Math 116: Homework N2

Due on Friday, October 12, in class

1. Compute the radius of convergence for the following power series:

\[ \sum_{n=0}^{\infty} (3 + (-1)^n) z^n; \]
\[ \sum_{n=0}^{\infty} (\cos n) z^n. \]

2. (Abel’s theorem)

a) (Problem 1.4.15 from the Stein-Shakarchi textbook) Prove that if a series \( \sum_{n=0}^{\infty} a_n \) converges (not necessarily absolutely) then

\[ \lim_{r \to 1, r < 1} \sum_{n=0}^{\infty} r^n a_n = \sum_{n=0}^{\infty} a_n. \]

*Hint:* Show first that \( \lim_{r \to 1, r < 1} \sum_{n=0}^{\infty} r^n a_n = (1 - r) \sum_{n=0}^{\infty} S_n r^n \), where \( S_n = \sum_{k=0}^{n} a_k \).

b) Show that the converse is not true, i.e. find an example of a divergent series \( \sum_{n=0}^{\infty} a_n \) such that there exists a finite limit \( \lim_{r \to 1, r < 1} \sum_{n=0}^{\infty} r^n a_n \).

3. (Problem 2.6.7 from the Stein-Shakarchi textbook) Let \( D = \{ |z| < 1 \} \) be the unit disc and \( f : D \to \mathbb{C} \) a holomorphic function. The diameter \( d \) of the set \( f(D) \) is defined as

\[ d := \sup_{z, w \in D} |f(z) - f(w)|. \]
Prove that
\[ d \geq 2|f'(0)|. \]

4. (Problem 2.6.8 from the Stein-Shakarchi’s textbook) Denote
\[ U := \{ z \in \mathbb{C}; -1 < \text{Im} \, z < 1 \}. \]

Let \( f : U \to \mathbb{C} \) be a holomorphic function which satisfies the inequality
\[ |f(z)| \leq A(1 + |z|)^\eta, \]
where \( A, \eta \in \mathbb{R} \) are positive real constants. Show that for each integer \( n \geq 0 \) there exists \( A_n > 0 \) such that
\[ |f^{(n)}(z)| \leq A_n(1 + |z|)^\eta \]
for every real \( z \).

5. (Removal of singularity theorem) Show that if a holomorphic function \( f : U \setminus a \to \mathbb{C} \) is bounded in a neighborhood of the point \( a \) then it extends holomorphically to the point \( a \).

Hint: First prove the extension property for the function \( f(z)(z - a) \).

6. (Open image theorem) Let \( U \subset \mathbb{C} \) be a connected domain. Show that for any non-constant holomorphic function \( f : U \to \mathbb{C} \) the image \( f(U) \subset \mathbb{C} \) is open.

Each problem (and subproblems 2a and 2b) is 10 points.