Math 116: Homework N8

Due on Wednesday, November 28, in class

1. a) Prove that any even (i.e. \( f(-z) = f(z) \)) elliptic function \( f \) with periods \( \omega_1 \) and \( \omega_2 \) and such that 0 is not a pole or zero, can be written in the form

\[
f(z) = C \prod_{1}^{r} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}, \quad a_j, b_j, C \in \mathbb{C}.
\]

b) How the formula should be corrected in case when some of \( a_k \) or \( b_k \) are zeroes?

Remark. Any function can be presented as a sum of an even and odd functions:

\[
f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2} = g(z) + h(z).
\]

On the other hand a ratio of two odd functions is even. Hence, any elliptic function \( f(z) \) can be written as

\[
f(z) = g(z) + \frac{h(z)}{\wp'(z)} \wp'(z),
\]

where \( g(z) \) and \( \frac{h(z)}{\wp'(z)} \) are even. Hence, Problem 1 allows us to express any elliptic function through the Weierstrass function and its derivative.

2. (Stein-Shakarchi 9.3.2) Suppose an elliptic function \( f : \mathbb{C}/\Lambda(\omega_1, \omega_2) \to \mathbb{C}P^1 \) has simple zeroes \( a_1, \ldots, a_r \) and poles \( b_1, \ldots, b_r \). Prove that

\[
\sum_{1}^{r} (a_j - b_j) = n\omega_1 + m\omega_2.
\]
Note that \(a_j\) and \(b_j\) are defined only up to multiple of \(\omega_1, \omega_2\).

**Hint.** Consider

\[
\int_{\partial P} \frac{zf'(z)}{f(z)} \, dz
\]

where \(P\) is a parallelogram spanned by \(\omega_1, \omega_2\) and chosen in such a way that there are no poles and zeroes on \(\partial P\).

3. Carry out a part of a proof of Theorem 11.20 in the lecture notes. Show that for every \(\tau \in \mathbb{H}\) there exists a fractional linear transformation

\[
\tau \mapsto \tilde{\tau} = \frac{n\tau + m}{k\tau + l}, \quad n\ell - km = 1, \quad k, n, m, l \in \mathbb{Z}
\]

such that

\[
\tilde{\tau} \in U = \{z \in \mathbb{H}; \ |\text{Re}z| \leq \frac{1}{2}, |z| \geq 1\}.
\]

4. Let \(u : \mathbb{R} \to \mathbb{R}\) be a continuous bounded function. Prove that the function

\[
U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} u(t) dt
\]

define a harmonic function in \(\mathbb{H}\) which extends continuously to the closure \(\overline{\mathbb{H}}\) as equal to \(u\) on \(\partial \mathbb{H} = \mathbb{R}\).

**Hint.** Use the Poisson-Schwarz formula for \(\mathbb{D}\) (Theorem 10.20 in lecture notes), and then apply a conformal map \(\mathbb{H} \to \mathbb{D}\) to transpose this formula from the disc to the upper half plane.

5. Consider a polynomial \(P(z) = z^4 + 2z^2 + 1\). Describe it as a branching cover \(\mathbb{C}P^1 \to \mathbb{C}P^1\), i.e. find all the branching points and their order.

Each (sub)problem is 10 points.