PRACTICE MIDTERM 1 (W/ SOLUTIONS)

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

  “On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

  Name: ________________________________

  Signature: ________________________________

The following boxes are strictly for grading purposes. Please do not mark.

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(1)  (a) Write down the linearization of a differentiable function \( f \) at a point \( a \).

Solution: The linearization of \( f \) near \( a \) is
\[
L(x) = f(a) + f'(a)(x - a).
\]

(b) Compute the linearization of the following functions at the specified points:

(i) \( f(x) = \cos(2x) - e^x \), \( a = 0 \)

Solution: We compute
\[
\begin{align*}
  f(0) &= \cos(0) - e^0 = 1 - 1 = 0 \\
  f'(x) &= -2\sin(2x) - e^x \\
  f'(0) &= -2\sin(0) - e^0 = -1.
\end{align*}
\]
So, near 0,
\[
f(x) \approx f(0) + f'(0)(x - 0) = 0 + (-1)(x - 0) = -x.
\]

(ii) \( f(x) = 3x^2 - \frac{1}{x} \), \( a = 1 \)

Solution: We compute
\[
\begin{align*}
  f(1) &= 3(1)^2 - \frac{1}{(1)} = 3 - 1 = 2 \\
  f'(x) &= 6x + \frac{1}{x^2} \\
  f'(1) &= 6(1) + \frac{1}{(1)^2} = 7
\end{align*}
\]
So, near 1,
\[
f(x) \approx f(1) + f'(1)(x - 1) = 2 + 7(x - 1).
\]
(2) Suppose you just baked what’s possibly the world’s largest pizza. When the Guinness Book judges arrive, they measure the area of your pizza to be $10,000\pi$ ft$^2$, with an error of at most $20\pi$ ft$^2$.

(a) What was the percent error in the measurement of the area?

**Solution:** We have

$$\text{Relative error} = \frac{\text{error in area}}{\text{total area}} = \frac{20\pi \text{ ft}^2}{10000\pi \text{ ft}^2} = \frac{2}{1000} = 0.2\%.$$ □

(b) What was the measured radius (in ft) of the pizza?

**Solution:** Using the formula for the area of a circle, we compute

$$A = \pi r^2$$

$$10000\pi \text{ ft}^2 = \pi r^2$$

$$\Rightarrow r = 100 \text{ ft}.$$ □

(c) Using differentials, estimate the possible error (in ft) the judges made in measuring the radius of the pizza.

**Solution:** The differential of the area $A$ is

$$dA = 2\pi r \, dr.$$ Substituting in the given information gives

$$20\pi \text{ ft}^2 = 2\pi(100 \text{ ft}) \, dr$$

$$\Rightarrow dr = \frac{1}{10} \text{ ft} = 0.1 \text{ ft}.$$ □
(3) Compute the following limits. If you use l’Hospital’s rule, make sure to justify its use by noting the type of limit in question; i.e., Type 0/0 or Type \( \infty/\infty \).

(i) \[ \lim_{x \to \infty} \frac{x^3 + 2x}{e^{3x}} \]

Solution:

\[
\lim_{x \to \infty} \frac{x^3 + 2x}{e^{3x}} = \lim_{x \to \infty} \frac{3x^2 + 2}{3e^{3x}} \\
= \lim_{x \to \infty} \frac{6x}{9e^{3x}} \\
= \lim_{x \to \infty} \frac{6}{27e^{3x}} \\
= 0.
\]

\( \square \)

(ii) \[ \lim_{x \to 2} \frac{x^3 - 4x}{x - 2} \]

Solution: We could directly compute this limit:

\[
\lim_{x \to 2} \frac{x^3 - 4x}{x - 2} = \lim_{x \to 2} \frac{x(x + 2)(x - 2)}{x - 2} = \lim_{x \to 2} x(x + 2) = 8.
\]

We could also use l’Hospital’s rule:

\[
\lim_{x \to 2} \frac{x^3 - 4x}{x - 2} = \lim_{x \to 2} \frac{3x^2 - 4}{1} = 8.
\]

\( \square \)

(iii) \[ \lim_{x \to 0^+} x^{2x} \]

Solution: Let \( y = x^{2x} \). Then

\[
\ln(y) = \ln(x^{2x}) = 2x \ln(x),
\]

and so

\[
\lim_{x \to 0^+} \ln(y) = 2 \lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{2 \ln(x)}{x} = \lim_{x \to 0^+} \frac{2}{x} = \frac{2}{0^+} = 0
\]

We thus have

\[
\lim_{x \to 0^+} y = e^0 = 1.
\]

\( \square \)
(4) (a) Write down the definition of an antiderivative of a continuous function $f$.

Solution: An antiderivative of $f$ is a function $F$ such that $F'(x) = f(x)$. □

(b) Write down a function $f$ for which the function

$$F(x) = 3x \cos(x) + e^x - 4$$

is an antiderivative.

Solution: Recall that $F$ is an antiderivative of $f$ if and only if $F'(x) = f(x)$. So, the desired function $f$ must be the derivative of the given function:

$$f(x) = F'(x) = 3 \cos(x) - 3x \sin(x) + e^x.$$ □

(c) Write down the general antiderivatives of the following functions.

(i) $f(x) = \frac{x^2 - x + 2}{x}$

Solution:

$$\int \frac{x^2 - x + 2}{x} \, dx = \int \left( x - 1 + \frac{2}{x} \right) \, dx = \frac{1}{2} x^2 - x + 2 \ln(|x|) + C.$$ □

(ii) $f(x) = -\cos(x) + 2 \sin(x)$

Solution:

$$\int (-\cos(x) + 2 \sin(x)) \, dx = -\sin(x) - 2 \cos(x) + C.$$ □

(iii) $f(x) = e^x + x^4$

Solution:

$$\int (e^x + x^4) \, dx = e^x + \frac{1}{5} x^5 + C.$$ □
(5) (a) Write down the definition of the definite integral of a continuous function $f$ over the interval $[a, b]$.

Solution:

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*) \Delta x,$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i \Delta x$, and $x_i^*$ is any sample point in $[x_{i-1}, x_i]$. □

(b) Write down, but do not evaluate, the Riemann sum for the integral $\int_0^3 x^5 - \cos(x) \, dx$ using right-hand endpoints.

Solution: Using $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and $x_i = 0 + i \Delta x = \frac{3i}{n}$, we have

$$\int_0^3 x^5 - \cos(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f \left( \frac{3i}{n} \right) \frac{3}{n}$$

$$= \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^n \left[ \left( \frac{3i}{n} \right)^5 - \cos \left( \frac{3i}{n} \right) \right].$$

□

(c) Estimate the integral $\int_1^3 x^2 \, dx$ using the Trapezoid Rule with $n = 2$ subintervals.

Solution: Using $\Delta x = \frac{3-1}{2} = 1$ and $x_0 = 1, x_1 = 2, x_2 = 3$, we compute

$$T_2 = \frac{1}{2} (f(x_0) + 2f(x_1) + f(x_2)) \Delta x$$

$$= \frac{1}{2} (f(1) + 2f(2) + f(3))$$

$$= \frac{1}{2} (1^2 + 2(2)^2 + 3^2)$$

$$= 9.$$
(6) (a) Write down the statement of the Evaluation Theorem.

Solution: If $f$ is a continuous function on the interval $[a, b]$ and $F$ is any antiderivative of $f$, then
\[ \int_a^b f(x) \, dx = F(b) - F(a). \]

(b) Using the Evaluation Theorem, compute the following definite integrals:

(i) $\int_{-1}^0 x - x^2 \, dx$

Solution:
\[
\int_{-1}^0 x - x^2 \, dx = \left[ \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-1}^0
\]
\[= \left( \frac{1}{2}(0)^2 - \frac{1}{3}(0)^3 \right) - \left( \frac{1}{2}(-1)^2 - \frac{1}{3}(-1)^3 \right)\]
\[= -\frac{5}{6}.\]

(ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 \cos(x) \, dx$

Solution:
\[
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 \cos(x) \, dx = [2 \sin(x)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}
\]
\[= 2 \sin(\pi) - 2 \sin\left( -\frac{\pi}{2} \right)\]
\[= 2.\]

(iii) $\int_1^e \frac{1}{x} + e^x \, dx$

Solution:
\[
\int_1^e \frac{1}{x} + e^x \, dx = [\ln(|x|) + e^x]_1^e
\]
\[= \ln(e) + e^e - (\ln(1) + e^1)\]
\[= 1 + e^e - e.\]
(7) (a) By interpreting the integral in terms of the area under a curve, compute \( \int_0^4 \sqrt{16 - x^2} \, dx \).

Solution: The equation \( y = \sqrt{16 - x^2} \) is the equation of the top half of the circle \( x^2 + y^2 = 16 \), which the circle of radius 4 centered at the origin. The given integral is thus one quarter of the area of this circle:

\[
\int_0^4 \sqrt{16 - x^2} \, dx = \frac{1}{4} \pi (4)^2 = 4\pi.
\]

(b) Suppose \( f \) is a continuous function such that \( \int_{-1}^3 f(x) \, dx = 5 \) and \( \int_2^3 2f(x) \, dx = 4 \). Compute \( \int_{-1}^2 f(x) \, dx \).

Solution: We use the equation

\[
\int_{-1}^2 f(x) \, dx + \int_2^3 f(x) \, dx = \int_{-1}^3 f(x) \, dx.
\]

Substituting in the given data gives

\[
\int_{-1}^2 f(x) \, dx + 2 = 5,
\]

and hence \( \int_{-1}^2 f(x) \, dx = 3 \).
(8) (a) Write down the statement of the Fundamental Theorem of Calculus.

Solution: Suppose \( f \) is a continuous function on the interval \([a, b]\). Define the function

\[
g(x) = \int_a^x f(t) \, dt
\]

for \( x \in [a, b] \). Then \( g'(x) = f(x) \) □

(b) Compute the derivatives of the following functions, by any means necessary.

(i) \( g(x) = \int_2^x t^3 + 1 - \cos(t) \, dt \)

Solution: By the Fundamental Theorem of Calculus, we have \( g'(x) = x^3 + 1 - \cos(x) \). □

(ii) \( g(x) = \int_0^x t^3 + e^t \, dt \)

Solution: If we let \( u = x^3 \), then we have

\[
g(u) = \int_0^u t + e^t \, dt.
\]

By the Fundamental Theorem of Calculus, we then have

\[
\frac{dg}{du} = u + e^u.
\]

By the Chain rule, we then have

\[
g'(x) = \frac{dg}{dx} = \frac{dg}{du} \frac{du}{dx} = (u + e^u)(3x^2) = (x^3 + e^{x^3})(3x^2).
\] □

(iii) \( g(x) = \int_{x^2}^2 \sqrt{t + e^t} \, dt \)

Solution: First observe that

\[
g(x) = -\int_{x^2}^2 \sqrt{t + e^t} \, dt.
\]

If we let \( u = x^2 \), then we have

\[
g(u) = -\int_2^u \sqrt{t + e^t} \, dt,
\]

and so the Fundamental Theorem of Calculus gives

\[
\frac{dg}{du} = -\sqrt{u + e^u}.
\]

Using the Chain Rule, we then have

\[
g'(x) = \frac{dg}{du} \frac{du}{dx} = -\sqrt{u + e^u}(2x) = -\sqrt{x^2 + e^{x^2}}(2x).
\] □