1. Let $U = \{ z \in \mathbb{C}; \ 0 < \text{Im} z < 1 \}$. Suppose that the group $\mathbb{Z}$ acts on $U$ by translations: $z \mapsto z + k$ for $k \in \mathbb{Z}$. Prove that $U/\mathbb{Z}$ is conformally equivalent to an annulus $A$ and compute its conformal modulus $m(A)$.

2. Let $f : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C}$ be an injective holomorphic function. Suppose that its Laurent expansion has the form

$$f(z) = z + \sum_{-\infty}^{-1} c_k z^k.$$ 

Prove that

$$\sum_{1}^{\infty} n|c_{-n}|^2 \leq 1.$$ 

What is the geometric meaning of this inequality?

3. Find the Laurent expansion of the function $e^{z + \frac{1}{z}}$ in $\mathbb{C} \setminus 0$.

4. Let $\rho \in (0, 1)$. Find a discrete subgroup $\Gamma_\rho$ in $\text{PSL}(2, \mathbb{R})$ acting freely on $\mathbb{H}$ such that $\mathbb{H}/\Gamma_\rho$ is conformally equivalent to the annulus $A(\rho, 1) = \{ \rho < |z| < 1 \}$.

5. Show that a discrete subgroup $G$ of $\text{PSL}(2, \mathbb{R})$ acts freely on $\mathbb{H}$ if and only if there are no elements $g \in G$ such that $g \neq \text{Id}$ but there is $k \in \mathbb{Z}$ such that $g^k = \text{Id}$.

6. Let $f(z)$ be a polynomial without multiple roots. Denote

$$S = \{(z, w) \in \mathbb{C}^2; w^2 = f(z)\}.$$
Show that $S$ is a Riemann surface and describe explicitly its coordinate charts.

All problems and subproblems are 10 points