \cos y \) and \( \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!} = \sin y \), and hence we get Euler’s formula

\[ e^{iy} = \cos y + i \sin y, \]

and furthermore,

\[ e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y), \]

i.e. \( |e^{x+iy}| = e^x, \arg(e^x) = y \). In particular, any complex number \( z = r(\cos \phi + i \sin \phi) \) can be rewritten in the form \( z = re^{i\phi} \). This is called the exponential form of the complex number \( z \).

Note that

\[ (e^{i\phi})^n = e^{in\phi}, \]

and hence if \( z = re^{i\phi} \) then \( z^n = r^n e^{in\phi} = r^n (\cos n\phi + i \sin n\phi) \).

Note that the operation \( z \mapsto iz \) is the rotation of \( \mathbb{C} \) counterclockwise by the angle \( \frac{\pi}{2} \). More generally a multiplication operation \( z \mapsto zw \), where \( w = \rho e^{i\theta} \) is the composition of a rotation by the angle \( \theta \) and a radial dilatation (homothety) \( \rho \) times.

**Exercise 1.1.**

1. Compute \( \sum_{0}^{n} \cos k\theta \) and \( \sum_{1}^{n} \sin k\theta \).

2. Compute \( 1 + \binom{n}{4} + \binom{n}{8} + \binom{n}{12} + \ldots \).

### 1.2 Complex linear function from the real perspective

A linear map \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) is by definition is required to satisfy two conditions:

\[ F(z_1 + z_2) = F(z_1) + F(z_2); \]

\[ F(\lambda z) = \lambda F(z), \]

\(^1\) Convergence of power series will be discussed later.
where $z_1, z_2, z$ are any vectors from $\mathbb{R}^2$ and $\lambda \in \mathbb{R}$ is a real number. Any such map is a multiplication by a $2 \times 2$-matrix:

$$F(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

A linear function of one complex variable is a linear map $F : \mathbb{C} \to \mathbb{C}$ which satisfies in addition, the condition

$$F(\lambda z) = \lambda F(z), \text{ for any complex number } \lambda. \quad (1.2.1)$$

Any such function has to satisfy $F(z) = F(1)z = cz$, where $c = a + ib = F(1)$. Equivalently,

$$F(x + iy) = (a + ib)(x + iy) = ax - by + i(ay + bx).$$

Thus, viewing a complex number $z = x + iy$ as a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ we get

$$F(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In other words, we proved the following

**Lemma 1.2.** A real linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ with a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is complex linear map $\mathbb{C} \to \mathbb{C}$ if and only if $a = d$ and $b = -c$.

In particular, the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the matrix of multiplication by $i$.

Note that

$$\det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2 + b^2 = |c|^2, \text{ where } c = a + ib. \quad (1.2.2)$$

In other words, the (real) determinant of the matrix of the multiplication by a complex number $c$ is equal to $|c|^2$.

We can also view a real linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ as a map $\mathbb{R}^2 \to \mathbb{C}$, i.e. as a complex-valued linear (in a real sense) function $F(x, y) = f_1(x, y) + if_2(x, y)$. If $F$ was given by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $f_1(x, y) = ax + by, f_2(x, y) = cx + dy$. We also have

$$F(x, y) = f_1(x, y) + if_2(x, y) = ax + by + i(cx + dy) = (a + ic)x + (b + id)y = Ax + By. \quad (1.2.3)$$
Note that $x = \frac{1}{2}(z + \bar{z}), y = -\frac{i}{2}(z - \bar{z})$, and hence
\[
F(x, y) = Ax + By = \frac{A}{2}(z + \bar{z}) - \frac{Bi}{2}(z - \bar{z}) = \frac{1}{2}(A - iB)z + \frac{1}{2}(A + iB)\bar{z} = \alpha z + \beta \bar{z}, \quad (1.2.4)
\]
where we denoted $\alpha := \frac{1}{2}(A - iB), \beta := \frac{1}{2}(A + iB)$. Note that the function $l_1(z) = \alpha z$ is complex linear, while the function $l_2(z) = \beta \bar{z}$ is complex anti-linear, which means that it is linear in the real sense, but satisfied the condition $l_2(\lambda z) = \bar{\lambda} l_2(z)$.

If $F$ is a complex linear map, then $\bar{F}$ is anti-linear and vice versa. In particular, every complex anti-linear map $F$ has the form $F(z) = a \bar{z}$ for a complex number $a$.

The following lemma summarizes the above discussion.

**Lemma 1.3.** Any linear in the real sense map $F : \mathbb{C} \to \mathbb{C}$ can be uniquely written as a sum $F = F_1 + F_2$, where $F_1$ is complex linear and $F_2$ is complex anti-linear. If $F$ is given by a matrix
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
then $F_1(z) = \alpha z, F_2(z) = \beta \bar{z}$, where $\alpha := \frac{1}{2}(A - iB), \beta := \frac{1}{2}(A + iB), A = a + ic, B = b + id$.

### 1.3 Differentiability and the differential

For any point $z = (x, y) \in \mathbb{R}^2$ we denote by $\mathbb{R}_z^2$ the space $\mathbb{R}^2$ with the origin, shifted to the point $z$. Though the parallel transport allows one to identify spaces $\mathbb{R}^2$ and $\mathbb{R}_z^2$ it will be important for us to think about them as different spaces.

Let $U$ be a domain in $\mathbb{R}^2$ and a map $f : U \to \mathbb{R}^2$ a function on on it. A vector-valued function $f : U \to \mathbb{R}^2$, where $U \subset \mathbb{R}^2$ a domain in $\mathbb{R}^2$, is called differentiable at a point $a \in U$ if near the point $a$ in can be well approximated by a linear function. More precisely, if there exists a linear map $A : \mathbb{R}_a^2 \to \mathbb{R}_{f(z)}^2$ such that
\[
f(a + h) - f(a) = A(h) + o(||h||)
\]
for any sufficiently small vector $h = (h_1, h_2) \in \mathbb{R}^2$, where the notation $o(t)$ stands for any vector-valued function such that $\frac{o(t)}{t} \to 0$. The linear function $A$ is called the differential of the function.

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$^2$ Complex-valued linear (in the real sense) functions on $\mathbb{R}^2$ form a 2-dimensional complex vector space. Formulas (1.2.3) and (1.2.4) say that the pairs of functions $(x,y)$ as well as the pair $(z, \bar{z})$ form a basis of this space.
The function $f$ at the point $a$ and is denoted by $d_a f$. In other words, $f$ is differentiable at $a \in U$ if for any $h \in \mathbb{R}^2$ there exists a limit
\[
d_a f(h) = \lim_{t \to 0} \frac{f(a + th) - f(a)}{t},
\]
and the limit $A(h)$ linearly depends on $h$. By identifying $\mathbb{R}^2$ and $\mathbb{R}^2_{f(a)}$ with $\mathbb{R}^2$ via the parallel transport we can associate with the linear map $d_a f$ its matrix $J_a(f)$, called the Jacobi matrix or derivative of the map $f$. If we denote by $u(x, y)$ and $v(x, y)$ the coordinate functions of the map $f$, then
\[
J(f) = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix}.
\]

The function $f$ is called differentiable on the whole domain $U$ if it is differentiable at each point of $U$.

### 1.4 Holomorphic functions, Cauchy-Riemann equations

Let us now view the map $f : U \to \mathbb{R}^2$ as a complex valued function $f(z) = u(z) + iv(z)$, $z = x + iy$.

The function $f$ is called differentiable at the point $a$ in a complex sense, or holomorphic at $a$ if the differential $d_a f$ is a complex linear map. The following theorem lists equivalent definitions of holomorphicity.

**Theorem 1.4.** The function $f = u + iv : U \to \mathbb{C}$ is holomorphic at a point $a \in U$ if one of the following equivalent conditions is satisfied:

1. $f(a + h) - f(a) = ch + o(|h|)$, for a complex number $c$;

2. There exists a limit $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$; the limits is denoted by $f'(a)$ and called a complex derivative at the point $a$;

3. The following Cauchy–Riemann equations are satisfied at the point $a$:
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.
\]

4. $\frac{\partial f}{\partial x}(a) := \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right) = 0$. 


**Proof.** Statement (1) is just a reformulation of the fact that the differential $d_a f$ is a complex linear map. Equivalence (1) and (2) is straightforward. According to Lemma 1.2, condition (3) just means that the Jacobi matrix is a matrix of a complex linear map.

To deal with condition (4) let us recall that according to Lemma 1.3 for any differentiable at $a$ function $f$ we can decompose the linear map $d_a f$ into a complex linear and anti-linear:

$$d_a f = \partial_a f + \bar{\partial}_a f,$$

so that we have $\partial_a f(h) = \alpha h, \bar{\partial}_a f(h) = \bar{\beta} \bar{h}$. Rephrasing Lemma 1.3 we have

$$\alpha = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) - i \frac{\partial f}{\partial y}(a) \right) h + \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right) \bar{h}.$$

If we introduce the notation

$$\frac{\partial f}{\partial z}(a) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) - i \frac{\partial f}{\partial y}(a) \right),$$

$$\frac{\partial f}{\partial \bar{z}}(a) = \frac{1}{2} \left( \frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a) \right),$$

then we can write

$$d_a f(h) = \frac{\partial f}{\partial z}(a) h + \frac{\partial f}{\partial \bar{z}}(a) \bar{h}.$$}

Hence, $f$ is holomorphic at a point $a$ if and only if

$$\frac{\partial f}{\partial \bar{z}}(a) = 0,$$}

and this condition is just another form of the Cauchy-Riemann equations (1.4.1).

It is important to note that $f$ is holomorphic at $a$, i.e. $\frac{\partial f}{\partial \bar{z}}(a) = 0$ then $f'(a) = \frac{\partial f}{\partial z}(a)$.

Note that the determinant $\det J(f)(a)$ for a holomorphic at $a$ map is equal to $|f'(a)|^2$, see formula (1.2.2).

The function $f : U \to \mathbb{C}$ is called holomorphic in $U$ if it is holomorphic at every point of $U$. This is equivalent to the condition $\frac{\partial f}{\partial \bar{z}} = 0$ in $U$.

The following proposition summarizes property of complex differentiation which are analogous to the corresponding facts in the real case.

**Proposition 1.5.** (1) If $f, g$ are holomorphic at $a \in \mathbb{C}$ the $f \pm g$ and $fg$ is holomorphic at $a$ at $(f \pm g)'(a) = f'(a) \pm g'(a), (fg)'(a) = f'(a)g(a) + f(a)g'(a)$; if $g(a) \neq 0$ then $\frac{f}{g}$ is holomorphic at $a$ and $\left( \frac{f}{g} \right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)};$

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(2) If \( f \) is holomorphic at \( a \) and \( g \) is holomorphic at \( f(a) \) then the composition \( g \circ f \) is holomorphic at \( a \) and 
\[
(g \circ f)'(a) = g'(f(a))f'(a).
\]

The proof of (1) repeats the corresponding proofs in the real case, while (2) is the chain rule with an additional observation that a composition of complex linear maps is itself complex linear.

1.5 Complex-valued differential 1-forms

Let us first recall some basics of the theory of real differential forms. For our purposes we will need only 1-forms on domains in \( \mathbb{R}^2 \). By definition a differential 1-form \( \lambda \) on a domain \( U \subset \mathbb{R}^2 \) is a field of linear functions \( \lambda_z : \mathbb{R}^2 \to \mathbb{R} \). Thus a differential 1-form is function of arguments of 2 kind: of a point \( z \in U \) and a vector \( h \in \mathbb{R}^2 \). It depends linearly on \( h \) and arbitrarily (but usually continuously and even differentiably) on \( z \), i.e. we have \( \lambda_z(h) = a_1(z)h_1 + a_2(z)h_2 \), where \( h_1, h_2 \) are Cartesian coordinates of \( h \in \mathbb{R}^2 \). Any differential 1-form can be multiplied by a function ("a field of scalars"):
\[
(f\lambda)_z(h) = f(z)\lambda_z(h).
\]

Given a real-valued function \( f : U \to \mathbb{R}^2 \) on \( U \) its differential \( df \) is an example of a differential form: 
\[
d_z(f)(h) = \frac{\partial f}{\partial x}h_1 + \frac{\partial f}{\partial y}h_2.
\]
In particular differentials \( dx \) and \( dy \) of the coordinate functions \( x, y \) are differential 1-forms, and any other differential form can be written as a linear combination of \( dx \) and \( dy \):
\[
\lambda = Pdx + Qdy,
\]
where \( P, Q : U \to \mathbb{R} \) are functions on the domain \( U \). In particular,
\[
df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.
\]

A differential 1-form \( \lambda \) is called exact if \( \lambda = df \). The function \( f \) is called the primitive of the 1-form \( \lambda \). The primitive is defined uniquely up to adding a constant.

Not every closed differential 1-form \( \lambda = Pdx + Qdy \) is exact. The necessary condition for exactness is that \( \lambda \) is closed which by definition means \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \). The necessity of this condition for exactness follows from the mixed derivatives equality (assuming that the coefficients \( P, Q \) are \( C^1 \)-smooth). Indeed, if \( P = \frac{\partial f}{\partial x} \) and \( Q = \frac{\partial f}{\partial y} \). Then
\[
\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.
\]
On the other hand the closedness of $\lambda$ is not sufficient for its exactness, as it is demonstrated by an example of a 1-form $d\phi$ on $\mathbb{R}^2 \setminus 0$ (written in polar coordinates). We will discuss a bit later the precise argument for that, but intuitively clear that the primitive of this form is not a univalent function on $\mathbb{R}^2 \setminus 0$. On the other hand, as we will see below any closed 1-form on $\mathbb{R}^2$, or more generally on any simply connected domain in $\mathbb{R}^2$ is exact.

We will also consider complex-valued differential 1-forms. A $\mathbb{C}$-valued differential 1-form is a field of $\mathbb{C}$-valued linear in the real sense functions, or simply it is an expression $\alpha + i\beta$, where $\alpha, \beta$ are usual real-valued differential 1-forms. All usual operations on complex valued 1-forms are defined in the same way as for real-valued forms, and in addition such forms can be multiplied by complex-valued functions.

Note that a complex-valued function (or 0-form) on a domain $U \subset \mathbb{C}$ is just a map $f = u + iv : U \to \mathbb{C}$. Its differential $df$ is the same as the differential of this map, but it also can be viewed as a $\mathbb{C}$-valued differential 1-form $df = du + idv$.

**Example 1.6.**

$$dz = dx + idy, d\bar{z} = dx - idy, zdz = (x + iy)(dx + idy) = xdx - ydy + i(xy + ydx),$$

We have

$$dx = \frac{1}{2}(dz + d\bar{z}), \quad dy = -i\frac{1}{2}(dz - d\bar{z}).$$

Hence, any complex valued 1-form $\lambda$ can be written as a linear combination of forms $dz$ and $d\bar{z}$:

$$\lambda = fdz + gd\bar{z},$$

which is a decomposition of $\lambda$ into a sum of complex linear and complex anti-linear parts.

**Lemma 1.7.** The form $\lambda = fdz + gd\bar{z}$ is closed if and only if

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial z}.$$

**Proof.**

$$\lambda = fdz + gd\bar{z} = f(dx + idy) + g(dx - idy) = (f + g)dx + i(f - g)dy.$$
The closedness of $\lambda$ means by definition that
\[
\frac{\partial(f + g)}{\partial y} = \frac{\partial(i(f - g))}{\partial x},
\]
which is equivalent to
\[
\frac{\partial f}{\partial y} - i \frac{\partial f}{\partial x} = - \frac{\partial g}{\partial y} - i \frac{\partial g}{\partial x}.
\]
Dividing both parts by $(-i)$ we get
\[
\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} - i \frac{\partial g}{\partial y},
\]
and hence
\[
\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right) = \frac{1}{2} \left(\frac{\partial g}{\partial x} - i \frac{\partial g}{\partial y}\right) = \frac{\partial g}{\partial z}.
\]

Let us express the differential $df$ of a complex valued function $f$ as a combination of differential forms $dz = dx + idy$ and $d\bar{z} = dx - idy$ parts.

**Lemma 1.8.** For any complex valued function $f = u + iv$ we have
\[
df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.
\]
In particular, when $f$ is holomorphic we have
\[
df = \frac{\partial f}{\partial z} dz = f'(z) dz.
\]

**Proof.** We have
\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{1}{2} \frac{\partial f}{\partial x} (dz + d\bar{z}) - i \frac{\partial f}{\partial y} (dz - d\bar{z})
\]
\[
= \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y}\right) dz + \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right) d\bar{z}
\]
\[
= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.
\]
1.6 Holomorphic 1-forms

A complex-valued 1-form \( \lambda \) is called \textit{holomorphic} if it is equal to \( f dz \) for a holomorphic function \( f \).

**Lemma 1.9.** The form \( f dz \) is closed in a domain \( U \) if and only if the function \( f \) is holomorphic in \( U \).

**Proof.** According to Lemma 1.7 closedness of \( f dz \) is equivalent to \( \frac{\partial f}{\partial \bar{z}} = 0 \), which in turn is equivalent to the holomorphicity. \( \square \)

**Example 1.10.** The holomorphic form \( \frac{dz}{z^n}, n \geq 1 \), on \( \mathbb{C} \setminus 0 \) is always closed. It is exact if and only if \( n > 1 \).

Indeed, If \( n > 1 \) then
\[
\frac{dz}{z^n} = d \left( \frac{dz}{(1-n)z^{n-1}} \right).
\]
If \( n = 1 \) we have in polar coordinates
\[
\frac{dz}{z} = \frac{d(re^{i\phi})}{re^{i\phi}} = \frac{e^{i\phi}dr + ire^{i\phi}d\phi}{r} = \frac{dr}{r} + id\phi = d(ln r) + id\phi,
\]
but we already discussed above that the form \( d\phi \) is not exact.

1.7 Integration of differential 1-forms along curves

Curves as paths

A \textit{path}, or \textit{parametrically given curve} in a domain \( U \subset \mathbb{R}^2 \gamma : [a, b] \to U \). We will assume in what follows that all considered paths are differentiable. Given a differential 1-form \( \alpha = Pdx + Qdy \) in \( U \) we define the \textit{integral of \( \alpha \) over \( \gamma \)} by the formula
\[
\int_\gamma \alpha = \int_a^b \gamma^* \alpha.
\]
Denoting the coordinate functions of \( \gamma(t) \) by \( x(t) \) and \( y(t) \) (i.e. \( \gamma(t) = (x(t), y(t)) \)) the pull-back differential form \( \gamma^* \alpha \) is by definition equal to
\[
\gamma^* \alpha = P(x(t), y(t))dx(t) + Q(x(t), y(t))dy(t) = (P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t))dt,
\]
so that

\[ \int_{\gamma} \alpha = \int_{a}^{b} \gamma^* \alpha = \int_{a}^{b} (P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)) dt. \]

An important property of the integral of a differential 1-form is that it does not depend on the parameterization of the curve.

**Proposition 1.11.** Let a path \( \tilde{\gamma} \) be obtained from \( \gamma : [a, b] \to U \) by a reparameterization, i.e. \( \tilde{\gamma} = \gamma \circ \phi \), where \( \phi : [c, d] \to [a, b] \) is an orientation preserving diffeomorphism. Then \( \int_{\tilde{\gamma}} \alpha = \int_{\gamma} \alpha \).

Thus the integral \( \int_{\gamma} \alpha \) depends only on the curve \( \gamma \) as an oriented submanifold and not on a particular parameterization which is compatible with the orientation. For instance, given a unit circle \( S^1 = \{ |z| = 1 \} \) oriented counter-clockwise we can compute \( \int_{S^1} d\phi = 2\pi \). Indeed, the circle can be parameterized by the angular coordinate \( \phi \in [0, 2\pi] \), and this parameterization is compatible with the counter-clockwise orientation. Hence, \( \int_{S^1} d\phi = \int_{0}^{2\pi} d\phi = 2\pi \).

**Exercise 1.12.** Compute \( \int_{S} \frac{1}{z} \)

**Solution.** Let us parameterize the circle by polar coordinates: \( z = e^{i\phi}, \phi \in [0, 2\pi] \). Then \( \frac{dz}{z} = id\phi \) and \( \int_{S} \frac{dz}{z} = \int_{S^1} d\phi = 2\pi i \).

### 1.8 Integrals of closed and exact differential 1-forms

**Theorem 1.13.** Let \( \alpha = df \) be an exact 1-form in a domain \( U \subset \mathbb{C} \). Then for any path \( \gamma : [a, b] \to U \) which connects points \( A = \gamma(a) \) and \( B = \gamma(b) \) we have

\[ \int_{\gamma} \alpha = f(B) - f(A). \]

In particular, if \( \gamma \) is a loop then \( \oint_{\gamma} \alpha = 0 \).

Similarly for an oriented curve \( \Gamma \subset U \) with boundary \( \partial \Gamma = B - A \) we have

\[ \int_{\Gamma} \alpha = f(B) - f(A). \]
Proof. We have \( \int_a^b \gamma^* df = \int_a^b df \circ \gamma = f(\gamma(b)) - f(\gamma(a)) = f(B) - f(A). \)

It turns out that closed forms are locally exact. A domain \( U \subset V \) is called star-shaped with respect to a point \( a \in V \) if with any point \( x \in U \) it contains the whole interval \( I_{a,x} \) connecting \( a \) and \( x \), i.e. \( I_{a,x} = \{ a + t(x - a); t \in [0, 1] \} \). In particular, any convex domain is star-shaped.

**Proposition 1.14.** Let \( \alpha \) be a closed 1-form in a star-shaped domain \( U \subset V \). Then it is exact.

**Proof.** Define a function \( F : U \to \mathbb{R} \) by the formula

\[
F(x) = \int_{I_{a,x}} \alpha, \ x \in U,
\]

where the intervals \( I_{a,x} \) are oriented from 0 to \( x \).

We claim that \( dF = \alpha \). Let us identify \( V \) with the \( \mathbb{R}^n \) choosing \( a \) as the origin \( a = 0 \). Then \( \alpha \) can be written as \( \alpha = \sum_{k=1}^{n} P_k(x)dx_k \), and \( I_{0,x} \) can be parameterized by

\[
t \mapsto tx, \ t \in [0, 1].
\]

Hence,

\[
F(x) = \int_{I_{0,x}} \alpha = \int_{0}^{1} \sum_{k=1}^{n} P_k(tx)x_k dt.
\] (1.8.1)

Differentiating the integral over \( x_j \) as parameters, we get

\[
\frac{\partial F}{\partial x_j} = \int_{0}^{1} \sum_{k=1}^{n} tx_k \frac{\partial P_k}{\partial x_j}(tx) dt + \int_{0}^{1} P_j(tx) dt.
\]

But \( d\alpha = 0 \) implies that \( \frac{\partial P_k}{\partial x_j} = \frac{\partial P_j}{\partial x_k} \), and using this we can further write

\[
\frac{\partial F}{\partial x_j} = \int_{0}^{1} \sum_{k=1}^{n} tx_k \frac{\partial P_j}{\partial x_k}(tx) dt + \int_{0}^{1} P_j(tx) dt = \int_{0}^{1} t \frac{dP_j(tx) dt}{dt} dt + \int_{0}^{1} P_j(tx) dt = \int_{0}^{1} (tP_j(tx)) | \left. \right|_0^1 - \int_{0}^{1} P_j(tx) dt + \int_{0}^{1} P_j(tx) dt = P_j(tx)
\]

Thus

\[
dF = \sum_{j=1}^{n} \frac{\partial F}{\partial x_j} dx_j = \sum_{j=1}^{n} P_j(x) dx = \alpha
\]

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1.9 Stokes/Green theorem

Given a bounded domain $U \subset \mathbb{C}$ with a smooth (or piece-wise smooth boundary) we will always orient it boundary $\partial U$ as follows. For each point $p \in \partial U$ take an outward normal vector $\nu$. Then $i\nu$ is tangent to $\partial U$ and defines its orientation. For instance, suppose $A$ is the annulus $1 \leq |z| \leq 2$. Its boundary is the union of two circles: $S_1 = \{|z| = 1\}$ and $S_2 = \{|z| = 2\}$. Then $A$ induces on the outer circle $S_2$ the counter-clockwise orientation, and on the inner circle $S_1$ the clockwise orientation.

The fundamentally important fact about integration of 1-forms is the following theorem which belongs to George Green and it is a special case of a more general result, called Stokes’ theorem (which was not actually proved by George Stokes!)

**Theorem 1.15.** Let $U \subset \mathbb{C}$ be a bounded domain with a smooth boundary, and $\alpha = Pdx + Qdy$ a differential 1-form on $U$ with $C^1$-smooth coefficients. Let us orient the curve $\partial U$ as the boundary of $U$. Then

$$\int_{\partial U} UPdx + Qdy = \int \int_U \lim_{U \to U} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy,$$

**Corollary 1.16.** Suppose a 1-form $\alpha$ is closed in a domain $U$. Then

$$\int_{\partial U} \alpha = 0.$$

Indeed, closedness of $\alpha = Pdx + Qdy$ just means that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

**Corollary 1.17.** Let $f$ be a function, holomorphic in the domain $U$ and smooth up to the boundary. Then

$$\int_{\partial U} f(z)dz = 0.$$

Indeed, according to Lemma 1.9 the holomorphic differential 1-form $f(z)dz$ is closed in $U$. 

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