1 Strategy

We prove the fundamental theorem of algebra, using only elementary techniques from calculus, point-set topology, and linear algebra; this proof apparently does not appear in the extensive literature on the subject [1], [2]. The exposition is essentially self-contained.

Theorem 1 (Fundamental Theorem of Algebra). Every non-constant polynomial with complex coefficients has a root in \( \mathbb{C} \).

This is the strategy of the proof. Let \( X_n \simeq \mathbb{C}^n \) be the space of degree \( n \) monic polynomials with complex coefficients, via the identification \((a_1, ..., a_n) \mapsto z^n + \sum_{i=1}^n a_i z^i\). Let \( D \subset X_n \) be the zero locus of the discriminant, and let \( R \subset X_n \setminus D \) be the set of polynomials with non-zero discriminant which have at least one complex root. We show:

1. \( X_n \setminus D \), the set of monic degree \( n \) polynomials with non-zero discriminant, is connected.

2. \( R \), the set of monic degree \( n \) polynomials with non-zero discriminant which have at least one root, is both open and closed in \( X_n \setminus D \). As \( R \) is nonempty it is thus equal to \( X_n \setminus D \), so every monic degree \( n \) polynomial with non-zero discriminant has a root.

3. By induction on \( n \), every polynomial with zero discriminant has a root.

2 Preliminaries

The following preliminary lemma is the only part of the argument that uses that the ground field is \( \mathbb{C} \), rather than \( \mathbb{R} \).

Lemma 1. Let \( V \subset \mathbb{C}^n \) be the zero locus of some polynomial \( p(x) = p(x_1, ..., x_n) \). Then \( \mathbb{C}^n \setminus V \) is path-connected, and thus connected.
Proof. Let \( y, z \in \mathbb{C}^n \setminus V \) be two points in the complement of \( V \). Consider the set \( S = \{ cy + (1 - c)z \mid c \in \mathbb{C} \} \subset \mathbb{C}^n \), which is a complex line connecting \( y \) and \( z \).
Then \( S \cap V \) is a finite set, as \( p(cy + (1 - c)z) \) is a polynomial in the single complex variable \( c \), and thus has at most finitely many zeros. In particular, \( S \setminus (S \cap V) \) is homeomorphic to the complex plane with finitely many points removed, and so is path connected. Thus there is a path in \( S \setminus (S \cap V) \) connecting \( y \) and \( z \). □

We will also need an easy lemma bounding the size of the roots of a monic polynomial in terms of its coefficients.

Lemma 2. Let \( \{ f_\alpha \} \) be a set of monic degree \( n \) polynomials whose coefficients all lie in some bounded region of \( \mathbb{C} \). Then there exists \( C > 0 \) such that if \( z \) is a zero of \( f_\alpha \) for some \( \alpha \), then \(| z | < C \).

Proof. This is immediate from the fact that
\[
\frac{f_\alpha(z)}{z^n} \to 1 \text{ as } |z| \to \infty
\]
uniformly in \( \alpha \). □

Finally, we introduce the resultant and discriminant. Let \( k \) be a field and let \( f, g \in k[x] \) be non-constant polynomials with coefficients in \( k \). Then there is a map
\[
\psi_{f,g} : k[x]/(f) \oplus k[x]/(g) \to k[x]/(fg)
\]
given by
\[
(a + (f), b + (g)) \mapsto ag + bf + (fg).
\]
By the chinese remainder theorem, this map is a \( k \)-vector space isomorphism if and only if \( \gcd(f, g) = 1 \). Define the resultant
\[
R_{f,g} = \det(\psi_{f,g}).
\]
Note that by the previous remark, \( R_{f,g} = 0 \) if and only if \( f, g \) have a common factor. Taking \( k = \mathbb{C}(a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m) \) with
\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, \quad g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0
\]
and choosing bases for \( k[x]/(f), k[x]/(g), k[x]/(fg) \) gives a formula for \( R_{f,g} \) as a polynomial in the coefficients of \( f, g \) for general polynomials \( f, g \) with complex coefficients.

Now let \( f \) be any polynomial of degree at least 2 with complex coefficients, and define the discriminant \( D_f = R_{f,f'} \), where \( f' \) is the derivative of \( f \). Note that \( D_f \) is a polynomial in the coefficients of \( f \). Furthermore, \( D_f = 0 \) if and only if \( f \) has a factor in common with its derivative.
3 The Proof

We now prove the fundamental theorem of algebra (Theorem 1).

Let $X_n \simeq \mathbb{C}^n$ be the space of degree $n$ monic polynomials with complex coefficients, via the identification $(a_1, \ldots, a_n) \mapsto z^n + \sum_{i=1}^{n} a_i z^i$; we endow $X_n$ with the analytic topology. Let $D \subset X_n, D := \{ f \in X_n \mid D_f = 0 \}$ be the set of polynomials $f$ with discriminant 0. Namely, $D$ consists of those polynomials which have a factor in common with their derivative. Note that $D$ is a closed subset of $X_n$, as it is the zero set of a polynomial. Define $R \subset X_n \setminus D$, by

$$R = \{ f \in X_n \setminus D \mid \exists z \in \mathbb{C} \text{ such that } f(z) = 0 \}.$$ 

That is, $R$ consists of those polynomials, with non-zero discriminant, which have a root in $\mathbb{C}$. Note that $R$ is non-empty; for example, it contains $z^n - 1$.

We claim that $R$ is open in $X_n \setminus D$, in the subspace topology. To see this, let $ev : \mathbb{C} \times (X_n \setminus D) \to \mathbb{C}$ be the evaluation map $(z, p) \mapsto p(z)$. Consider $f \in R$; by definition, $f$ has a root $t$, so $ev(t, f) = 0$. Furthermore, $(\frac{\partial}{\partial t} ev)(t, f) = f'(t)$ is non-zero, as otherwise $(z - t)$ divides both $f$ and $f'$, and thus $D_f = 0$, which contradicts the fact that $f \notin D$.

Thus, by the implicit function theorem, there exists an open neighborhood $U \subset X_n \setminus D$ with $f \in U$, and a function $r : U \to \mathbb{C}$ such that $r(f) = t$ and $g(r(g)) = ev(r(g), g) = 0$ for all $g \in U$. That is, we have found a neighborhood $U$ of $f$ and a function on $U$ parametrizing roots of polynomials in $U$; in particular, all of the polynomials in $U$ have a root. Thus $U \subset R$, and so $R$ is open.

Now, we claim $R$ is closed in $X_n \setminus D$. Let $f_k \to f$ in $X_n \setminus D$, with $f_k \in R$ for all $k$; we wish to show that $f$ has a root in $\mathbb{C}$. As each $f_k \in R$, there exists $z_k \in \mathbb{C}$ with $f_k(z_k) = 0$. By Lemma 2, the $z_k$ are bounded, and so there exists a convergent subsequence $z_{k_n} \to z$. So replacing $\{f_j\}, \{z_j\}$ by subsequences, we may assume $z_j \to z$. We claim $f(z) = 0$, and thus $f \in R$. Indeed, we have

$$|f(z) - f_k(z_j)| \leq |f(z) - f(z_j)| + |f(z_j) - f_k(z_j)|. \quad (*)$$

Taking $j, k$ large, we may make the right hand side of $(*)$ arbitrarily small, by the continuity of $f$ and the fact that the $f_k$ converge to $f$ pointwise. Now taking $j = k$ large, $f_k(z_j) = 0$, so we may make $|f(z)|$ arbitrarily small. Thus $f(z) = 0$ as desired.

So $R$ is both open and closed in $X_n \setminus D$. But by Lemma 1, $X_n \setminus D$ is connected, so $R = X_n \setminus D$. In particular, every polynomial of degree $n$ with non-zero discriminant has a root.

It remains only to show that those degree $n$ polynomials $f$ with zero discriminant have a root. But such polynomials $f$ have a factor $g$ in common with their derivatives $f'$. The degree of $g$ is less than that of $f$, and so we are done by induction on $n$, as the degree 1 case is trivial.
References
