PICARD GROUPS OF MODULI PROBLEMS II

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1. Recap

Let’s briefly recall what we did last time. I discussed the stack \(BG_m\), as classifying line bundles—by analyzing the sense in which line bundles may be specified locally (e.g., descent data), we arrived at an implicit definition of a stack. I also described the definition of a quotient stack, by analogy with the case of an étale categorical quotient \(X \to X/G\). Let’s recall a few equivalent ways of thinking about the definition of a stack quotient. Throughout, \(G\) is a finite group.

Recall that a map \(X \to Y/G\) was specified by an étale cover \(U \to X\), a map \(\phi: U \to Y\), and an element \(g \in \Gamma(U \times_X U, \mathcal{G})\) so that \(g\pi_1^*\phi = \pi_2^*\phi\). The element \(g\) was required to satisfy a cocycle condition—that is, \(\pi_1^*g \cdot \pi_2^*\phi = \pi_3^*\phi\), where \(\pi_i: U \times_X U \to U \times_X U\) are the projections to the \((i,j)\)-factors. Note that this data is the same as specifying an étale-locally trivial \(G\)-torsor \(\mathcal{G}\) on \(X\) and a \(G\)-equivariant map \(\mathcal{G} \to Y\). [Exercise: Convince yourself that this is true!] An isomorphism \((U, \phi, g)\) and \((U', \phi', g')\) is given by étale covers \(V \to U, V \to U'\) so that the pullbacks of \(\phi, \phi', g, g'\) agree on \(V\) (with an evident equivalence relation on isomorphisms for refinements of étale covers).

Another way of thinking about this is as follows—\(Y\) represents some sheaf \(h_Y\). We can define a pre-sheaf of groupoids \(h_{Y/G}\) by setting the objects of \(h_{Y/G}(T) = h_Y(T)\), with morphisms between two \(T\)-points \(x\) and \(y\) given by

\[
\text{Hom}(x, y) := \{ g \in G \mid gx = y \}.
\]

The stack \(h_{Y/G}\) is given by sheafifying \(h_{Y/G}\) in the étale topology (where I mean sheafification in the homotopical sense of a sheaf of groupoids).

Example 1. Consider the stack \(\mathbb{A}^1/G_m\), where \(G_m\) acts by scaling (this is a stack in the smooth topology, not the étale topology). A map \(X \to \mathbb{A}^1/G_m\) is given by a line bundle \(\mathcal{L}\) on \(X\) and a section \(s \in \Gamma(X, \mathcal{L})\). Note that \(BG_m = pt/G_m\) is a closed substack of \(\mathbb{A}^1/G_m\) (via \(pt \to 0\)). Thus the map \(X \to \mathbb{A}^1/G_m\) induces a map \(V(s) \to BG_m\), which classifies the conormal bundle of \(V(s)\).

Example 2. \(BG = pt/G\) classifies étale-locally trivial \(G\)-torsors, for \(G\) a finite group (or étale group scheme, really).

2. The Picard Group of \(BG\)

The ultimate goal of this note is to compute \(\text{Pic}(\mathcal{M}_{1,1})\), following Mumford. We will now do a warm-up, by defining and computing \(\text{Pic}(BG)\), for a finite group \(G\). [Can you guess what the answer is?] We work over a field \(k\).

Suppose \(G\) is a finite group. Let us discuss what it means to give a line bundle on \(BG\). If \(X\) is a scheme, and \(\mathcal{L}\) is a line bundle on \(X\), a line bundle on \(X\) is specified by giving a line bundle \(\mathcal{L}\) on any fppf (or étale) cover \(U \to X\), as well as descent data—namely, if \(\pi_i: U \times_X U \to U\) are the projections, an isomorphism \(f: \pi_1^*\mathcal{L} \cong \pi_2^*\mathcal{L}\), satisfying the cocycle condition.

Remark 1. Note that we may view a line bundle as giving a lot more data than the above—namely if \(\mathcal{L}\) is a line bundle on \(X\), and \(f: T \to X\) is any morphism, we get a line bundle \(f^*\mathcal{L}\) on \(T\). Furthermore, if we have a commutative triangle

\[
\begin{array}{ccc}
T' & \xrightarrow{g} & T \\
\downarrow{f'} & & \downarrow{f} \\
X & \xrightarrow{1} & X
\end{array}
\]
there is a canonical isomorphism $f^*L \cong g^*f^*L$; these isomorphisms are compatible with compositions in the obvious sense.

Now, here are two (equivalent) ways of thinking of a line bundle on $BG$. As above, we may view such a line bundle as associating to each map $T \to BG$ (e.g. a $G$-torsor $\mathcal{G}$ on $T$) a line bundle $L_{\mathcal{G}}$—furthermore, for each $h : T' \to T$ over $BG$, we need to specify an isomorphism $h^*L_G \cong L_{G'}$, (where $T' \to BG$ is specified by a $G$-torsor $\mathcal{G}'$), compatible with compositions. Note that there are automorphisms $T \to T$ over $BG$, which are the identity on $T$—namely, automorphisms of $\mathcal{G}$, aka $G$ itself! So in particular, we have a map $\chi_G : G \to \text{Aut}(L_{\mathcal{G}}) = G_m$ for each $\mathcal{G}$.

On the other hand, we may specify a line bundle $L$ on an étale cover; that is, every line bundle on $BG$ comes from a line bundle and descent data on $pt$. Let us think about what it means to specify a line bundle as associating to each map $T \to BG$ a (set-theoretic) map $\text{Tot}(L_{\mathcal{G}})$, compatible with compositions. Note that there are automorphisms $T \to T$ over $BG$, which are the identity on $T$—namely, automorphisms of $\mathcal{G}$, aka $G$ itself! So in particular, we have a map $\chi_G : G \to \text{Aut}(L_{\mathcal{G}}) = G_m$ for each $\mathcal{G}$.

Remark 2. An identical argument shows that

$$\text{Pic}(BG_m) = H^1(G_m, G_m) = \mathbb{Z} = H^2(\mathbb{C}P^\infty, \mathbb{Z})(!)$$

as one might expect from the discussion above.

Remark 3. These results hold true even if $k$ is not algebraically closed—one must apply Hilbert 90 in the argument, however. Note that the computation of the Picard group depends on the number of roots of unity $k$ contains, and so varies depending on the characteristic of $k$ and whether or not it is algebraically closed.

3. $M_{1,1}$

Like Mumford, I’ll work over a field $k$ of characteristic different from 2 or 3. (Olsson and Fulton work out the Picard group of $M_{1,1}$ over a quite general base, if you’re interested.) We define:

Definition 4. A family of elliptic curves over $S$ is a smooth projective morphism $\pi : X \to S$ whose geometric fibers are curves of genus 1, and with a section $\epsilon : S \to X$ (the identity section).

The moduli stack $M_{1,1}$ is described as follows: $M_{1,1}(T)$ is the groupoid of families of elliptic curves over $T$, with the evident notion of isomorphism. At this point, it is not at all obvious that $M_{1,1}$ is algebraic—that is, we wish to find an étale cover of $M_{1,1}$. We note first that the map

$$M_{1,1} \to M_{1,1} \times M_{1,1}$$

is representable. This follows from the representability of the Isom functor, due to Grothendieck. Indeed, if $T$ is a scheme, and $X_1, X_2$ are families of elliptic curves over $T$, the pullback of the diagram

$$\begin{array}{ccc}
T & \xrightarrow{(x_1, x_2)} & M_{1,1} \\
\downarrow & & \downarrow
\end{array}$$

has $T'$-points given by $\text{Isom}_T(X_1, X_2)(T')$. 2
Remark 5. Note that in the analytic setting (that is, if we work in the analytic topology, where covers are given by surjective local homeomorphisms), \( \mathcal{M}_{1,1} \) admits a presentation as a quotient stack. Let \( \mathbb{H} \) be the upper-half plane. Then the functor of points of \( \mathbb{H} \) (in complex-analytic spaces) is given by
\[
\mathbb{H}(T) = \{ \phi : \mathbb{Z} \to C_T; s_1, s_2 \in \Gamma(T, \mathbb{Z}) \}/ \sim
\]
where \( C_T \) is the trivial \( \mathbb{C} \)-local system on \( T \), \( \mathbb{Z} \) is a rank 2 \( \mathbb{Z} \), local system, \( s_1 \) and \( s_2 \) are trivializing global sections to \( \mathbb{Z} \) with \( \phi(s_1)/\phi(s_2) \in \mathbb{H} \), and \( \phi \) is the inclusion of a lattice in \( \mathbb{C} \) on each fiber. The universal family over \( \mathbb{H} \) is given by the map over \( \mathbb{H} \)
\[
\mathbb{Z}^2 \times \mathbb{H} \to \mathbb{C} \times \mathbb{H}
\]
\[
(n, m) \mapsto n + \tau m
\]
where \( \tau \) is the coordinate on \( \mathbb{H} \).

Taking the cokernel of this map (over \( \mathbb{H} \)) gives an elliptic curve over \( \mathbb{H} \), with identity section given by the zero section. \( SL(2, \mathbb{Z}) \) acts naturally on \( \mathbb{H} \) via
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a \tau + b}{ct + d},
\]
(This action is induced by the natural action of \( SL(2, \mathbb{Z}) \) on the moduli problem, given by moving \( s_1, s_2 \) around.)

I claim that the quotient stack \( \mathbb{H}/SL(2, \mathbb{Z}) \) is \( \mathcal{M}_{1,1} \). Namely, we may view \( \mathbb{H} \) as representing the functor \( \mathbb{H}(T) = \{ \text{families of elliptic curves on } T \text{ with trivialized homology basis} \}/ \sim \), where by “trivialized homology basis” for a family of elliptic curves \( X \to T \), I mean, a choice of trivialization of the homology local system \( \mathcal{H}_1(X/T, \mathbb{Z}) \). Now a map \( X \to \mathcal{M}_1,1 \) lifts analytically-locally to a map to \( \mathbb{H} \), (by choosing an open cover of \( X \) where the homology local systems are trivial), and the lifts are a torsor for \( SL(2, \mathbb{Z}) \), giving the claim.

Now we may compute \( \text{Pic}(\mathcal{M}_{1,1}) \) analytically. Namely, \( \text{Pic}(\mathbb{H}) = \{ 1 \} \), so \( \text{Pic}(\mathcal{M}_{1,1}) = \text{Hom}(SL(2, \mathbb{Z}), \mathbb{C}^*) \). The abelianization of \( SL(2, \mathbb{Z}) \) is \( \mathbb{Z}/12\mathbb{Z} \), so \( \text{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12\mathbb{Z} \).

Let’s get back to the algebraic setting. Now we ask: what does it mean for a map \( U \to \mathcal{M}_{1,1} \), induced by a family of elliptic curves \( \mathfrak{X} \to U \), to be an étale cover? Last time, we said that this meant that for any pullback diagram
\[
\begin{array}{ccc}
U \times_{\mathcal{M}_{1,1}} T & \longrightarrow & U \\
\downarrow & & \downarrow \chi \\
T & \longrightarrow & \mathcal{M}_{1,1}
\end{array}
\]
the map \( U \times_{\mathcal{M}_{1,1}} T \to T \) is an étale surjection. Now a \( T' \)-point of \( U \times_{\mathcal{M}_{1,1}} T \) is given by a maps \( f : T' \to U, g : T' \to T \), and an isomorphism \( f^* \mathfrak{X} \cong g^* \mathfrak{Y} \) over \( T' \). There exist families \( \mathfrak{Y} \) containing every isomorphism class of elliptic curve, e.g. the family over \( \mathbb{A}^1 \setminus \{ 0, 1 \} \) (with parameter \( \lambda \)) defined by
\[
y^2 = x(x-1)(x-\lambda).
\]
So \( U \cong \mathcal{M}_{1,1} \) is a surjection if and only if the the family \( \mathfrak{X} \to U \) has every isomorphism class of elliptic curve among its fibers.

Now recall that a morphism is étale if and only if it is locally of finite presentation and \textit{formally} étale—that is, a lfp map \( X \to Y \) is étale if and only if every diagram
\[
\begin{array}{ccc}
\text{Spec}(A/I) & \longrightarrow & X \\
\downarrow & \searrow & \downarrow \exists \phi \\
\text{Spec}(A) & \longrightarrow & Y
\end{array}
\]
admits a unique dotted arrow as above, making the diagram commute, where \( A \) is an Artinian-local \( k \)-algebra and \( I \) is a square-zero ideal. Thus we say \( U \cong \mathcal{M}_{1,1} \) is (finite) étale if it satisfies the diagram above, and if each isomorphism-class of elliptic curve appears at most finitely-many times in \( \mathfrak{X} \).
Let’s unwind this a bit more (that is, we’ll give a more explicit description of the small étale site of $M_{1,1}$. Suppose an elliptic curve is given by the equation

$$y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

where $\alpha_1 \neq \alpha_2 \neq \alpha_3$. We set $\lambda = \frac{\alpha_3 - \alpha_1}{\alpha_2 - \alpha_1}$ and

$$j = -64 \left( \frac{(\lambda - 2)(2\lambda - 1)(\lambda + 1)}{\lambda(\lambda - 1)} \right)^2.$$

What does this mean? The elliptic curve as presented admits a 2-to-1 map to $\mathbb{P}^1$, given by projection to the $x$ coordinate. This map is ramified over $\alpha_1, \alpha_2, \alpha_3, \infty$; composing with a fractional linear transformation we may arrange that it is ramified over $0, 1, \infty, \lambda$. Permuting the $\alpha_i$’s leads to a different $\lambda$—namely one of $\lambda, 1 - \lambda, \frac{1}{\lambda}, 1 - \frac{1}{\lambda}, \lambda, 1 - \frac{1}{\lambda}$.

Note that the values of $\lambda$ for which this permutation action has non-trivial stabilizer are $2, 1/2, -1, -\omega, -\omega^2$, where $\omega$ is a primitive cube root of 1. In this case $j = 0, 1728$.

Now every elliptic curve has at least one non-trivial automorphism, given by inversion—in the presentation above, this is given by sending $y$ to $-y$. In the cases where $j = 0, 1728$, there are extra automorphisms, given as follows. We may represent the case $j = 0$ by the curve

$$y^2 = x(x^2 - 1),$$

which has automorphism group $\mathbb{Z}/4\mathbb{Z}$, generated by the automorphism

$$x \mapsto -x$$
$$y \mapsto iy.$$

The second case is represented by the curve

$$y^2 = x^3 - 1$$

which has automorphism group $\mathbb{Z}/6\mathbb{Z}$, generated by

$$x \mapsto \omega x$$
$$y \mapsto -y.$$

Let $A = \frac{27}{4} \cdot \frac{1728 - j}{j}$, and consider the family

$$y^2 = x^3 + A(x + 1)$$

over $\mathbb{A}_k^1 \setminus \{0, 1\}$. The fiber over some point $s$ is the curve with $j$-invariant $s$. This is étale of degree 2 over $M_{1,1}$, as elliptic curves admit no infinitesimal deformations (fixing the identity) and each curve in this family has automorphism group $\mathbb{Z}/2\mathbb{Z}$. But it is not a cover, as it misses the curves with $j$-invariant 0, 1728.

That said, if $X/S$ is any family of elliptic curves, there is a natural map $X \to \mathbb{A}_k^1$ sending each closed point $s \in S$ to the $j$-invariant of $X_s$.

**Remark 6.** It is not obvious that this map is algebraic; here’s a sketch. Namely, let $\epsilon : S \to X$ be the identity section. Then there is a diagram

$$\begin{array}{ccc}
X & \longrightarrow & \mathbb{P}(\pi_* \mathcal{O}(2\epsilon)) \\
\downarrow & & \downarrow \\
S & \longrightarrow & \\
\end{array}$$

We may choose a Zariski-cover of $S$ so that $\mathbb{P}(\pi_* \mathcal{O}(2\epsilon))$ is trivialized over each open in our cover; then the $j$-invariant may be defined as above on each open set. It’s easy to see that it glues together to a map on all of $S$. 

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Unfortunately the map \( j : S \to \mathbb{A}^1 \) does not determine \( \mathcal{X}/S \). Note that if \( S \) is connected and has varying modulus, the map \( S \to \mathbb{A}^1 \) is étale over \( \mathbb{A}^1 \setminus \{0, 1728\} \), and ramified to degree 2 if \( j = 0 \), and degree 3 if \( j = 1728 \). Let’s add another invariant of \( \mathcal{X}/S \) which, along with the \( j \)-invariant, does determine the family. Namely, suppose \( S \to \mathcal{M}_{1,1} \) is étale and \( S \) is connected. Then we may take the fiber product

\[
\begin{array}{ccc}
T & \longrightarrow & \mathcal{A}^1_j \setminus \{0, 1728\} \\
\downarrow & & \downarrow \\
S & \longrightarrow & \mathcal{M}_{1,1}
\end{array}
\]

Over the locus \( \{ s \in S \mid j(\mathcal{X}_s) \neq 0, 1728 \} \), \( T \to S \) is étale of degree 2. So normalizing \( S \) in the function field of \( T \) (if \( T \) is connected), we obtain a 2 : 1 cover of \( S \), ramified exactly where \( j = 0, 1728 \) (if \( T \) is not connected, we replace it with two copies of \( S \)). We claim that the map \( j : S \to \mathbb{A}^1 \) and \( T/S \) determine the family \( \mathcal{X} \).

**Sketch Proof.** Let \( S_0 \) be the locus where \( T/S \) is étale, and let \( T' \to S_0 \) be an étale cover trivializing \( T/S \) (which as before extends uniquely to a ramified map \( T'' \to S \)). Then the map \( T' \to \mathbb{A}^1_j \setminus \{0, 1728\} \) determines a family of elliptic curves over \( T' \), with trivial monodromy—thus this family extends uniquely to a family with trivial monodromy on \( T'' \). Furthermore, the descent data associated to \( T'/S_0 \) extending uniquely, so we obtain a natural family of elliptic curves on \( S \) associated to \( j, T/S \).

Now we may give a more concrete description of the small étale site over \( \mathcal{M}_{1,1} \):

- An étale map \( S \to \mathcal{M}_{1,1} \), where \( S \) is a connected smooth curve, is a dominant map \( j : S \to \mathbb{A}^1_j \), étale over \( \mathbb{A}^1_j \setminus \{0, 1728\} \), and ramified to degree 2 if \( j = 0 \), and degree 3 if \( j = 1728 \), plus a double covering \( T/S \) ramified exactly where \( j = 0, 1728 \).
- A cover is a collection of such maps, so that the associated \( j \) maps are jointly surjective.

### 4. Algebraic Computation of the Picard Group

We now compute \( \text{Pic}(\mathcal{M}_{1,1}) \). Recall that, as before, an element of \( \text{Pic}(\mathcal{M}_{1,1}) \) is a way of (functorially) associating, to each family of elliptic curves \( \mathcal{X}/S \), a line bundle \( \mathcal{L}(\mathcal{X}/S) \) over \( S \). Here is a natural, and important, example:

**Example 3 (The Hodge Bundle).** To each \( \pi : \mathcal{X} \to S \) a family of elliptic curves, we associate the line bundle \( R^1\pi_*\mathcal{O}_X \). This is a line bundle by cohomology and base change, and is obviously functorial under pullback.

Now we wish to associate some invariant to each line bundle on \( \mathcal{M}_{1,1} \). Now note that each elliptic curve admits an automorphism—namely inversion—so given a line bundle \( \mathcal{L} \) on \( \mathcal{M}_{1,1} \), and a map \( f : S \to \mathcal{M}_{1,1} \), we obtain a natural automorphism \( \mathcal{L}(\text{inversion}) \) on \( f^*\mathcal{L} \), which squares to the identity. (Indeed, it is easy to see that this is independent of \( f \), as this homomorphism is locally constant.) Thus, we obtain a map \( \mathbb{Z}/2\mathbb{Z} \to \mathbb{G}_m \). Furthermore, our two distinguished elliptic curves, with \( j \)-invariant 0, 1728, yield analogous homomorphisms \( \mu_4 \to \mathbb{G}_m, \mu_6 \to \mathbb{G}_m \). Putting these all together, we obtain a natural homomorphism

\[
\mu_{12} = (\mu_4) \times \mu_6 \to \mathbb{G}_m,
\]

for each line bundle—that is, a canonical map

\[
\text{Pic}(\mathcal{M}_{1,1}) \to \text{Hom}(\mu_{12}, \mathbb{G}_m).
\]

We claim this map is an isomorphism (in characteristics different from 2, 3). Furthermore, we claim that the Hodge bundle above maps to generator of \( \text{Hom}(\mu_{12}, \mathbb{G}_m) \).

It’s easy to check that the Hodge bundle is a generator. We may compute the homomorphism \( \mu_{12} \to \mathbb{G}_m \) induced by the Hodge bundle as follows. Let \( C_0/\text{Spec}(k) \) be the curve with \( j \)-invariant 0, defined by the equation

\[
y^2 = x^3 - x
\]

and \( C_{1728}/\text{Spec}(k) \) the curve defined by

\[
y^2 = x^3 - 1.
\]
We must compute the action of the automorphism groups of these curves on the Hodge bundle, which in this case is just $H^1(C, O_X)$, which by Serre duality is dual to $H^0(C, \omega_C)$, which has generator $dx/y$.

Now the automorphism group of $C_0$ is generated by $x \mapsto -x, y \mapsto iy$, which thus has action $dx/y \mapsto idx/y$ (that is, multiplication by a primitive 4-th root of unity). The automorphism group of $C_{1728}$ is generated by $x \mapsto \omega x, y \mapsto -y$, which has action $dx/y \mapsto -\omega dx/y$ (that is, multiplication by a primitive 6-th root of unity). Thus the induced map $\mu_{12} \to \mathbb{G}_m$ hits a primitive 12-th root of unity, and is thus a generator of the character group of $\mu_{12}$. We have shown that the map $\text{Pic}(\mathcal{M}_{1,1}) \to \text{Hom}(\mu_{12}, \mathbb{G}_m)$ is surjective.

We now show that the map is injective. Suppose $L$ is a line bundle mapping to zero in $\text{Hom}(\mu_{12}, \mathbb{G}_m)$. It suffices to trivialize $L$ on any surjective étale cover of $\mathcal{M}_{1,1}$, so that the associated descent data is trivial (by which I mean, the descent data commutes with our chosen isomorphism $L|_S \cong \mathcal{O}_S$). Just to be explicit, let’s use the $\lambda$-family over $S = \mathbb{A}^1_s \setminus \{0,1\}$, defined by

$$y^2 = x(x-1)(x-\lambda),$$

which is a degree 12 cover of $\mathcal{M}_{1,1}$. All line bundles on $S$ are trivial, so we choose an isomorphism $\phi : \mathcal{O}_S \to L|_S$, which is in $\Gamma(S, L|_S)^G$, where $G$ is the group acting on $S$ generated by $\lambda \mapsto \lambda^{-1}, \lambda \mapsto 1 - \lambda$ (this fixed space is non-empty by the existence of the $j$-invariant). Now we have descent data $\pi^*_1L \sim \pi^*_2L$, where $\pi_i : S \times_{\mathcal{M}_{1,1}} S \to S$ are the projections. We need to check that

$$\begin{array}{ccc}
\pi^*_1L & \longrightarrow & \pi^*_2L \\
\downarrow \phi & & \downarrow \phi \\
\pi^*_1\mathcal{O}_S & \longrightarrow & \pi^*_2\mathcal{O}_S
\end{array}$$

commutes.

But we may check this on the level of stalks, whence it is precisely the triviality of the map $\text{Hom}(\mu_{12}, \mathbb{G}_m)$. 

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