Problem 1. Note that \( f(z) \) has a pole exactly at those points where \( \sin \pi z \) has a zero, i.e., at all integer points \( n \in \mathbb{Z} \). At such points \( \sin \pi z \) has a double zero, so that \( f(z) \) has a double pole, and we are required to compute

\[
\lim_{z \to n} \frac{d}{dz} \frac{(z - n)^2}{(\sin \pi z)^2}
\]

The function \((z - n)^2 \sin \pi z\) takes on the same values at \( n + w \) and \( n - w \) for all \( w \in \mathbb{C} \) such that it is defined at \( n + w \), so that if \( g \) is the function extending \((z - n)^2 f(z)\) holomorphically to \( n \), we have

\[
g'(n) = \lim_{w \to 0} \frac{w^2 f(n + w) - g(n)}{w} = - \lim_{w \to 0} \frac{w^2 f(n - w) - g(n)}{-w} = -g'(n),
\]

from which it follows that \( g'(n) = 0 \), i.e., the residue at \( n \) of \( \frac{1}{(\sin \pi z)^2} \) is 0.

Problem 2. a) First recall that \( \sin^2(x) = (1 - \cos(2x))/2 \), so that

\[
\sin^2(z) = \text{Re}(\frac{1 - e^{2iz}}{2}).
\]

For convenience, let \( f(z) = (1 - e^{2iz})/2z^2 \). Note that \( f(z) \) has a double pole at 0, and that

\[
\lim_{z \to 0} \frac{d}{dz} \frac{z^2 (1 - e^{2iz})}{2z^2} = -ie^{2i\pi 0} = -i.
\]

Let \( \epsilon > 0 \) and \( R > 0 \), and define a contour \( C \) in the complex plane as follows: first, \( C \) goes in a straight line \( C_1 \) from \( \epsilon \) to \( R \). Next, \( C \) goes in a counterclockwise circular arc \( C_2 \) of radius \( R \) from \( R \) to \(-R \). Then \( C \) travels in a straight line \( C_3 \) from \(-R \) to \(-\epsilon \) and finally in a clockwise circular arc of
radius $\epsilon$ from $-\epsilon$ to $\epsilon$. (This can be written more explicitly using parametric equations, but the expression would be quite long.) So we have

$$0 = \int_{C} f(z)dz = \int_{\epsilon}^{R} f(x)dx + \int_{C_2} f(z)dz + \int_{-R}^{-\epsilon} f(x)dx + \int_{C_4} f(z)dz$$

since $C$ bounds a region with no poles. If $z$ is on the arc $C_2$, then $\text{Im}(z) > 0$ and $|z| = R$, so that $|1 - e^{2iz}| \leq 2$ and $|z^2| = R^2$, so that $|f(z)| \leq R^{-2}$ and hence

$$|\int_{C_2} f(z)dz| \leq \int_{C_2} |f(z)|dz \leq \pi R^{-1}.$$ 

Also, by the residue theorem

$$\int_{C_4} f(z)dz = -\frac{2\pi i}{2}(-i) = -\pi.$$ 

Finally, we have

$$\int_{\epsilon}^{R} f(x)dx + \int_{-R}^{-\epsilon} f(x)dx = 2\int_{\epsilon}^{R} \frac{\sin^2(x)}{x^2}dx$$

since $\text{Re}(f(x))$ is an even function and $\text{Im}(f(x))$ is an odd function. Hence the above calculations can be combined to show

$$|\pi - 2\int_{\epsilon}^{R} \frac{\sin^2(x)}{x^2}dx| \leq \pi R^{-1}.$$ 

Taking the limit as $\epsilon \to 0$ and $R \to \infty$ shows that

$$\int_{0}^{\infty} \frac{\sin^2(x)}{x^2}dx = \frac{\pi}{2}.$$ 

b) Let $f(z) = ze^{iz}/(z^2 + 1)$, so if $x \in \mathbb{R}$ then $\text{Im}(f(x)) = x \sin x/(x^2 + 1)$. Note that $f(z)$ has simple poles at $i$ and $-i$, so that

$$\text{Res}_i f(z) = \frac{ie^{-1}}{2i} = \frac{1}{2e}.$$ 

Let $r > 1$ and $R > 1$, and define the contour $C$ to be the rectangle from $-R$ to $R$ to $R + ir$ to $R - ir$ and back to $-R$. Since $r > 1$, $C$ encloses the point $i$ and no other pole of $f(z)$, so by the residue theorem,

$$\int_{C} f(z)dz = \frac{i\pi}{e}.$$ 

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Let \( L_1 \) be the line from \(-R\) to \( R\), and similarly for \( L_2, L_3, \) and \( L_4 \), counted counterclockwise from \( L_1 \). First,

\[
\int_{L_1} f(z) \, dz = i \int_{-R}^{R} \frac{x \sin x}{x^2 + 1} \, dx = 2i \int_{0}^{R} \frac{x \sin x}{x^2 + 1} \, dx
\]

since \( \text{Re}(f(z)) \) is an odd function and \( \text{Im}(f(z)) \) is even. Next, note that \(|(R + it)^2 + 1| \geq R^2 - 1\) and similarly \(|(-R + it)^2 + 1| \geq R^2 - 1\) for all \( 0 \leq t \leq r \). Since \(|e^{iz}| \leq 1\) for all \( z \) with positive imaginary part, this shows that

\[
|\int_{L_2} f(z) \, dz + \int_{L_4} f(z) \, dz| \leq \int_{L_2} |f(z)| \, dz + \int_{L_4} |f(z)| \, dz \leq 2r \frac{R + r}{R^2 - 1}.
\]

If \( z = t + ir \) for \(-R \leq t \leq R\), then

\[
|f(z)| \leq \frac{(R + r)e^{-r}}{R^2 - 1},
\]

so that

\[
|\int_{L_3} f(z) \, dz| \leq 2R \frac{(R + r)e^{-r}}{R^2 - 1}.
\]

Letting \( r = \log R \), we combine all of these equations to obtain

\[
2i \int_{0}^{R} \frac{x \sin x}{x^2 + 1} \, dx - \frac{i \pi}{e} \leq 2(\log R) \frac{R + \log R}{R^2 - 1} + 2R \frac{R + \log R}{R^3 - R}.
\]

Taking the limit as \( R \to \infty \) yields

\[
\int_{0}^{\infty} \frac{x \sin x}{x^2 + 1} \, dx = \frac{\pi}{2e}.
\]

**Problem 3.** Suppose without loss of generality that \( f \) has an isolated singularity at 0. There are three possibilities: \( f \) has a removable singularity, a pole, or an essential singularity. In the first case, we know that \( f \) extends holomorphically to 0, so that \( e^{f(z)} \) also extends holomorphically to 0 since the composition of holomorphic functions is holomorphic. Next suppose that \( f \) has a pole at 0, so that the function \( 1/f(z) \) can be extended to a function \( g(z) \) which is holomorphic at 0. If \( U \) is any open set containing 0 on which \( g(z) \) is well-defined, then the open mapping theorem implies that \( g(U) \) is
open. Since $0 \in g(U)$, there is some $\epsilon > 0$ such that $D_\epsilon \subset g(U)$, where $D_\epsilon$ is the disc of radius $\epsilon$ centered at 0. From this it follows that

$$\{z \in \mathbb{C} : |z| > \epsilon^{-1}\} \subset f(U \setminus \{0\}).$$

In particular, $f(U \setminus \{0\})$ contains all sufficiently large positive and negative real numbers.

Since $U$ was arbitrary in the above argument, for each $n$ there exists $\epsilon_n > 0$ such that

$$\{z \in \mathbb{C} : |z| > \epsilon_n^{-1}\} \subset f(D_{1/n} \setminus \{0\}).$$

For each $n$, let $z_n, w_n \in D_{1/n}$ be such that $f(z_n) = \epsilon_n^{-1} + 1$ and $f(w_n) = -\epsilon_n^{-1} - 1$. Then $\{z_n\}$ and $\{w_n\}$ both converge to 0, and we have

$$\lim_{n \to \infty} e^{f(z_n)} = \infty,$$

whereas

$$\lim_{n \to \infty} e^{f(w_n)} = 0$$

since $\{z_n\}$ and $\{w_n\}$ are sequences of real numbers converging to $\infty$ and $-\infty$, respectively. Hence $e^{f(z)}$ is unbounded in every neighborhood of 0, but there is no well-defined limit as $z \to 0$, so that $e^{f(z)}$ has an essential singularity at 0 (and in particular not a pole).

Finally, consider the case that $f$ has an essential singularity at 0. By the Casorati-Weierstrass theorem, we can take sequences $\{z_n\}$ and $\{w_n\}$ as in the previous case to show that $e^{f(z)}$ has an essential singularity at 0.

**Problem 4.** Our goal is to apply Rouché’s theorem. First, note that $|z^4 - 6z| \geq 5$ whenever $|z| = 1$, and $|z^4 - 6z| \geq 4$ whenever $|z| = 2$ (this follows from the triangle equality). In particular, $|z^4 - 6z| > 3$ for all $z$ on $\partial A$, so Rouché’s theorem implies that $z^4 - 6z + 3$ has the same number of zeros as $z^4 - 6z$ in $A$. Now $z^4 - 6z = z(z^3 - 6)$, and $z^3 - 6$ has exactly three roots, all of modulus $6^{1/3}$ (namely $6^{1/3}$, $e^{2\pi i/3}6^{1/3}$, and $e^{4\pi i/3}6^{1/3}$), and $1 < 6^{1/3} < 2$, so that $z^4 - 6z$ has exactly three zeros in $A$, and hence the same is true of $z^4 - 6z + 3$.

**Problem 5.** Let $f(z) = \pi \cot \frac{\pi z}{u+z}$. Compute

$$\cot \frac{\pi z}{u+z} = \frac{\cos \pi z}{\sin \pi z} = \frac{(e^{\pi iz} + e^{-\pi iz})/2}{(e^{\pi iz} - e^{-\pi iz})/2i} = \frac{e^{2\pi iz} + 1}{i e^{2\pi iz} - 1}.$$
Now $e^{2\pi iz} - 1$ has a simple zero at every integer, and can be expanded in a power series around any $n \in \mathbb{Z}$ via

$$e^{2\pi iz} - 1 = 2\pi i(z - n) + \frac{1}{2}(2\pi i(z - n))^2 + \frac{1}{3!}(2\pi i(z - n))^3 + \ldots$$

Hence $f(z)$ has a simple pole at each $n \in \mathbb{Z}$, and we have

$$\lim_{z \to n} (z - n)f(z) = i\pi \frac{2}{2\pi i(u + n)^2} = \frac{1}{(u + n)^2}$$

by the above power series computation and our expression for $\cot \pi z$.

The function $f(z)$ also has a double pole at $-u$, and the residue at this pole is

$$\lim_{z \to -u} \frac{d}{dz}(u + z)^2 \pi \cot \pi z = \lim_{z \to -u} \frac{d}{dz} \pi \cot \pi z.$$  

Now

$$\frac{d}{dz} \pi \cot \pi z = -\frac{\pi^2}{\sin^2(\pi z)},$$

so that in fact the residue of $f(z)$ at $-u$ is

$$-\frac{\pi^2}{\sin^2(\pi u)}.$$  

If $N \geq |u|$ is an integer and $C_N$ is the circle of radius $N + 1/2$ centered at 0 and oriented counter-clockwise, then the residue theorem implies that

$$\int_{C_N} f(z)dz = 2\pi i \left( \sum_{n=-N}^{N} \frac{1}{(u - n)^2} - \frac{\pi^2}{\sin^2(\pi u)} \right).$$

To complete the proof, we must compute this integral in a different way as well.

Combining the above expressions, we know that

$$f(z) = \frac{\pi i(e^{2\pi iz} + 1)}{(e^{2\pi iz} - 1)(u + z)^2}.$$  

If $|z| = N + 1/2$, then $|(u + z)^2| \geq (N + 1/2 - |u|)^2$ by the triangle inequality. Write $z = a + bi$, where $a, b \in \mathbb{R}$ and $a^2 + b^2 = (N + 1/2)^2$. Then

$$|e^{2\pi iz} - 1|^2 = (e^{-2\pi b} \cos(2\pi a) - 1)^2 + e^{-4\pi b} \sin^2(2\pi a)$$

$$= e^{-4\pi b} - 2e^{-2\pi b} \cos(2\pi a) + 1.$$
If $N + 1/4 \leq |a| \leq N + 1/2$, then $\cos(2\pi a) \leq 0$, so that

$$|e^{2\pi iz} - 1|^2 \geq 1.$$ 

Note that $N + 1/4 \leq |a| \leq N + 1/2$ if and only if $|b| \leq \sqrt{N/2 + 3/16}$. If $|b| > \sqrt{N/2 + 3/16}$, then in particular $|b| > 1/4$, so by the above equation,

$$|e^{2\pi iz} - 1|^2 \geq (e^{-2\pi b} - 1)^2 \geq (e^{-\pi/2} - 1)^2.$$ 

since $|\cos(2\pi a)| \leq 1$ and $(e^{\pi/2} - 1)^2 > (e^{-\pi/2} - 1)^2$. Because $(e^{-\pi/2} - 1)^2 < 1$, this shows that

$$|e^{2\pi iz} - 1| \geq 1 - e^{-\pi/2}$$

for all $z$ with $|z| = N + 1/2$. Also,

$$\frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1} = 1 + \frac{2}{e^{2\pi iz} - 1},$$

so this shows that for all $|z| = N + 1/2$,

$$|f(z)| \leq \frac{\pi i}{(N + 1/2 - |u|)^2} + \frac{2\pi i}{(1 - e^{-\pi/2})(N + 1/2 - |u|)^2}.$$ 

Hence

$$|\int_{C_N} f(z)dz| \leq 2\pi N\left(\frac{\pi i}{(N + 1/2 - |u|)^2} + \frac{2\pi i}{(1 - e^{-\pi/2})(N + 1/2 - |u|)^2}\right).$$

Taking the limit as $N \to \infty$ then gives 0. Combining this with our earlier calculation of the integral shows that

$$\frac{\pi}{\sin^2(\pi u)} = \sum_{n=-\infty}^{\infty} \frac{1}{(u - n)^2}.$$ 

**Problem 6.** To show that an entire function $f$ is a polynomial, it suffices to show that all derivatives of high enough order vanish at 0. By the Cauchy estimates, given $R > 0$ and $n \geq 1$ we have

$$|f^{(n)}(0)| \leq \max_{|z|=R} \frac{|f(z)|n!}{R^n} \leq n!(AR^{k-n} + BR^{-n}).$$

If $n > k$, then taking the limit $R \to \infty$ shows that $f^{(n)}(0) = 0$, i.e., $f$ is a polynomial.