Math 116 Homework 5 Solutions

Friday, November 2, 2018

Problem 1. First, if \( z \in D_+ \), then \( z = x + iy \) for \( y > 0 \) and \( x^2 + y^2 < 1 \).
So
\[
\text{Im} \left( -\frac{1}{2} \left( z + \frac{1}{z} \right) \right) = -\frac{1}{2} \text{Im} \left( \frac{z(x^2 + y^2) + \overline{z}}{x^2 + y^2} \right) > 0
\]
since \( \text{Im}(z(x^2 + y^2) + \overline{z}) < 0 \). Now suppose that \( f(w) = f(z) \) for some \( z, w \in D_+ \), so that \( z + 1/z = w + 1/w \) and hence \( z^2w + w = w^2z + z \). Rearranging yields \( wz(z - w) = z - w \), which implies \( z = w \) since \( |wz| < 1 \). Hence \( f \) is injective.

Next let \( w \in \mathbb{H} \). If \( w = -\frac{1}{2}(z + 1/z) \), then \( z^2 + 2wz + 1 = 0 \), and the quadratic formula shows that
\[
z = -\frac{2w \pm \sqrt{4w^2 - 4}}{2} = -w \pm \sqrt{w^2 - 1},
\]
where \( \sqrt{w^2 - 1} \) is any fixed choice of square root for \( w^2 - 1 \) (of which there are exactly two). If \( \alpha = -w + \sqrt{w^2 - 1} \) and \( \beta = -w - \sqrt{w^2 - 1} \), notice that
\[
\alpha \beta = w^2 - (w^2 - 1) = 1,
\]
so that \( \alpha = c\beta \) for some \( c \in \mathbb{R}_{\geq 0} \). By interchanging \( \alpha \) and \( \beta \), we may assume \( c \geq 1 \). Note that in fact \( c > 1 \) since \( \alpha + \beta = w \not\in \mathbb{R} \), so that \( |\alpha| < 1 \). Also, since \( \text{Im}(\alpha + \beta) = \text{Im}(-2w) < 0 \), it follows that \( \alpha \in D_+ \). Hence \( f \) is surjective, and since it is a bijective holomorphic map, it is conformal.

Problem 2. Let \( \phi: \mathbb{H} \rightarrow \mathbb{D} \) be given by \( \phi(z) = \frac{z + 1}{z + i} \), with inverse map \( \psi: \mathbb{D} \rightarrow \mathbb{H} \). Then for all \( w \in \mathbb{D} \) we have \( |F(\psi(w))| \leq 1 \), so by the Schwarz lemma we in fact have \( |F(\psi(w))| \leq |w| \). Letting \( w = \phi(z) \) for \( z \in \mathbb{H} \) this yields \( |F(z)| \leq \left| \frac{z + 1}{z + i} \right| \), as desired.
Problem 3. Let $g : \mathbb{D} \to \mathbb{D}$ be the biholomorphic map given by

$$g(z) = \frac{z - z_1}{1 + \overline{z}_1 z},$$

so that $g(z_1) = 0$. Then let $h = g \circ f \circ g^{-1}$, so that $h$ is an automorphism of $\mathbb{D}$ and $h(0) = g(f(g^{-1}(0))) = g(f(z_1)) = g(z_1) = 0$. Moreover, if $w = g(z_2)$, then $g^{-1}(w) = z_2$ and it follows easily that $h(w) = w$. Of course, since $h$ is biholomorphic and $z_1 \neq z_2$, it follows that $w \neq 0$. But then the Schwarz lemma implies that $h(z) = z$ for all $z \in \mathbb{D}$. But then $f = g^{-1} \circ h \circ g = g^{-1} \circ g$, which is the identity map on $\mathbb{D}$.

Problem 4. Suppose that $f(\beta) = 0$, and let $g : \mathbb{H} \to \mathbb{D}$ be given by

$$g(z) = \frac{z - \beta}{z - \overline{\beta}}.$$  

It is not hard to see that this map is well-defined: $|z - \beta| < |z - \overline{\beta}|$ since $\beta, z \in \mathbb{H}$. Also, $g(\beta) = 0$. Consider now the automorphism $h = f \circ g^{-1}$ of $\mathbb{D}$. Then $h(0) = f(g^{-1}(0)) = f(\beta) = 0$. By the known classification of automorphisms of the disk, there is some $\theta \in \mathbb{R}$ such that $h(z) = e^{i\theta}z$ for all $z \in \mathbb{D}$. Then indeed

$$f(z) = f(g^{-1}(g(z))) = h(g(z)) = e^{i\theta} \frac{z - \beta}{z - \overline{\beta}},$$

as desired.

Problem 5. It will be useful to make a few observations before launching into the computations. If $f(z) = \frac{az + b}{cz + d}$ is any linear fractional transformation, then if $z_0 \in \mathbb{C}$ is a fixed point of $f(z)$ we have $z_0(cz_0 + d) - (az_0 + b) = 0$, i.e.,

$$p(z) = cz_0^2 + (d - a)z_0 - b = 0.$$ 

The discriminant of $p(z)$ is $\Delta = (d - a)^2 + 4bc$. This $\Delta$ controls the behavior of the fixed points of $f(z)$ in a certain sense. First of all, if $c = 0$ and $b = 0$ then either $f(z)$ is the identity or $f(z)$ has fixed points 0 and $\infty$. If $c = 0$, $b \neq 0$, and $a = d$ then $f(z)$ has $\infty$ as a unique fixed point. If $c = 0$, $b \neq 0$, and $a \neq d$, then $f(z)$ has fixed points $\infty$ and $\frac{b}{a-\alpha}$. If $c \neq 0$ and $\Delta = 0$, then $f(z)$ has a unique fixed point, namely $\frac{a-d}{2c}$. (This is a consequence of the quadratic formula.) Finally, if $c \neq 0$ and $\Delta \neq 0$, then $f(z)$ has exactly two fixed points, also given by the quadratic formula.
a) Since $f(z) = \frac{az + b}{cz + d}$ is parabolic, $\Delta = 0$, i.e., $(d - a)^2 + 4bc = 0$. If $c = 0$, this implies that $d = a$, so that indeed $w = f(z) = z + \frac{b}{a}$. Since by the above $z_0 = \infty$, we have established the second part of the question. Now suppose instead that $c \neq 0$; by scaling, we may assume then that $c = 1$. The unique fixed point of $f(z)$ is $z_0 = a - \frac{d}{2}$, and we have $z_0^2 = -b$, so that

$$f(z) - z_0 = \frac{az + b - z_0(z + d)}{z + d} = \frac{(a - z_0)z + (b - dz_0)}{z + d}.$$  

Notice that $a - z_0 = a - \frac{a - d}{2} = \frac{a + d}{2}$ and

$$b - dz_0 = b - \frac{d(a - d)}{2} = b + \left(\frac{a - d}{2} - \frac{a + d}{2}\right) \frac{a - d}{2} = -\frac{a + d}{2}z_0$$

since $z_0^2 + b = 0$. If $\alpha = \frac{a + d}{2} = z_0 + d$, then together this yields

$$(a - z_0)z + (b - dz_0) = \alpha z - \alpha z_0.$$  

Now suppose for contradiction $a = -d$, so that $\alpha = 0$ and $z_0 = a$. Then $4a^2 = 4b$, so that $a^2 = b$ and hence $ad - b = 0$, contradicting assumption that $f(z)$ is an automorphism of $\mathbb{C}P^1$. Thus in any case $\alpha \neq 0$. So we obtain

$$\frac{1}{f(z) - z_0} = \frac{\alpha^{-1}(z + d)}{z - z_0} = \frac{(z + d)/(z_0 + d)}{z - z_0} = \frac{1 + (z - z_0)/(z_0 + d)}{z - z_0} = \frac{1}{z - z_0} + \frac{1}{z_0 + d^2}$$

as desired.

b) Suppose first that $z_2 = \infty$. By the opening discussion, it follows that $c = 0$. By scaling, we may assume $d = 1$. If $b = 0$, then also $a \neq 1$ and so $z_1 = 0$. Thus

$$f(z) - z_1 = az = a(z - z_1).$$

If $b \neq 0$, then again $a \neq 1$ and $f(z) = az + b$, so the other fixed point is $z_1 = -b/(a - 1)$. Thus

$$f(z) - z_1 = az + b + \frac{b}{a - 1} = a\left(z + \frac{b}{a - 1}\right) = a(z - z_1)$$

as desired.
since \( b + \frac{b}{a-1} = \frac{ab}{a-1} \). For future reference, note that if we do not assume \( d = 1 \), essentially the same computation shows that \( f(z) - z_1 = \frac{a}{d}(z - z_1) \).

Now suppose \( z_1, z_2 \neq \infty \), so by the opening discussion we have \( c \neq 0 \), so we may scale to assume \( c = 1 \). If \( \rho^2 = \Delta \) then the fixed points of \( f(z) \) are \( z_1 = \frac{1}{2}((a - d) + \rho) \) and \( z_2 = \frac{1}{2}((a - d) - \rho) \). So if \( w = f(z) = \frac{az + b}{z + d} \) we have

\[
\frac{w - z_1}{w - z_2} = \frac{2w - (a - d) - \rho}{2w - (a - d) + \rho}.
\]

Calculate \( 2w - (a - d) = \frac{(a+d)z-ad+2b+d^2}{z+d} \), so that

\[
\frac{w - z_1}{w - z_2} = \frac{(a + d)z - ad + 2b + d^2 - \rho(z + d)}{(a + d)z - ad + 2b + d^2 + \rho(z + d)}.
\]

Notice that \( \Delta = (d - a)^2 + 4b = d^2 - 2ad + a^2 + 4b \), so

\[
d^2 - ad + 2b = \frac{1}{2}(\Delta + d^2 - a^2) = \frac{1}{2}(\rho^2 + d^2 - a^2),
\]

so that

\[
\frac{w - z_1}{w - z_2} = \frac{2(a + d)z + \rho^2 + d^2 - a^2 - 2(z + d)\rho}{2(a + d)z + \rho^2 + d^2 - a^2 + 2(z + d)\rho}
= \frac{2(a + d - \rho)z - 2(a + d - \rho)z_1}{2(a + d + \rho)z - 2(a + d + \rho)z_2}
\]

since

\[
2(a + d - \rho)z_1 = a^2 - d^2 + 2d\rho - \rho^2
\]

and similarly for \( 2(a + d + \rho)z_2 \). Thus finally we obtain the desired result,

\[
\frac{w - z_1}{w - z_2} = \frac{a + d - \rho}{a + d + \rho} \left( \frac{z - z_1}{z - z_2} \right).
\]

For future reference, note that the particular constant in this expression did not depend on the scaling \( c = 1 \), and also that the constant is only well-defined up to inversion, as we could have interchanged \( z_1 \) and \( z_2 \).

c) The opening discussion shows that \( f(z) \) is parabolic if and only if \( \Delta = (d - a)^2 + 4bc = 0 \). Since \( bc = ad - 1 \) by assumption, this is equivalent to \( (d + a)^2 = 4 \), i.e., \( a + d = \pm 2 \).
Now suppose that $f(z)$ is not parabolic, i.e., $\Delta \neq 0$. Suppose first that $z_2 = \infty$. In part b), we saw that in this case $k = a/d$ and $c = 0$, from which it follows that $ad = 1$. From this we find that $a = \alpha d$ for some $\alpha \in \mathbb{R}_{\geq 0}$. If $a + d \in \mathbb{R}$, then indeed $\alpha = 1$, so that $|a + d| < 2$ and $|k| = 1$ although $k \neq 1$ (since $a \neq d$ by our condition on the discriminant). If $a + d \notin \mathbb{R}$, then $\alpha \neq 1$, so $|k| = \alpha \neq 1$ and hence $f(z)$ is loxodromic.

In the second case, suppose that $z_1, z_2 \neq \infty$. In part b), we saw that in this case $c \neq 0$ and if $z_1 = \frac{1}{2c}((a - d) + \rho)$ and $z_2 = \frac{1}{2c}((a - d) - \rho)$, then $\rho^2 = \Delta$ and

$$k = \frac{a + d - \rho}{a + d + \rho}.$$ 

If $a + d \in \mathbb{R}$, then $\Delta = (d - a)^2 + 4bc = (a + d)^2 - 4 \in \mathbb{R}$. If $|a + d| < 2$, then $\Delta < 0$ and hence $\rho$ is pure imaginary and nonzero, so that $a + d - \rho$ and $a + d + \rho$ are complex conjugates and neither is real. Hence $|k| = 1$ and $k \neq 1$, so that $f(z)$ is elliptic. If instead $|a + d| > 2$, then $\Delta > 0$, so $\rho \in \mathbb{R}$. Since $\rho < a + d$, we have $k > 0$, and thus $f(z)$ is hyperbolic.

Finally suppose that $a + d \notin \mathbb{R}$. We would like to show that $k \notin \mathbb{R}_{>0}$ and $|k| \neq 1$. Note that

$$a + d - \rho = k(a + d + \rho)$$

by definition. Rearranging gives

$$(k - 1)(a + d) = -(k + 1)\rho,$$

so that in particular $k \neq \pm 1$ since $\rho \neq 0$ and $a + d \neq 0$. Hence

$$a + d = -\frac{k + 1}{k - 1}\rho.$$ 

Let $\alpha = -\frac{k + 1}{k - 1} \in \mathbb{R}$ and square both sides to obtain

$$(a + d)^2 = \alpha^2 \Delta,$$

so

$$\Delta + 4 = \alpha^2 \Delta$$

and thus

$$(\alpha^2 - 1)\Delta = 4.$$ 

If we assume $k > 0$, it follows that $\alpha^2 > 1$ and hence $\Delta > 0$, contradicting assumption that $a + d \notin \mathbb{R}$ since again $\Delta = (a + d)^2 - 4$. Thus $f(z)$ is not
hyperbolic. Otherwise, assume $|k| = 1$ and $k \neq 1$. Then $k = e^{i\theta}$ for some $0 < \theta < 2\pi$, so

$$\alpha = -\frac{(e^{i\theta} + 1)(e^{-i\theta} - 1)}{|e^{i\theta} - 1|^2} = \frac{e^{i\theta} - e^{-i\theta}}{|e^{i\theta} - 1|^2} = 2i \frac{\sin \theta}{|e^{i\theta} - 1|^2}$$

and in particular $\alpha$ is pure imaginary, so that $\alpha^2 \leq 0$. From the above equation, it follows that $\Delta < -4$, so that $(a + d)^2 \in \mathbb{R}$ and hence $a + d$ must also be pure imaginary since it is not real. Moreover, $\rho$ must be pure imaginary since $\rho^2 = \Delta$. But then from the equation

$$a + d = \alpha \rho,$$

it follows that $a + d \in \mathbb{R}$, a contradiction. Hence $f(z)$ is not elliptic, and hence it must be loxodromic.