1. Prove that
\[ \frac{2}{\sqrt{2}} \frac{2}{\sqrt{2} + \sqrt{2}} \frac{2}{\sqrt{2} + \sqrt{2} + \sqrt{2}} \cdots = \frac{\pi}{2}. \]

*Hint.* Use Problem 5 in Homework 9.

2. Prove that
\[ \frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 + 4n^2\pi^2}. \]

3. Show that the map
\[ t \mapsto \left( \frac{t^2}{1 + t^2}, \frac{t^3}{1 + t^2} \right) \]
defines a biholomorphism of \( \mathbb{C} \setminus \{0, i, -i\} \) onto the Riemann surface \( S \subset \mathbb{C}^2 \), where
\[ S = \{(z, w); \ w^2 = \frac{z^3}{1 - z}, \ z \neq 0\}. \]

4. Consider a conformal map \( f : \mathbb{H} \to \mathbb{D} \) given by the formula
\[ f(z) = \frac{z - i}{z + i}. \]

Find the set of points \( z \in \mathbb{H} \) such that
\[ \lim_{\epsilon \to 0} \frac{\text{Area}(f(D_\epsilon(z)))}{\text{Area}(D_\epsilon(z))} > 1. \]
5. Let $f : \mathbb{D} \to \mathbb{D}$ be a conformal automorphism such that $f(a) = 0$, $a \in \mathbb{D}, a \neq 0$. Denote by $S_\theta$ the semi-circle 
\[ S_\theta := \{ e^{i\phi}; \phi \in (\theta, \theta + \pi) \}. \]
Show that $f(S_\theta)$ is a semi-circle $S_{\theta'}$ for some $\theta'$ if and only if $a = \pm re^{i\theta}$.

6. Let $U = \mathbb{D} \setminus \{ z = x + iy; \ y = 0, x > 0 \}$. Find a conformal equivalence $f : U \to \mathbb{D}$.

7. Let $u : \mathbb{R} \to \mathbb{R}$ be a continuous $2\pi$-periodic function. Find a harmonic function
\[ f : \mathbb{C} \setminus \{|z| \leq R\} \to \mathbb{R} \]
such that

- $\lim_{|z| \to R, z \to Re^{i\theta}} f(z) = u(\theta)$;
- $\lim_{|z| \to \infty} f(z) = C < \infty$.

8. Suppose $z_1, z_2, z_3 \in \mathbb{C}/\Lambda$ are distinct zeroes of $\wp'(z)$. Prove that
\[ \wp(z_1) + \wp(z_2) + \wp(z_3) = 0. \]

The actual final will consist of 5 problems.