MATH 21 SECTION NOTES

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1. MARCH 29

1.1. Administrative miscellany. These weekly sections will be for some review and many example problems, in general. Attendance will be taken as per class policy. We will be using playing cards for immediate feedback; black means yes or true, red means no or false, and the back of the card means I have no idea (but this is a cop-out!).

In general for administrative things the course website, http://web.stanford.edu/class/math21/ is quite comprehensive. These notes will be posted at http://math.stanford.edu/~ebwarner/

1.2. Sequences. For our purposes, a sequence is just a list of numbers, which we notate with something like \( \{a_n\} \), or if we want to be pedantic about indices, \( \{a_n\}_{n=1}^{\infty} \). (The starting index, usually zero or one, is just a matter of convention and is usually chosen so as to make the notation in any particular situation as simple as possible.)

Often sequences are given by explicit formulas. For example, \( a_n = n, n \geq 1 \) defines the sequence 1, 2, 3, 4, 5, ..., while \( a_n = \frac{1}{n^2}, n \geq 1 \) defines the sequence 1, \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \). Even better than writing out the first few terms, we can graph sequences just as we graph functions (the functions that we are familiar with are just maps from the real numbers to the real numbers; sequences are maps from the natural numbers to the real numbers so here the “x-axis” is just a bunch of points going off to the right).

There’s no reason a sequence has to be given by an explicit formula (whatever that means). Another way of uniquely specifying a sequence is by a recurrence relation; the usual example is the Fibonacci numbers, which are defined as follows:

\[
F_0 = 1, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for all} \quad n \geq 2.
\]

That is, we fix the first two values and declare that each subsequent value is the sum of the previous two. This clearly specifies the sequence 1, 1, 2, 3, 5, 8, 13, ... (In fact, in this case one can without too much difficulty derive an explicit formula for the Fibonacci sequence, but with more complicated recurrence relations this need not be the case. And for some purposes a recurrence relation may be more useful than an explicit formula.)

A sequence \( \{a_n\} \) has a limit \( C \) if as \( n \) gets larger and larger, \( a_n \) gets closer and closer to \( C \). (A more precise definition is in section 8.2 of the text, but right now we’re just building intuition.) We write

\[
\lim_{n \to \infty} a_n
\]

to denote this limit, if it exists. Of course it may not, which makes the above notation occasionally slightly frustrating.

Here are some examples:
• \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \) because as \( n \) gets larger and larger the square of its reciprocal gets closer and closer to zero.

• \( \lim_{n \to \infty} n \) does not exist because there is no real number such that as \( n \) gets larger and larger it gets closer and closer to that number. However, this limit does fail to exist in a particular way: as \( n \) gets larger and larger, the sequence eventually grows larger than any value (and stays there), so in this situation we write \( \lim_{n \to \infty} n = \infty \).

• \( \lim_{n \to \infty} (-1)^n \) does not exist: it oscillates forever between the values \(-1\) and \(1\), and so never gets close to any one value.

• \( \lim_{n \to \infty} \sin(2\pi n) \) does exist; if we evaluate the sequence on any \( n \) we get zero and the limit of the zero sequence is zero! So while the real-valued function \( \sin(2\pi x) \) oscillates and consequently does not have a limit as \( x \to \infty \), the sequence \( \sin(2\pi n) \) does have a limit as \( n \to \infty \) because it just so happens that all of the oscillation happens away from the integers. (Note the confusing notation, which doesn’t properly distinguish between sequences and functions! Usually the use of a letter like \( i, j, k, \ell, m, n \) is a signal that we have a sequence, while a letter like \( t, u, v, x, y, z \) is a signal that we are considering a real-valued function.)

1.3. Series. We use the above concept of a limit to define what we mean by the sum of infinitely many real numbers, which is not \emph{a priori} at all well-defined. Namely, we define the sum of the sequence \( \{a_n\}_{n=1}^{\infty} \), which is usually notated

\[
\sum_{n=1}^{\infty} a_n,
\]

to be the limit of the sequence of partial sums. The sequence of partial sums is the sequence \( a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \ldots \); it is the sequence whose \( n \)th entry is the sum of the first \( n \) terms (which is well-defined; we certainly know how to add finitely many numbers together!). An infinite sum like this is called a \emph{series}.

Because limits of sequences might not exist, the value of a series might not exist. Here are some examples:

• \( \sum_{n=1}^{\infty} 0 = 0 \), by definition, the limit of the sequence of partial sums, which is \( 0, 0 + 0, 0 + 0 + 0, 0 + 0 + 0 + 0, \ldots \). Obviously all terms of this sequence are zero, so the limit is also zero. Thus under our definition of the sum of a sequence, infinitely many zeroes sum to zero.

• \( \sum_{n=1}^{\infty} 1 = \) is the limit of the sequence \( 1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1, \ldots \), which is just the sequence \( 1, 2, 3, 4, \ldots \), which has no limit (we say goes to infinity). Therefore this sum is not defined (and we could say it is infinite).

• \( \sum_{n=1}^{\infty} (-1)^n \) has partial sums \(-1, -1 + 1 = 0, -1 + 1 - 1 = -1, \) and so on. That is, the sequence of partial sums oscillates forever between \(-1\) and \(0\), and consequently has no limit, so this sum is not defined.

In the 9:30 section, we had time to sketch an argument that \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) does exist, even if we can’t easily figure out what the precise value is. We will come back to this in more detail later.
2. April 5

2.1. Series and decimals. We should now be moderately familiar with geometric series, which are one of the only kinds of series we can evaluate exactly. As an example, let’s calculate

\[ S = \sum_{n=1}^{\infty} \frac{3}{10^n}. \]

The usual formula for the sum of a geometric series (proved in lecture, and worth reviewing!) is

\[ \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r| < 1. \]

If \( |r| \geq 1 \) it is easy to see that the series does not converge. Here we want to apply it to a series starting at \( n = 1 \), so we peel off the first term, which is \( r^0 = 1 \):

\[ \sum_{n=0}^{\infty} r^n = 1 + \sum_{n=1}^{\infty} r^n, \]

so

\[ \sum_{n=1}^{\infty} r^n = \frac{1}{1-r} - 1 = \frac{r}{1-r}. \]

Back to our example: we can pull out the constant multiple of 3 (for the same reason we can pull out a constant multiple from an integral: both are defined in terms of limits, and we can pull out constants from limits in the same way). So, using the formula we derived above,

\[ S = 3 \sum_{n=1}^{\infty} \frac{1}{10^n} = 3 \sum_{n=1}^{\infty} \left( \frac{1}{10} \right)^n = 3 \cdot \frac{\frac{1}{10}}{1 - \frac{1}{10}} = 3 \cdot \frac{1}{9} = \frac{1}{3}. \]

Great! This sort of calculation should get to the point where it is quick and easy.

But here’s an even quicker way to see what the sum is. Let’s write the partial sums in decimal notation. We have \( S_1 = 0.3, S_2 = 0.33, S_3 = 0.333, \) and so on. Clearly, in the limit we should have

\[ S = 0.\overline{3} = \frac{1}{3}, \]

where we recognize the decimal expansion in question as precisely one third.

The underlying lesson here is that “decimals” really are a way of writing a real number as a very particular kind of series! A lot of the intuition we have for decimals is really intuition about series. For example,

\[ \pi = 3.14159\ldots, \]

which is the same as writing the series

\[ \pi = 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \frac{9}{100000} + \ldots. \]

Of course this series converges, because its sum is \( \pi \)! We can’t write down a formula for the terms of this series, because the digits of \( \pi \) are highly irregular, but it is a convergent series nonetheless. If the decimal is repeating, then we can write down the series explicitly and it will be a geometric series.
Example: calculate \( S = \sum_{n=1}^{\infty} \frac{9}{100n} \). We can solve this using the geometric series formula, getting \( \frac{1}{11} \), or we can do the following: in decimal notation, the number is \( S = 0.09 \).

How can we recover a fraction from a repeating decimal? (Last time we just recognized \( 0.3 \) as one third because it is so common.) In general, here’s a trick: notice that

\[
\frac{S}{100} = 0.009999999\ldots
\]

so

\[
S - \frac{S}{100} = 0.09 = \frac{9}{100}
\]

exactly (all other digits cancel). Solving for \( S \), we have

\[
\frac{99}{100} S = \frac{9}{100} \implies S = \frac{9}{99} = \frac{1}{11}.
\]

In fact in general a repeating decimal with \( n \) repeating digits is equal to the fraction with those \( n \) digits as numerator and \( 10^n - 1 \) in the denominator. For example, \( 0.1\overline{6} = \frac{16}{99} \) and \( 0.1\overline{53} = \frac{153}{999} \).

Two last facts about decimals. One, there is of course nothing special about decimals, as opposed to expansions in any other base (binary, ternary, etc.); they are all still series in disguise. Two, decimal expansions are slightly nonunique, which causes a lot of confusion when people realize that \( 0.\bar{5} = 1 \) if they think that real numbers are defined as decimals. They’re not! Decimals are merely a (sometimes) convenient way of writing down or approximating real numbers. In fact \( 0.\bar{5} = 1 \) is “essentially” the only sort of non-uniqueness you get in decimal expansions, and it is entirely harmless.

2.2. Telescoping series. Here’s another rare form of series that we can calculate exactly. Consider

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right).
\]

This is certainly not a geometric series, so to see what is going on we have no choice but to start writing down partial sums. We get

\[
S_1 = \left( 1 - \frac{1}{2} \right),
\]

\[
S_2 = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right),
\]

\[
S_3 = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right),
\]

and so on. But notice that if we “regroup” the terms, the \( \frac{1}{2} \) terms cancel, and the \( \frac{1}{3} \) terms cancel, and clearly this will continue as we take more and more terms. In fact for each partial sum the middle terms all cancel, leaving exactly

\[
S_N = 1 - \frac{1}{N+1}.
\]
By definition, the sum is the limit of the partial sums, which is
\[ \lim_{N \to \infty} \left( 1 - \frac{1}{N+1} \right) = 1. \]
So not only does this series converge, it has a particularly simple sum!

As another example, consider
\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right). \]
Here (check!) all of the terms in each partial sum except four cancel, and we have
\[ S_N = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}. \]
Taking the limit as \( N \to \infty \), the sum is therefore \( \frac{3}{2} \).

Telescoping sums are rare, so you probably won’t meet any “in the wild.”

2.3. A conceptual question. True or false:

If \( \sum_{n=0}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0. \)

(In other words, if a series converges, then its terms must go to zero.)

This is true; the heuristic thinking should be something like “if the partial sums are converging to a particular value, then the terms with very large index had better not be very big.” And the proof is not so difficult: we write each term as a difference of partial sums as follows:
\[ a_n = S_n - S_{n-1} \]
(why is this true?). Then taking the limit as \( n \to \infty \), we find that \( \lim_{n \to \infty} S_n = S \) (because the series converges) and \( \lim_{n \to \infty} S_{n-1} = S \) (because this is the same sequence, just indexed slightly differently). So
\[ \lim_{n \to \infty} a_n = S - S = 0, \]
as desired.

As we will learn shortly, the converse is very much false! That is, there are plenty of sequences \( a_n \) whose terms go to zero but such that \( \sum_{n=0}^{\infty} a_n \) does not converge. The simplest example is given by \( a_n = \frac{1}{n} \) (the harmonic series).

3. April 12

3.1. The tests we know so far. Here are some of the tests we know so far:

- Divergence test: if the limit of the terms is not zero, the series diverges.
- Integral test: a (nonnegative, monotonically decreasing) series converges if and only if the “obvious” corresponding integral does.
- ‘Geometric series’ test: if the series is geometric, then it converges if and only if the ratio of terms is less than 1 in absolute value (and we can even calculate the sum!).
- \( p \)-test: if the series is of the form \( \sum_{n=N}^{\infty} \frac{1}{n^p} \), it converges if and only if \( p > 1 \).
- Comparison test: if a series with nonnegative terms is bounded termwise by a convergent series, it is convergent. Similarly, if a series is greater termwise than a divergent series with nonnegative terms, it is divergent.
• Limit comparison test.

Since the limit comparison test is so new, let’s go over it again. It states that if \(a_n\) and \(b_n\) are all nonnegative terms and

\[
0 < \lim_{n \to \infty} \frac{a_n}{b_n} < \infty
\]

(so in particular the limit in question exists), then \(\sum a_n\) and \(\sum b_n\) either both converge or both diverge. The limit need not be 1, although it often will be (and you can always arrange it to be such by multiplying one of the series by a constant, of course); the important thing is that it exists and is not zero. We tend to use this test if there is a series that “looks like” one we know how to deal with except for a smaller extra term.

For example, take

\[
\sum_{n=1}^{\infty} \frac{1}{n + \log n}.
\]

We know that the harmonic series \(\sum_{n=1}^{\infty} \frac{1}{n}\) diverges, and we hope that the extra logarithm is irrelevant. So we apply the limit convergence test to \(a_n = \frac{1}{n + \log n}\) and \(b_n = \frac{1}{n}\). We have

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{n + \log n} = \lim_{n \to \infty} \frac{n}{n + \log n} = \lim_{n \to \infty} \frac{1}{1 + \log n} = 1.
\]

So the limit comparison test applies, and since we already know that \(\sum b_n\) diverges, we conclude that \(\sum a_n\) does as well. Note that we could not have used the comparison test in any straightforward way here, because \(a_n < b_n\) for all \(n\) so knowing that \(\sum b_n\) diverges is not helpful.

Notice that the two comparison tests don’t tell you anything unless you already know the convergence of another series; we use them to bootstrap from knowledge about a simple series to knowledge about a more complicated one.

3.2. Examples. These examples alternate between easy and hard.

• \(\sum_{n=1}^{\infty} \frac{1}{n^{1.7}}\). This series converges by the \(p\)-test because \(\pi > 1\).

• \(\sum_{n=2}^{\infty} \frac{1}{n \log n}\). We use the integral test because \(\frac{1}{x \log x}\) has an elementary antiderivative, using the substitution \(u = \log x\):

\[
\int_2^{\infty} \frac{dx}{x \log x} = \int \frac{du}{u} = \log u = \log \log x,
\]

so

\[
\int_2^{\infty} \frac{dx}{x \log x} = [\log \log x]_{x=2}^{\infty} = \infty.
\]

The integral does not converge, so the sum does not converge either.

• \(\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}\). We have \(\frac{1}{n^{0.2}} > \frac{1}{n}\) for each \(n\), and the series \(\sum \frac{1}{n}\) diverges, so by the comparison test so does this one.

• \(\sum_{n=1}^{\infty} \sin \left( \frac{1}{n^2} \right)\). Here we should notice first that we’re evaluating sine on values that grow closer and closer to zero, so the convergence of the sum should only depend on what the sine function does very close to zero. And in fact \(\sin x < x\) for every nonnegative \(x\), so \(\sin \frac{1}{n^2} < \frac{1}{n^2}\), and by the comparison
test the series converges. Alternatively, we could use the limit comparison
test with $\sum \frac{1}{n^2}$ if we remember that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, so certainly
$$\lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1,$$
and the limit comparison test applies.

- $\sum_{n=2}^{\infty} \frac{1}{\log n}$. Here we can just use the straightforward comparison test with
the harmonic series $\frac{1}{n}$; certainly termwise the given sum dominates the
harmonic series so because the harmonic series diverges so does this one.

4. April 19

4.1. **Ratio test.** The only new tool we have for determining convergence of series
is the ratio test, which is useful when you see terms like $k!$ or $2^k$ because then the
ratios $\frac{a_{n+1}}{a_n}$ will be particularly simple. The rule itself is simple: calculate
$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$
If $L < 1$, the series converges. If $L > 1$ (including if the limit goes to $\infty$), the
series diverges. If $L = 1$ or does not exist in another way, the ratio test doesn’t
tell you anything. This makes a lot of sense, given what we know about geometric
series (where the ratio is constant); “morally” the ratio test is kind of like a limit
comparison test applied to a geometric series.

Example: determine the convergence of
$$\sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!}.$$
This sum is tailor-made for the ratio test. We calculate
$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{2^{n+1} (n+1)!} \frac{2^n n!}{(2n)!} = \frac{(2n + 2) (2n + 1)}{2(n + 1)} = 2n + 1.$$
Clearly the limit is infinite, so by the ratio test this series diverges.

Bad example: determine the convergence of
$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$
using the ratio test. We get
$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)^2} = \frac{n^2}{(n + 1)^2}.$$
The limit as $n \to \infty$ is 1, which is unhelpful: the ratio test doesn’t tell us anything.
In fact the ratio test will fail for any series whose $n$th term is a ratio of polynomials
in $n$ in precisely the same way. Fortunately, we know that this series converges by
other methods ($p$-test, integral test).
4.2. Some exercises. What test should I use to determine the convergence of the following series?

- \[ \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}. \]
- \[ \sum_{n=1}^{\infty} \frac{1}{(3n+1)!(n^2)}. \]
- \[ \sum_{n=1}^{\infty} \frac{4^{n+1}}{(3n+1)!-n} \]
- \[ \sum_{n=2}^{\infty} \frac{\log(n)^{\log(n)}}{n!} \]

Answers, respectively: the integral test (since we have a u-substitution we can use!), the ratio test, either the ratio test directly or the limit comparison test with the previous example, and direct comparison. To see the last one, the hardest part is the following algebraic manipulation:

\[ \log(n)^{\log(n)} = \left( e^{\log(\log n)} \right)^{\log(n)} = e^{\log(n) \log(\log n)} = n^{\log(\log n)}. \]

Now you can compare the terms of the sum to, say, \( \frac{1}{n^2} \).

As an easy-ish exercise for yourself, carry out the tests as indicated above to see whether the series converge or not!

5. April 26

5.1. Exam recap. The majority of both sections was spent going over the trickier problems from the exam, so see the exam solutions for any lingering questions.

5.2. Alternating series. Alternating series are quite easy; you should be happy if you see one! For our purposes, an alternating series is a series of the form

\[ \sum_{n=A}^{\infty} (-1)^n a_n, \]

where \( a_n \geq 0 \) for all \( n \). There is a quite general and (in practice) easy to apply convergence test for such series: they converge if (1) \( \lim_{n \to \infty} a_n = 0 \) and (2) the sequence of terms \( \{a_n\} \) is eventually monotonically decreasing. We could state (and prove) more general results for more interesting sign patterns than \( + - + - \cdots \), if we wanted, but we won’t.

Example: does \( \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{n!}\right) \) converge? Yes! Its terms, when we ignore the \( (-1)^n \), go to zero (albeit rather slowly), and are monotonically decreasing. Note that without the \( (-1)^n \), this series would diverge due to the \( p \)-test, so in some sense “alternatingness” helps us converge.

This is a general fact: if \( \sum |a_n| \) converges, then \( \sum a_n \) does too. If we have a series for which the sum of the absolute value of the terms converges, we call it absolutely convergent; if a series is convergent but not absolutely convergent (like the example above) we call it conditionally convergent (as in, its convergence is conditional on the fact that it has both positive and negative terms).

So instead of just having two possibilities for the convergence of a series (convergence or divergence) we now have three (absolute convergence, conditional convergence, or divergence). This is actually sometimes important: there are theorems that hold for absolutely convergent series that do not hold for conditionally convergent series. As an example, any rearrangement of the terms of an absolutely convergent series yields the same sum (as one would hope!) but this dramatically fails to be true for conditionally convergent series.
As a postscript, a word about condition (2) in the alternating series test. In basically all examples that anyone is going to throw at you, it will be easily satisfied, but it is actually necessary. Consider the series $\sum (-1)^n a_n$, where $a_n = 2/n$ if $n$ is even and $a_n = 1/n$ if $n$ is odd. We have $\lim a_n = 0$, but the sequence isn’t eventually monotonic (it bounces up and down forever as it approaches zero). And in fact I’ve cooked this series up in such a way that it does not converge: every even term contributes a lot in the positive direction and every odd term contributes not so much in the negative direction, and this systematic bias leads to divergence (as a good exercise, prove this!). The monotonicity condition (2) rules out this sort of bias.

6. May 3

6.1. General blather about power series. A power series is a particular kind of series depending on a variable (which we’ll call $x$), that can be written in the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

where each $a_n$ and $c$ are real numbers. The number $c$ is the center of the power series, for reasons that will become clearer.

As a quick check, let’s see a couple of examples of what is and isn’t a power series. The series

$$\sum_{n=0}^{\infty} e^{-nx}$$

is not a power series; it is a perfectly nice series depending on a variable $x$ (and converging for $x > 0$; actually, it is a geometric series for each such value of $x$ and therefore we could actually find the sum explicitly if we wanted to) but it is not of the correct form. On the other hand, the series

$$\sum_{n=0}^{\infty} x^{2n}$$

is a power series, even though it does not look exactly like the template. If we wanted, however, we could put it in the desired form by letting $a_n = 1$ for $n$ even and $a_n = 0$ for $n$ odd (and of course $c = 0$).

In general, the point of this definition is that a power series is a limit of polynomials: the partial sums are

$$S_0 = a_0,$$

$$S_1 = a_0 + a_1(x - c),$$

$$S_2 = a_0 + a_1(x - c) + a_2(x - c)^2,$$

..., each of which is a polynomial (of increasing degree). The hope is that by writing our favorite functions as power series, we can manipulate them almost as if they were polynomials (which are very easy to deal with!). This hope will be realized when we learn about Taylor series, but in the meantime we can figure out how to manipulate power series in general.

Fact: given a power series, there always exists an $R$, the radius of convergence, which may be zero or $\infty$, such that the series converges when $|x - c| < R$ (that
is, when the distance between \(x\) and the center is less than \(R\) and diverges when 
\(|x - c| > R\). (If \(R = \infty\) we interpret this as meaning that the series converges for 
all \(x\).) In practice, we can determine \(R\) by using the ratio test. This fact tells us 
nothing about what happens at the boundary points, but those are usually easy 
enough to deal with separately.

6.2. Some examples. For all of the following, we want to determine the exact 
interval of convergence, which means finding the radius of convergence and figuring 
out what happens at the boundary points (if they exist). I’ll just show the examples 
and a very brief explanation of the answers; if you want more details see the 
textbook or the solutions to Homework 5.

- \(\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n} (x - 2)^n\). By the ratio test, the radius of convergence is 2. The 
endpoints both diverge by the divergence test. Therefore the interval of 
convergence is \((2, 6)\).

- \(\sum_{n=1}^{\infty} \frac{1000^n}{n} x^n\). By the ratio test, the radius of convergence is \(\frac{1}{1000}\); the 
upper endpoint yields the harmonic series which diverges and the lower 
endpoint yields the alternating harmonic series which converges (by the al-
ternating series test). Therefore the interval of convergence is 
\([-\frac{1}{1000}, \frac{1}{1000}]\).

New fact: if we differentiate (respectively, integrate) a power series term by term 
(which is easy, because each term is a polynomial!!) we get the right answer, at least 
away from the boundaries. That is, the power series we get has the same radius of 
convergence as the original power series, and is equal to the derivative (resp. an 
antiderivative) where it converges. This is powerful for at least two reasons: one, 
we can create new power series formulas from old ones without too much work, and 
two, even if we have a function without an elementary antiderivative, if we express 
it as a power series we can still write down a power series that converges to said 
antiderivative.

- If the function defined by the power series in the previous example is denoted \(f(x)\), we consider the derivative \(f'(x)\) and check its convergence. (By 
the above fact we know the radius of convergence already, but let’s check it. 
We’d have to check the endpoints anyway.) Differentiating term by term, we get 
\[f'(x) = \sum_{n=1}^{\infty} \frac{1000^n}{n} \cdot n x^{n-1} = \sum_{n=1}^{\infty} 1000^n x^{n-1}.\]
The ratio test gives us \(R = \frac{1}{1000}\), as we expected. Testing the boundaries, 
we find that the power series diverges at both by the divergence test. So 
our interval of convergence is \((-\frac{1}{1000}, \frac{1}{1000})\), which is slightly different from 
that of \(f(x)\). In general, differentiating can hurt convergence.

- Now let’s take an antiderivative:
\[F(x) = \sum_{n=1}^{\infty} \frac{1000^n}{n(n+1)} x^{n+1}.\]
Again by the ratio test the radius of convergence is the same, but now the 
series converges at both boundary points \(x = \pm\frac{1}{1000}\) (one by the alternating 
series test, and one by a comparison to \(\sum \frac{1}{n^n}\)). In general, integrating can 
help with convergence.
7. May 10

7.1. **Generalities on finding Taylor series.** In general we have two tools for finding the Taylor series expansion of a function around a given point. The tool we have already used (which I’ll call “bootstrapping”) is to start with a Taylor series expansion we already know – usually the geometric series – and manipulate it until we get what we want. We can do this, as we’ve seen, by taking derivatives and antiderivatives, plugging in certain different functions of \( x \) for \( x \), and multiplying by powers of \( x \).

The other tool is Taylor’s formula, which has the advantage of being perfectly general and the disadvantage that it is often quite difficult to calculate in nontrivial situations.

7.2. **Taylor’s formula.** The formula for the Taylor series expansion of a function around \( a \) is

\[
 f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.
\]

Here \( f^{(n)}(a) \) means the \( n \)th derivative of the function, evaluated at \( x = a \). Clearly the right side is a power series. If we only consider the first \( N + 1 \) terms of the sum, we will get a polynomial of degree \( N \) that “best approximates” the function \( f(x) \) near \( x = a \) in the sense that it is rigged up to share the same value and first \( N \) derivatives as the function itself.

Let’s do an example by finding the Taylor series expansion of \( f(x) = \cos x \) around \( \frac{\pi}{2} \). We first have to calculate all of the derivatives of \( f(x) \). This is usually the hard part, but fortunately we know what happens with \( \cos x \); the derivatives go \( -\sin x, -\cos x, \sin x, \) and then they repeat. We have to evaluate these derivatives at \( a = \frac{\pi}{2} \), getting \( -\sin \frac{\pi}{2} = -1 \), \( -\cos \frac{\pi}{2} = 0 \), and \( \sin \frac{\pi}{2} = 1 \) (and of course \( \cos \frac{\pi}{2} = 0 \) as well). Therefore by Taylor’s formula our series looks like

\[
 f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2} \cdot (x-a)^2 + \frac{f'''(a)}{6} \cdot (x-a)^3 + \ldots
 = 0 + (-1) \left( x - \frac{\pi}{2} \right) + 0 + \frac{1}{6} \left( x - \frac{\pi}{2} \right)^3 + \ldots
\]

We can write this with summation notation as

\[
 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \left( x - \frac{\pi}{2} \right)^{2n+1}.
\]

Finally, we can check where it converges by the usual procedure (ratio test!); in so doing we find it converges everywhere. It really makes sense to write

\[
 \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \left( x - \frac{\pi}{2} \right)^{2n+1}
\]
for all $x$. Furthermore, if we cut off the sum after $N$ terms (due to finite computer time, for example) we know that this truncated sum will be quite accurate near $x = \frac{\pi}{2}$.

But wait, there’s even more we can get out of this example! This formula should look suspiciously familiar. In fact, it is just $-1$ times the series we know converges to the sine function, if we shift that series by $\frac{\pi}{2}$ (i.e., replace $x$ by $x - \frac{\pi}{2}$). So we have shown, using power series, that $\cos(x) = -\sin \left( x - \frac{\pi}{2} \right)$, a basic trigonometric identity. Of course in this case it would have been easier to just draw a triangle, but that isn’t necessarily the case for all functional identities.

Here’s another example: finding the Taylor series of $\frac{1}{1 - x}$ around $x = 5$. Now we have to differentiate $(1 - x)^{-1}$ arbitrarily many times; the first few are $f'(x) = (1 - x)^{-2}$, $f''(x) = 2(1 - x)^{-3}$, and $f'''(x) = 2 \cdot 3 (1 - x)^{-4}$. The pattern is pretty clear; we have $f^{(n)}(x) = n! (1 - x)^{-n-1}$. Plugging this into Taylor’s formula, we get

$$\sum_{n=0}^{\infty} \frac{n!(1 - 5)^{-n-1}}{n!} (x - 5)^n = \sum_{n=0}^{\infty} \frac{(x - 5)^n}{(-4)^{n+1}}.$$  

If we wanted, we could check the radius of convergence and see that here it is 4. This makes sense: it could not possibly be greater than 4, because then it would converge at $x = 1$, for which the original function $\frac{1}{1 - x}$ is not defined. In other words, it converges wherever it can, subject to the facts we know about power series.

Finally, here’s a more conceptual example. What is the Taylor series of $1 + x - x^2$ at $x = 0$? The answer is immediate: the Taylor series is just $1 + x - x^2$. This is already a power series centered at $x = 0$; it so happens that most of its terms are zero. It clearly converges everywhere (it’s a finite sum!).

That was a little silly, so let’s use Taylor’s formula to find the power series for $f(x) = 1 + x - x^2$ at $x = 1$ instead. We have $f'(x) = 1 - 2x$, $f''(x) = -2$, and all higher derivatives zero, so $f(1) = 1$, $f'(1) = -1$, $f''(1) = -2$, and all higher terms are zero. Thus Taylor’s formula gives us

$$\frac{f(1)}{0!}(x - 1)^0 + \frac{f'(1)}{1!}(x - 1)^1 + \frac{f''(1)}{2!}(x - 1)^2 = 1 - (x - 1) - (x - 1)^2.$$  

Whatever this function is, it surely must be the same as our original $f(x)$ (we just wrote it in a different form). And in fact

$$1 - (x - 1) - (x - 1)^2 = 1 - x + 1 - x^2 + 2x - 1 = 1 + x - x^2.$$  

So (at least in this example) Taylor’s formula gives consistent answers when we apply it to polynomials, as it must.

7.3. Sometimes we shouldn’t use Taylor’s formula. If we can avoid it, it’s best to use the power series manipulation techniques from last week rather than using Taylor’s formula directly. Applying Taylor’s formula requires calculating all of the derivatives of a function, which can be a problem if the function is reasonably complicated.

For example, let’s say we are asked to find the power series expansion of $x \cos(x^2)$ at $x = 0$. Instead of using Taylor’s formula, let’s use the fact that we already know the power series expansion for $\cos(x)$ and bootstrap from there. That is,

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$
so plugging in $x^2$ for $x$ on both sides yields
\[
\cos(x^2) = \sum_{n=0}^{\infty} \frac{x^{4n}}{(2n)!}
\]
and finally multiplying by $x$ on both sides yields
\[
x \cos(x^2) = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{(2n)!},
\]
which is the formula we want. We can now use the ratio test to verify that this power series converges everywhere.

7.4. **A more conceptual exam-style question.** Say we have a power series
\[
\sum_{n=0}^{\infty} a_n (x - 2)^n
\]
and we are given the information that the power series converges at $x = 0$.

- Does this series converge at $x = 3$?
- Does the series $\sum (-1)^n a_n$ converge?
- Does the series $\sum na_n$ converge?

It may be somewhat surprising that we can actually answer these questions. Let’s first discuss what we know about the original power series. It is centered at 2 and converges at 0, so its radius of convergence must be at least 2. From that, we conclude that it converges when $x \in (0, 4)$, at least: actually, we know it converges when $x = 0$ as well because that was given to us, but we have no idea what happens at $x = 4$. So the first question is answerable; the series must converge at $x = 3$.

The second question is really quite similar to the first, but in a slight disguise. To get the series $\sum (-1)^n a_n$ from our power series, we can just plug in $x = 1$. That is, the question just asks whether our original series converges at $x = 1$. By the same logic as above, the answer is yes.

The third question is trickier again. How can we possibly figure out what happens when we multiply each term by $n$? We use one more useful fact about power series: when we take a derivative or antiderivative, the radius of convergence does not change. The derivative of our original power series is
\[
\sum_{n=0}^{\infty} na_n (x - 2)^{n-1}.
\]
Now plug in $x = 3$, so $(x - 2)^{n-1} = 1^{n-1} = 1$ and we get $\sum na_n$, as desired. Therefore this question is asking whether the derivative of our original power series converges at $x = 3$. Said derivative must have radius of convergence at least 2, by the above useful fact, so the answer is again yes.

8. **May 17**

8.1. **Review of absolute versus conditional convergence.** Since this was a tricky topic on the second exam, let’s review two examples, one fair and one unfair.

First consider
\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{1 + \log n}.
\]
Does it converge absolutely, converge conditionally, or diverge? We can either check absolute convergence or conditional convergence first; arbitrarily, let’s check absolute convergence. Thus we are asking whether the series formed via taking the...
absolute value of each term, $\sum \frac{1}{1 + \log n}$, converges. By comparison with the harmonic series, say, we see that it does not. Now let’s check conditional convergence. We can apply the alternating series test, for this is an alternating series. The function $1 + \log n$ is increasing, so $\frac{1}{1 + \log n}$ is decreasing, so it passes one component of the alternating series test. The function $1 + \log n$ goes to infinity, so $\frac{1}{1 + \log n}$ goes to zero, so it passes the other component as well. Therefore the series does converge, but not absolutely; i.e., it is conditionally convergent.

Now let’s look at an unfairly tricky example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n \sin n}.$$

To even know that this is well-defined (i.e., we’re not dividing by zero anywhere), we have to know that $\pi$ is an irrational number, but that’s OK. What can we say about the (conditional or absolute) convergence of this series? First let’s check absolute convergence. We know that $|\sin n| \leq 1$ for all $n$, so $\frac{1}{|\sin n|} \geq 1$, so $\frac{1}{n |\sin n|} \geq \frac{1}{n}$. Thus a comparison test with the harmonic series shows that this sum does not converge absolutely.

Does it converge conditionally? We might originally think that we could use the alternating series test because the sum is certainly changing in sign fairly regularly, but unfortunately it is not actually an alternating series in the sense that we have used, which must change sign every term. In fact the only way of seeing that this sum actually diverges is to use the divergence test, but here that is quite difficult: while it is true that $\lim_{n \to -\infty} \frac{1}{n \sin n}$ does not exist, showing this is more difficult than naively calculating the corresponding limit of a function (“replacing $n$ by $x$”) because $\frac{1}{\pi \sin x}$ is not well-defined for all large enough $x$. (If you’re interested in trying to prove it yourself, look up something called Dirichlet’s approximation theorem and apply it to the irrational number $\pi$.)

8.2. Sums of series via power series. One more thing that we can do with power series, which is implicit in a lot of what we’ve done so far but deserves to be made explicit, is use them to calculate the exact sum of some series by plugging in specific values of $x$ for which the power series converges.

For example, one could ask for an exact value for the sum $\sum_{n=0}^{\infty} \frac{1}{n!}$. To solve, we note that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all $x$, so just plugging in $x = 1$ yields

$$e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

That is, the sum in question is exactly $e$. This is not a terrible way to define the number $e$, actually; if you look carefully you will find that the textbook never actually does define what $e$ is precisely!

Another example: what is $\sum_{n=0}^{\infty} \frac{n}{2^n}$? To solve this, we recall the geometric series

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n.$$
and differentiate both sides to get that extra \( n \) in each term of the series:

\[
\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}.
\]

Now if we plug in \( x = \frac{1}{2} \), the right side is exactly the series we wanted and the right side is 4, so the sum of the above series is exactly 4.

Some more examples to work out: find the sums of \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \), \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \), and \( \sum_{n=0}^{\infty} \frac{n}{x^n} \). This last example you may remember from the challenge problem on the first exam; now we have a much more systematic method to figure out what is going on with it!

8.3. A warning. There’s something I would feel negligent if I didn’t mention, which is a caution about using Taylor series on “arbitrary” functions: basically, they may converge, but not to the function you started with!

Here’s a trivial example to help us understand why this must be true: consider the function

\[
f(x) = \begin{cases} 
0 & \text{when } x < 1; \\
x - 1 & \text{when } x \geq 1.
\end{cases}
\]

This is a continuous function that is zero everywhere less than (or equal to) \( x = 1 \) and then increases linearly from there. Let’s calculate its Taylor series at the origin via Taylor’s formula. This is really easy: at the origin, the function is zero, its derivative is zero, its second derivative is zero, its third derivative is zero, and so on forever: derivatives only measure what happens in an arbitrarily small neighborhood of a point, and in a neighborhood of the origin our function \( f \) is just the zero function. Therefore Taylor’s formula yields the power series \( \sum_{n=0}^{\infty} 0 \frac{x^n}{n!} = 0 \), which certainly converges everywhere. But when \( x > 1 \), it doesn’t converge to the function \( f \) we started with!

You might think that the problem is with the “unreasonable” abrupt turn the function took at \( x = 1 \), and this is basically right, but how do we define “unreasonable”? Actually, although it would take us too far astray to demonstrate this, there are infinitely differentiable functions whose Taylor series are bad in the above way as soon as you leave the center point! The way we deal with this is to declare that functions that \( \text{are} \) good, in that they are represented by their own Taylor series, are called analytic and studied for their own particular properties. Essentially all of the functions that we deal with in this class are analytic wherever they are defined, so Taylor series work like we expect them to.

In retrospect, it was totally unreasonable to think that we could take an arbitrary function and hope to recapture all of its values (at least within some radius of convergence) just given the “local” data of the derivatives at a single point. Functions for which you can do this are special, and it is fortunate that many of the functions we know and love (trig functions, exponential functions, and so on) are special in this way.

9. May 24

9.1. Separable differential equations. Unlike most equations you’ve seen, a solution to a differential equation is a function. For example, consider the very
The simple differential equation
\[ \frac{dy}{dt} = t. \]

In words, this equation says that there is a function \( y \) depending on \( t \) whose first derivative is equal to \( t \); our job is to find that function. In this case, that’s very easy, since we know what functions have derivative equal to \( t \): they are the antiderivatives of the function \( t \), and all of them look like \( \frac{t^2}{2} + C \) for some constant \( C \). Essentially, we just took antiderivatives of both sides of the original equation.

We can’t always do this. For example, we might be asked to find solutions to
\[ \frac{dy}{dt} = 2y. \]

That is, we’re looking for a function whose derivative is twice the original function. The solution here is to separate variables and then integrate; i.e., multiply both sides by \( \frac{1}{y} \) to get
\[ \frac{1}{y} \frac{dy}{dt} = 2 \]
and then integrate in \( t \), getting
\[ \int \frac{1}{y} \frac{dy}{dt} dt = \int 2 \, dt. \]

By the change of variables formula, this is just
\[ \int \frac{1}{y} \, dy = \int 2 \, dt. \]

Integrating yields
\[ \log |y| = 2t + C \]
(we arbitrarily put the ambiguity constant \( C \) on the right side, but it doesn’t matter). Finally, exponentiate, getting
\[ y = \pm e^C e^{2t}, \]
which we can rewrite as
\[ y = Ae^{2t} \]
where \( A \neq 0 \) is a nonzero constant. Have we found all of the solutions? Almost! When we divided by \( y \) at the beginning, we implicitly assumed that \( y \neq 0 \). Checking shows that \( y = 0 \) is actually a perfectly good solution too, so the whole set of solutions is
\[ y = Ae^{2t} \]
with \( A \) an arbitrary constant.

In general, if we can write our differential equation in the form
\[ g(y) \frac{dy}{dt} = h(t) \]
where \( g \) and \( h \) are two functions that we know how to integrate, we can solve by separation of variables.

Not all simple equations are “separable” in this way. For example, consider
\[ \frac{dy}{dt} = y + t. \]
No matter what you do, you won’t be able to force this equation into the above general form. This doesn’t mean that the equation doesn’t have a reasonable solution, but it does mean that you won’t be expected to solve it in this class.

9.2. Direction fields. Sometimes more useful than an explicit solution is a qualitative understanding of what the solutions are doing. For any first order differential equation

$$\frac{dy}{dt} = f(y, t)$$

we can draw a direction field by taking every possible value of $y$ and $t$, plugging them into $f$, and recording that value as the slope of a little mark we write over the point $(t, y)$ of a graph. Whatever solutions exist, whenever they pass through $(t, y)$ they must have the same slope as our little mark, because that’s exactly what $\frac{dy}{dt} = f(y, t)$ represents. See the text for many examples. If $f$ is independent of $t$, then the direction field will be the same for every value of $t$; similarly, if $f$ is independent of $y$ then the direction field will be the same for every value of $y$.

If $f$ is independent of $t$, so our equation is

$$\frac{dy}{dt} = g(y),$$

then there will in general be equilibrium solutions, which are solutions for which the slope is zero for all time (i.e., the solution $y$ is constant). To find equilibrium solutions we just search for solutions for which $\frac{dy}{dt} = 0$, which means we are solving $g(y) = 0$. For example, the equation $\frac{dy}{dt} = \cos y$ has equilibrium solutions whenever $\cos y = 0$, which occurs whenever $y = \frac{n\pi}{2}$ with $n$ an odd integer.

Equilibria can be stable or unstable, which in practice means that solutions “close by” to the equilibrium in question either settle down towards the equilibrium or zoom away from it. It is easy to identify whether an equilibrium is stable or not from the direction field: are the little marks pointing towards or away from the equilibrium as time increases?

If your differential equation is modeling a real phenomenon, you will probably never see any actual unstable equilibria: an air molecule will come along and bump your system, causing the state to zoom away from its precarious equilibrium. Think a rigid pendulum standing on its end, for instance.

9.3. A predator-prey model. The relevant pages in Chapter D of the book (pp. 37-39) cover this in essentially the same way as I did, so you might as well go there. Note that the “direction field” we write down in this situation is somewhat different than the direction fields from before, since we’re trying to trace a function meandering in the fox-hare ($F-H$) plane. Therefore the slope of our marks should be $\frac{dF}{dH}$, which we can write using the chain rule as

$$\frac{dF}{dH}$$

and therefore calculate (since neither of these quantities depend explicitly on $t$).

Explicitly,

$$\frac{dF}{dH} = -aF + bFH$$

Incidentally, it is not so hard to see that this is actually a separable equation. Exercise: find the solutions! (Don’t bother trying to solve for $F$ or $H$.) In this
model, we can actually find fairly explicitly what the solutions look like. The direction field is probably more useful, though: it gives us the crucial prediction that the fox and hare populations should be periodic, with the fox population trailing the rabbit population by about a quarter period.

For more information, the wikipedia page for the Lotka-Volterra equations is informative.