PRACTICE MIDTERM 2 (W SOLUTIONS)

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

  “On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

Name: ____________________________

Signature: ____________________________

The following boxes are strictly for grading purposes. Please do not mark.

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(1) Compute the following:

(a) \( \int \frac{x^2 + 2x - 1}{x} \, dx \)

Solution: 
\[
\int \frac{x^2 + 2x - 1}{x} \, dx = \int \left( x + 2 - \frac{1}{x} \right) \, dx = \frac{1}{2} x^2 + 2x - \ln |x| + C.
\]

(b) \( \int_0^2 \frac{x + 1}{x^2 + 2x + 7} \, dx \)

Solution: Let \( u = x^2 + 2x + 7 \). Then \( du = (2x + 2) \, dx = 2(x + 1) \, dx \), and so
\[
\int_0^2 \frac{x + 1}{x^2 + 2x + 7} \, dx = \frac{1}{2} \int_7^{15} \frac{1}{u} \, du = \frac{1}{2} \left[ \ln |u| \right]_7^{15} = \frac{1}{2} (\ln(15) - \ln(7)).
\]

(c) \( \int e^{\frac{1}{x^2}} \, dx \)

Solution: Let \( u = \frac{1}{x} \). Then \( du = -\frac{1}{x^2} \, dx \), and so
\[
\int e^{\frac{1}{x^2}} \, dx = - \int e^u \, du = -e^u + C = -e^{\frac{1}{x^2}} + C.
\]

(d) \( \int_5^6 \sqrt{t - 5} \, dt \)

Solution: Let \( u = t - 5 \). Then \( du = dt \), and so
\[
\int_5^6 \sqrt{t - 5} \, dt = \int_0^1 \sqrt{u} \, du = \left[ \frac{2}{3} u^{3/2} \right]_0^1 = \frac{2}{3}.
\]
(2) Compute the following antiderivative using Integration by Parts:

$$\int t^2 e^{2t} \, dt.$$ 

**Solution:** Let $u = t^2$, $dv = e^{2t} \, dt$. Then $du = 2t \, dt$, $v = \frac{1}{2} e^{2t}$, and so

$$\int t^2 e^{2t} \, dt = \frac{1}{2} t^2 e^{2t} - \int te^{2t} \, dt.$$ 

In the new integral, let $u = t$, $dv = e^{2t} \, dt$. Then $du = dt$, $v = \frac{1}{2} e^{2t}$, and so

$$\int t^2 e^{2t} \, dt = \frac{1}{2} t^2 e^{2t} - \left( \frac{1}{2} te^{2t} - \frac{1}{2} \int e^{2t} \, dt \right)$$

$$= \frac{1}{2} t^2 e^{2t} - \left( \frac{1}{2} te^{2t} - \frac{1}{4} e^{2t} \right) + C.$$ 

$\square$
(3) Compute 
\[ \int x e^{x^2} \cos(x^2 + 1) \, dx. \]

**Hint:** First try making a substitution, and then use Integration by Parts.

**Solution:** Let \( u = x^2 \). Then \( du = 2x \, dx \), and so
\[ \int x e^{x^2} \cos(x^2 + 1) \, dx = \frac{1}{2} \int e^u \cos(u + 1) \, du. \]

We now integrate by parts. Let \( w = e^u \), \( dv = \cos(u + 1) \, du \). Then \( dw = e^u \, du \), \( v = \sin(u + 1) \), and so
\[ \int e^u \cos(u + 1) \, du = e^u \sin(u + 1) - \int e^u \sin(u + 1) \, du. \]

For this new integral, let \( w = e^u \), \( dv = \sin(u + 1) \, du \). Then \( dw = e^u \, du \), \( v = -\cos(u + 1) \), and so
\[ \int e^u \cos(u + 1) \, du = e^u \sin(u + 1) - \left( -e^u \cos(u + 1) + \int e^u \cos(u + 1) \, du \right) \]
\[ = e^u \sin(u + 1) + e^u \cos(u + 1) - \int e^u \cos(u + 1) \, du. \]

Thus,
\[ \int e^u \cos(u + 1) \, du = \frac{1}{2} e^u \sin(u + 1) + \frac{1}{2} e^u \cos(u + 1) + C. \]

So, we’ve computed
\[ \int x e^{x^2} \cos(x^2 + 1) \, dx = \frac{1}{2} \int e^u \cos(u + 1) \, du \]
\[ = \frac{1}{4} e^u \sin(u + 1) + \frac{1}{4} e^u \cos(u + 1) + C \]
\[ = \frac{1}{4} x^2 \sin(x^2 + 1) + \frac{1}{4} e^{x^2} \cos(x^2 + 1) + C. \]

\[ \Box \]
(4) Compute

\[ \int \sin^4(\theta) \, d\theta. \]

*Hint:* Try using the double-angle formulas:

\[
\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \\
\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}
\]

*Solution:* We use both of the double-angle formulas. Observe that

\[
\sin^4(\theta) = (\sin^2(\theta))^2 = \left(\frac{1 - \cos(2\theta)}{2}\right)^2
\]

\[
= \frac{1}{4} - \frac{1}{2} \cos(2\theta) + \frac{1}{4} \cos^2(2\theta)
\]

\[
= \frac{1}{4} - \frac{1}{2} \cos(2\theta) + \frac{1}{4} \cdot \frac{1 + \cos(4\theta)}{2}
\]

\[
= \frac{3}{8} - \frac{1}{2} \cos(2\theta) + \frac{1}{8} \cos(4\theta).
\]

We can now compute the desired integral:

\[
\int \sin^4(\theta) \, d\theta = \int \frac{3}{8} - \frac{1}{2} \cos(2\theta) + \frac{1}{8} \cos(4\theta) \, d\theta
\]

\[
= \frac{3}{8} \theta - \frac{1}{4} \sin(2\theta) + \frac{1}{32} \sin(4\theta) + C.
\]
(5) Prove
\[ \int_{0}^{2} \frac{x^4}{\sqrt{4-x^2}} \, dx = 16 \int_{0}^{\pi/2} \sin^4(\theta) \, d\theta. \]

**Hint:** Try the substitution \( x = 2 \sin(\theta). \)

**Solution:** Let \( x = 2 \sin(\theta). \) Then \( dx = 2 \cos(\theta) \) and \( \sqrt{4-x^2} = 2 \cos(\theta). \)

To change the bounds of integration, observe that when \( x = 0, \) we have \( \sin(\theta) = 0, \) and so \( \theta = 0. \)

Similarly, when \( x = 2, \) we have \( 2 \sin(\theta) = 2, \) and so \( \theta = \pi/2. \)

So,
\[ \int_{0}^{2} \frac{x^4}{\sqrt{4-x^2}} \, dx = \int_{0}^{\pi/2} \frac{(2 \sin(\theta))^4}{2 \cos(\theta)} \cdot 2 \cos(\theta) \, d\theta = 16 \int_{0}^{\pi/2} \sin^4(\theta) \, d\theta. \]

\[ \square \]
(6) Using the method of partial fractions, compute 
\[ \int \frac{x + 11}{x^2 + 2x - 3} \, dx. \]

**Solution:** First observe the denominator factors:
\[ \frac{x + 11}{x^2 + 2x - 3} = \frac{x + 11}{(x + 3)(x - 1)}. \]
If we wish to use the method of partial fractions, we should therefore find \( A, B \) such that
\[ \frac{x + 11}{(x + 3)(x - 1)} = \frac{A}{x + 3} + \frac{B}{x - 1}. \]
Clearing denominators, the above equation becomes:
\[ x + 11 = A(x - 1) + B(x + 3) \]
\[ = (A + B)x + (-A + 3B). \]
Matching coefficients, we have
\[ A + B = 1 \]
\[ -A + 3B = 11. \]
The first equation gives \( B = 1 - A \). Substituting this into the second equation gives
\[ -A + 3(1 - A) = 11. \]
Solving this equation for \( A \) gives \( A = -2. \) Then \( B = 1 - A = 3. \)
So, have have
\[ \int \frac{x + 11}{x^2 + 2x - 3} \, dx = \int \frac{-2}{x + 3} + \frac{3}{x - 1} \, dx \]
\[ = -2 \int \frac{1}{x + 3} \, dx + 3 \int \frac{1}{x - 1} \, dx \]
\[ = -2 \ln |x + 3| + 3 \ln |x - 1| + C. \]
\[ \square \]
(7) For each of the following, compute the integral or prove it diverges:

(a) \( \int_{1}^{\infty} \frac{\ln(x)}{x} \, dx \)

**Solution:** We must compute

\[
\lim_{t \to \infty} \int_{1}^{t} \frac{\ln(x)}{x} \, dx.
\]

Let \( u = \ln(x) \). Then \( du = \frac{1}{x} \, dx \), and so

\[
\int_{1}^{t} \frac{\ln(x)}{x} \, dx = \int_{0}^{\ln(t)} u \, du = \left[ \frac{1}{2} u^2 \right]_{0}^{\ln(t)} = \frac{1}{2} (\ln(t))^2.
\]

So,

\[
\lim_{t \to \infty} \int_{1}^{t} \frac{\ln(x)}{x} \, dx = \lim_{t \to \infty} \frac{1}{2} (\ln(t))^2 = \infty.
\]

Thus, the integral diverges.

(b) \( \int_{0}^{1} \frac{1}{x^3} \, dx \)

**Solution:** By definition,

\[
\int_{0}^{1} \frac{1}{x^3} \, dx = \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{x^3} \, dx
\]

\[
= \lim_{a \to 0^+} \left[ -\frac{1}{2x^2} \right]_{a}^{1}
\]

\[
= \lim_{a \to 0^+} \left( -\frac{1}{2} + \frac{1}{2a^2} \right)
\]

\[
= \infty.
\]

So, this integral also diverges.