1. Prove that
\[
\frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2 + \sqrt{2}}} \cdot \frac{2}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}} \cdots = \frac{\pi}{2}
\]

According to Problem 5 in Homework 9 we have
\[
\prod_{1}^{\infty} \frac{1}{\cos \frac{z}{2^n}} = \frac{z}{\sin z}. \tag{1}
\]

Using formula \( \cos \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{2}} \) we prove by induction that
\[
\cos \frac{\pi}{2^n} = \sqrt{\frac{2 + \sqrt{2 + \cdots + \sqrt{2}}}{n-1}}
\]

Indeed, \( \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \) and if
\[
\cos \frac{\pi}{2^{n-1}} = \sqrt{\frac{2 + \sqrt{2 + \cdots + \sqrt{2}}}{n-2}}
\]

then
\[
\cos \frac{\pi}{2^n} = \sqrt{1 + \sqrt{\frac{2 + \sqrt{2 + \cdots + \sqrt{2}}}{2}}\frac{n-2}{n-1}} = \sqrt{\frac{2 + \sqrt{2 + \cdots + \sqrt{2}}}{n-1}}
\]
Taking $z = \frac{\pi}{2}$ and using (1) we get the required identity.

2. Prove that

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 + 4n^2 \pi^2}.$$ 

First we note that the right-hand $h(z)$ side converges uniformly on compact sets outside poles, and at the poles $\pm 2\pi ni$ the rest of the expansion besides the term $\frac{2z^2}{z^2 + 4n^2 \pi^2}$ converges as well. Hence, the sum is a meromorphic function on $\mathbb{C}$. The residues at the poles are equal $\pm 2\pi in$. The left-hand side has the same (simple) poles with the same residues. Hence

$$f(z) = \frac{z}{e^z - 1} - h(z)$$

is a holomorphic function. Foreover, $f(z) = 0$, and hence $\tilde{f}(z) := \frac{f(z)}{z}$ is holomorphic as well.

Let us show that $\tilde{f}(z)$ is bounded. Indeed, $\frac{1}{e^z - 1}$ is $2\pi i$-periodic and

$$\lim_{x \to +\infty} \frac{1}{e^x + iy - 1} = 0, \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{e^x + iy - 1} = 1,$$

i.e. $\frac{1}{e^z - 1}$ is bounded at infinity. Let us show that $\frac{h(z)}{z}$ is also $2\pi i$-periodic

We have

$$\tilde{h}(z) := \frac{h(z)}{z} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + 4n^2 \pi^2} = \frac{1}{z} - \frac{1}{2} + \sum_{n\neq 0} \frac{z}{2n\pi i(z - 2n\pi i)}.$$

Next,

$$S_N(z) := \frac{1}{z} + \sum_{n\neq 0, |n| \leq N} \frac{z}{2n\pi i(z - 2n\pi i)} = \frac{1}{z} + \sum_{n\neq 0, |n| \leq N} \left( \frac{1}{2n\pi i} + \frac{1}{z - 2n\pi i} \right)$$

$$= \sum_{-N}^{N} \frac{1}{z + 2n\pi i}.$$ 

Hence,

$$S_N(z + 2\pi i) - S_N(z) = \frac{1}{z + 2(N + 1)\pi i} = \frac{1}{z - 2N\pi i} \to 0,$$

and therefore, $\tilde{h}(z)$ is $2\pi i$-periodic. Now it is straightforward to see that $|\tilde{h}(x + iy)|$ stays bounded when $|x|$ grows. Hence, $\tilde{f}(z)$ is bounded, and thus in view of the Liouville theorem it is constant. But $\tilde{f}(0) = 0$ and hence $\tilde{f}(z)$, and together with it $f(z)$, vanishes.
3. Show that the map
\[ t \mapsto \left( \frac{t^2}{1 + t^2}, \frac{t^3}{1 + t^2} \right) \]
defines a biholomorphism of \( \mathbb{C} \setminus \{0, i, -i\} \) onto the Riemann surface \( S \subset \mathbb{C}^2 \), where
\[ S = \{(z, w); \quad w^2 = \frac{z^3}{1 - z}, \quad z \neq 0\}. \]

We first see that
\[ h(t) = \left( \frac{t^2}{1 + t^2}, \frac{t^3}{1 + t^2} \right) \in S \]
for all \( t \neq \pm i \).

Next, we check that the map is surjective. The map \( t \mapsto \frac{t^2}{1 + t^2} \) is a double cover of \( \mathbb{C} \setminus 0 \). Indeed, when extended as a map \( \mathbb{CP}^1 \to \mathbb{CP}^1 \) it is a meromorphic function with a multiplicity 2 zero at 0 and two single poles \( \pm i \). Hence, when the poles and the zero are removed we get a double cover of \( \mathbb{C} \setminus 0 \). Note that \( h_1(t) = h_1(-t) \). But \( h_2(t) = -h_2(t) \). Hence, to points which are mapped by \( h_1 \) to the same \( z \) are mapped by \( h \) onto two branches of the curve \( S \).

4. Consider a conformal map \( f : \mathbb{H} \to \mathbb{D} \) given by the formula
\[ f(z) = \frac{z - i}{z + i}. \]

Find the set of points \( z \in \mathbb{H} \) such that
\[ \lim_{\epsilon \to 0} \frac{\text{Area}(f(D_\epsilon(z)))}{\text{Area}(D_\epsilon(z))} > 1. \]

We have
\[ \lim_{\epsilon \to 0} \frac{\text{Area}(f(D_\epsilon(z)))}{\text{Area}(D_\epsilon(z))} = |f'(z)|^2. \]
Hence, we just need to analyze the modulus derivative of the map \( f(z) = \frac{z-i}{z+i} \). We have 

\[
f'(z) = \frac{2i}{(z+i)^2} \quad \text{and} \quad |f'(z)|^2 = \frac{4}{|z+i|^4}.
\]

The inequality \(|z+i| < \sqrt{2}\) defines a disc of radius \( \sqrt{2} \) centered at the point \(-i\). Hence, the required set is the part of this disc which is in \( \mathbb{H} \).

5. Let \( f : \mathbb{D} \to \mathbb{D} \) be a conformal automorphism such that \( f(a) = 0 \), \( a \in \mathbb{D}, a \neq 0 \). Denote by \( S_\theta \) the semi-circle 

\[ S_\theta := \{ e^{i\phi}; \phi \in (\theta, \theta + \pi) \}. \]

Show that \( f(S_\theta) \) is a semi-circle \( S_{\theta'} \) for some \( \theta' \) if and only if \( a = \pm re^{i\theta} \).

The map \( f \) has the form 

\[ f(z) = e^{i\alpha} \frac{a - z}{1 - \bar{a}z}. \]

The factor \( e^{i\alpha} \) can be dropped because it does not affect the property we need to prove.

We need to find out for which \( \theta \) we have \( f(-z) = -f(z) \), \( z = e^{i\theta} \). This is equivalent to 

\[(a - z)(1 + \bar{a}z) + (z + a)(1 - \bar{a}z) = 0,
\]

or \( z^2 = \frac{a}{\bar{a}} \). If \( a = re^{i\phi} \) then we get \( 2\theta = 2\phi \), or \( \theta = \phi \) or \( \theta = \phi + \pi \).

6. Let \( U = \mathbb{D} \setminus \{ z = x + iy; y = 0, x > 0 \} \). Find a conformal equivalence \( f : U \to \mathbb{D} \).

First consider a transformation 

\[ z \mapsto \sqrt{z}, \]

where we choose a branch of \( \sqrt{ } \) on \( \mathbb{C} \setminus \{ z = x + iy; y = 0, x > 0 \} \), i.e. for \( z = e^{i\theta} \), \( \theta \in (0, 2\pi) \). \( f_1 \) maps \( U \) onto the half-disc \( U_1 := \mathbb{D} \cap \mathbb{H} \). Next we consider the transformation 

\[ z \mapsto z + 1, \]

which translates the half-disc \( U_1 \) to \( U_2 \). Next, take an inversion 

\[ z \mapsto -\frac{1}{z}, \]
The image $f_3(U_2)$ is a quadrant

$$\quad U_3 := \{\operatorname{Re} z \frac{1}{2}, \operatorname{Im} z > 0\}.$$ 

Next we translate the corner of the quadrant to the origin by

$$z \mapsto z - \frac{1}{2},$$

and using

$$z \mapsto z^2$$

we get $\mathbb{H}$ as the image. It remains to apply

$$z \mapsto \frac{z - i}{z + i}$$

to map $\mathbb{H}$ onto $\mathbb{D}$. The required conformal equivalence $f$ is the composition $f_6 \circ \cdots \circ f_1$.

7. Let $u : \mathbb{R} \to \mathbb{R}$ be a continuous $2\pi$-periodic function. Find a harmonic function

$$f : \mathbb{C} \setminus \{|z| \leq R\} \to \mathbb{R}$$

such that

- $\lim_{|z| > R, z \to R e^{i\theta}} f(z) = u(\theta)$;
- $\lim_{|z| \to \infty} f(z) = C < \infty$.

The map $h(z) = \frac{R}{z}$ maps the complement of the disc $D_R$ of radius $R$ to the unit disc $\mathbb{D}$, and the inverse map is given by the same formula.

So the function $u(Re^{i\theta})$ on $\partial D_R$ corresponds to the function $u(e^{-i\theta})$ on $\partial D$. Let $g(z)$ be the solution of Dirichlet problem for $\mathbb{D}$ with the boundary data $\tilde{u}(\theta) = u(-\theta)$. Then $f(z) = g(Rz^{-1})$ is the solution for the complement of $D_R$ for the boundary data $\tilde{u}(Re^{i\theta}) = u(\theta)$.

Let us use Poisson formula for $\mathbb{D}$:

$$g(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2} \tilde{u}(e^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \phi) + r^2} u(-\theta) d\theta.$$
Then to get the formula for \( f(z) = f(re^{i\phi}) \) we just need to substitute \( r \mapsto \frac{R}{r} \) and \( \phi \mapsto -\phi \):

\[
\begin{align*}
  f(re^{i\phi}) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - (Rr^{-1})^2}{1 - 2(Rr^{-1}) \cos(\theta + \phi) + (Rr^{-1})^2} u(-\theta) d\theta \\
  &= \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - R^2}{r^2 - 2Rr \cos(\theta + \phi) + R^2} u(-\theta) d\theta 
\end{align*}
\]

8. Suppose \( z_1, z_2, z_3 \in \mathbb{C}/\Lambda \) are distinct zeroes of \( \wp'(z) \). Prove that

\[
\wp(z_1) + \wp(z_2) + \wp(z_3) = 0.
\]

Recall the differential equation for \( \wp \):

\[
(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3.
\]

Thus the equality \( \wp'(u) = 0 \) is equivalent to the fact that \( \wp(u) \) is one of the roots of the polynomial \( g(u) = 4u^3 - g_2u - g_3 \). Denote the roots by \( e_1, e_2, e_3 \). Thus by assumption \( e_j = \wp(z_j) \), where \( \wp'(z_j) = 0 \), \( j = 1, 2, 3 \). Let us factor this polynomial \( g \):

\[
g(u) = 4(u - e_1)(u - e_2)(u - e_3).
\]

By Vieta’s theorem \( e_1 + e_2 + e_3 \) is equal the coefficient with \( u^2 \) multiplied by \(-\frac{1}{4}\). But this coefficient in \( g \) is 0. Hence,

\[
\wp(z_1) + \wp(z_2) + \wp(z_3) = 0.
\]