7 Semi-Discrete Central Schemes

Our goal in this section is to solve convection-diffusion equations of the form

\[
\frac{\partial}{\partial t} u(x, t) + \frac{\partial}{\partial x} f(u(x, t)) = \frac{\partial}{\partial x} Q[u(x, t), u_x(x, t)],
\]

where \( Q \) satisfies the (weakly) parabolic condition \( \nabla_s Q(u, s) \geq 0 \) for all \( u \) and \( s \), \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), \( f(u) = (f^1, \ldots, f^d) \) and \( u(x, t) = (u_1(x, t), \ldots, u_N(x, t)) \).

References: second order method [40], third order method [36].

Stability requires small time steps of the order \( \Delta t \sim \Delta x^2 \), and that results with a substantial accumulation of numerical dissipation over all the time steps that are required in order to get to the desired final time. A possible way to overcome this difficulty is to use a semi-discrete formulation and an ODE solver. This will change the order of the numerical dissipation from \( O\left(\frac{\Delta x^2}{\Delta t}\right) \) to \( O\left(\frac{\Delta x^{2r}}{\Delta t}\right) \) where \( r \) is the order of accuracy of the method.

A semi-discrete formulation is obtained in the limit \( \Delta t \to 0 \). Unfortunately, it is impossible to directly consider this limit with central schemes that are written in their “standard” formulation. To demonstrate this point, consider, e.g., the Lax-Friedrichs scheme, which for the conservation law \( u_t + f(u)_x = 0 \) is

\[
u^{n+1}_j = \frac{1}{2} (u^{n+1}_{j+1} + u^{n+1}_{j-1}) - \frac{\lambda}{2} (f(u^{n+1}_{j+1}) - f(u^{n+1}_{j-1})). \tag{7.99}\]

Rewriting (7.99) in its viscous form, we have

\[
\frac{u^{n+1}_j - u^n_j}{\Delta t} + \frac{f(u^{n+1}_{j+1}) - f(u^n_{j-1})}{2\Delta x} = \frac{1}{2\Delta t} \left[ (u^{n+1}_{j+1} - u^n_{j}) - (u^n_{j} - u^n_{j-1}) \right]. \tag{7.100}\]

As \( \Delta t \to 0 \), (7.100) becomes

\[
\frac{\partial u}{\partial t} + f\left(\frac{u^{n+1}_{j+1} - u^n_{j-1}}{2\Delta x}\right) \sim \frac{\Delta x^2}{2\Delta t} u_{xx} \to \infty,
\]

(for a fixed \( \Delta x \)).

7.1 A Local Central Scheme for Conservation Laws

In order to develop a semi-discrete central scheme for \( u_t + f(u)_x = 0 \), we will add information about the local speed of propagation. Say we are given a piecewise linear or polynomial reconstruction at \( t^n \). How do we evolve this reconstruction in time? We don’t use the exact speed of propagation, as that would amount to solving a Riemann problem. Rather we estimate the maximum possible value of the speed of propagation at each discontinuity.

We estimate the local speed of propagation at the cell boundaries \( x_{j+\frac{1}{2}} \), disregarding the direction of propagation. (A more careful estimation of the local speed of propagation that is based on different speeds in each direction will lead to a semi-discrete
Figure 7.12: A schematic representation of the semi-discrete formulation: the cell averages $\bar{u}^n_j$ are evolved to the intermediate cell averages $\tilde{w}^n_j$ on an unequally spaced grid, which are averaged to obtain $\bar{w}^{n+1}_j$.

formulation that enjoys even less numerical dissipation compared with the scheme we write below. Ref [Kurganov-Noelle-Petrova]). The upper bound of the local speed of propagation at $x_{j+\frac{1}{2}}$ is

$$a^n_{j+\frac{1}{2}} = \max \left\{ \rho \left( \frac{\partial f}{\partial u} (u^-_{j+\frac{1}{2}}) \right), \rho \left( \frac{\partial f}{\partial u} (u^+_{j+\frac{1}{2}}) \right) \right\}$$

where

$$\rho (A) = \max_{i} |\lambda_i (A)|.$$  

Due to the finite speed of propagation, the information coming from $x^n_{j+\frac{1}{2}}$ can, at most, reach the following two points:

$$x_{j+\frac{1}{2},L} = x_{j+\frac{1}{2}} - a^n_{j+\frac{1}{2}} \Delta t,$$

and

$$x_{j+\frac{1}{2},R} = x_{j+\frac{1}{2}} + a^n_{j+\frac{1}{2}} \Delta t.$$  

Hence, $x^n_{j+1/2,L}$ and $x^n_{j+1/2,R}$ separate between smooth and non-smooth regions. The non-smooth parts of the solution are contained inside the narrow control volume of width $2a^n_{j+1/2}\Delta t$.

Let's now assume a piecewise linear reconstruction of the form $\tilde{u}(x,t) = \sum_j [\bar{u}^n_j + (u_x)^n_j (x-x_j)] \chi_j(x)$. An exact integration over the control volume followed by approxi-
mation of the integrals with midpoint quadrature results in
\[
w_{j+\frac{1}{2}}^{n+1} = \frac{1}{2} \left( u_j^n + u_{j+1}^n \right) + \frac{1}{4} \left( \Delta x - a_{j+\frac{1}{2}}^n \Delta t \right) \left[ \left( u_x \right)_j^n - \left( u_x \right)_{j+1}^n \right] \\
- \frac{1}{2a_{j+\frac{1}{2}}^n} \left[ f \left( u_{j+\frac{1}{2}, r}^{n+\frac{1}{2}} \right) - f \left( u_{j+\frac{1}{2}, l}^{n+\frac{1}{2}} \right) \right],
\]
and
\[
w_j^{n+1} = u_j^n + \Delta t \left( a_{j-\frac{1}{2}}^n - a_{j+\frac{1}{2}}^n \right) \left( u_x \right)_j^n \\
- \frac{\lambda}{1 - \lambda \left( a_{j-\frac{1}{2}}^n + a_{j+\frac{1}{2}}^n \right)} \left[ f \left( u_{j+\frac{1}{2}, l}^{n+\frac{1}{2}} \right) - f \left( u_{j-\frac{1}{2}, r}^{n+\frac{1}{2}} \right) \right].
\]
We use the obvious notation \( u_{j+\frac{1}{2}, r}^{n+\frac{1}{2}} = u \left( x_{j+\frac{1}{2}, r}, t^{n+\frac{1}{2}} \right) \), etc. The midpoint values can be obtained from the Taylor expansions
\[
u_{j+\frac{1}{2}, l}^n = u_j^n - \Delta t f_x \left( u_{j+\frac{1}{2}, l}^n \right),
\]
where from the reconstruction we get
\[
u_{j+\frac{1}{2}, l}^n = u_j^n + \Delta x \left( u_x \right)_j^n \left( \frac{1}{2} - \lambda a_{j+\frac{1}{2}}^n \right).
\]
At this point we have the cell averages at time \( t^{n+1} \) on a non-uniform grid. We now project on the original grid points \( x_j \) by integrating over the cell \( I_j = \left[ x_{j-1/2}, x_{j+1/2} \right] \), which is composed of three parts: a left part \( \left[ x_{j-1/2}, x_{j-1/2, r} \right] \) of length \( a_{j-1/2}^n \Delta t \), a centered part on which the solution is constant, \( \left[ x_{j-1/2, r}, x_{j+1/2, l} \right] \). The centered interval is of length \( \Delta x - a_{j-1/2}^n + a_{j+1/2}^n \Delta t \). The last part, the one to the right, is \( \left[ x_{j+1/2, l}, x_{j+1/2} \right] \), and is of length \( a_{j+1/2}^n \Delta t \).
\[
u_{j+1}^n = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \bar{w} \left( \xi, t^{n+1} \right) d\xi \\
= \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}, r}} \bar{w} \left( \xi, t^{n+1} \right) d\xi + \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}, r}}^{x_{j+\frac{1}{2}, l}} \bar{w} \left( \xi, t^{n+1} \right) d\xi \\
+ \frac{1}{\Delta x} \int_{x_{j+\frac{1}{2}, l}}^{x_{j+\frac{1}{2}}} \bar{w} \left( \xi, t^{n+1} \right) d\xi.
\]
Now in the interval \( \left[ x_{j-\frac{1}{2}, r}, x_{j+\frac{1}{2}, l} \right], \ \bar{w} \left( \xi, t^{n+1} \right) = \bar{w}_{j-\frac{1}{2}}^{n+1} + \left( w_x \right)_{j-\frac{1}{2}}^{n+1} \left( \xi - x_{j-\frac{1}{2}} \right) \) where \( (w_x)^{n+1}_{j-\frac{1}{2}} \) are the reconstructed slopes computed from the \( \bar{w}_{j}^{n+1} \) via, for example, min-mod limiters. In the interval \( \left[ x_{j-\frac{1}{2}, r}, x_{j+\frac{1}{2}, l} \right], \ \bar{w} \left( \xi, t^{n+1} \right) = \bar{w}_{j}^{n+1}. \) In the interval
\[
\left[ x_{j+\frac{1}{2}, L}, x_{j+\frac{1}{2}} \right], \ \bar{w} (\xi, t^{n+1}) = \bar{w}_{j+\frac{1}{2}}^{n+1} + (w_x)_{j+\frac{1}{2}}^{n+1} \left( \xi - x_{j+\frac{1}{2}} \right). \text{ Then}\]
\[
\frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \bar{w} (\xi, t^{n+1}) \, d\xi = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left[ \bar{w}_{j+\frac{1}{2}}^{n+1} + (w_x)_{j+\frac{1}{2}}^{n+1} \left( \xi - x_{j-\frac{1}{2}} \right) \right] \, d\xi
\]
\[
= \frac{1}{\Delta x} \left[ \Delta t \bar{w}_{j+\frac{1}{2}}^{n+1} + \frac{1}{2} (w_x)_{j+\frac{1}{2}}^{n+1} \left( \Delta t \right)^2 \right]
\]
\[
= \lambda a_{j+\frac{1}{2}} \bar{w}_{j+\frac{1}{2}}^{n+1} + \frac{\Delta x}{2} \left( \lambda a_{j+\frac{1}{2}} \right)^2 \left( w_x \right)_{j+\frac{1}{2}}^{n+1},
\]

Similarly,
\[
\frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \bar{w} (\xi, t^{n+1}) \, d\xi = \left[ 1 - \lambda \left( a_{j-\frac{1}{2}} + a_{j+\frac{1}{2}} \right) \right] \bar{w}_{j+\frac{1}{2}}^{n+1}
\]
\[
\frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \bar{w} (\xi, t^{n+1}) \, d\xi = \lambda a_{j+\frac{1}{2}} \bar{w}_{j+\frac{1}{2}}^{n+1} + \frac{\Delta x}{2} \left( \lambda a_{j+\frac{1}{2}} \right)^2 \left( w_x \right)_{j+\frac{1}{2}}^{n+1},
\]
so we have
\[
\begin{aligned}
\left( u_{j+\frac{1}{2}} \right)^{n+1} & = \lambda a_{j+\frac{1}{2}} \bar{w}_{j+\frac{1}{2}}^{n+1} + \left[ 1 - \lambda \left( a_{j-\frac{1}{2}} + a_{j+\frac{1}{2}} \right) \right] \bar{w}_{j+\frac{1}{2}}^{n+1} \\
& \quad + \lambda a_{j+\frac{1}{2}} \bar{w}_{j+\frac{1}{2}}^{n+1} + \frac{\Delta x}{2} \left[ \left( \lambda a_{j+\frac{1}{2}} \right)^2 \left( w_x \right)_{j+\frac{1}{2}}^{n+1} - \left( \lambda a_{j+\frac{1}{2}} \right)^2 \left( w_x \right)_{j+\frac{1}{2}}^{n+1} \right].
\end{aligned}
\] (7.101)

Finally, the semi-discrete central scheme is obtained in the limit as \( \Delta t \to 0 \), i.e.
\[
\frac{du_j}{dt} = \lim_{\Delta t \to 0} \frac{u_{j+1}^n - u_j^n}{\Delta t} = \left( u_{j+\frac{1}{2}} \right)^{n+1} - \left( u_{j+\frac{1}{2}} \right)^n
\]
\[
= -\frac{1}{2\Delta x} \left[ f \left( u_{j+\frac{1}{2}}^n \right) - f \left( u_{j-\frac{1}{2}}^n \right) \right] + \frac{1}{2\Delta x} \left\{ a_{j+\frac{1}{2}} \left( u_{j+\frac{1}{2}}^n - u_{j+\frac{1}{2}}^n \right) - a_{j-\frac{1}{2}} \left( u_{j-\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n \right) \right\}.
\] (7.102)

With a piecewise-linear reconstruction at every time step, (7.102) is a second-order scheme.

Remarks.

1. The corresponding first-order scheme is due to Rusanov. By setting the derivatives in (7.101) to zero, we have
\[
\left( u_{j+\frac{1}{2}} \right)^{n+1} = u_j^n - \frac{\lambda}{2} \left[ f \left( u_{j+1}^n \right) - f \left( u_{j-1}^n \right) \right]
\]
\[
+ \frac{1}{2} \left[ \lambda a_{j+\frac{1}{2}} (u_{j+1}^n - u_{j}^n) - \lambda a_{j-\frac{1}{2}} (u_{j+1}^n - u_{j}^n) \right].
\]

Hence
\[
\frac{\partial u_j^n}{\partial t} = \lim_{\Delta t \to 0} \frac{u_{j+1}^n - u_j^n}{\Delta t} = -\frac{1}{2\Delta x} \left[ f \left( u_{j+1}^n \right) - f \left( u_{j-1}^n \right) \right]
\]
\[
+ \frac{1}{2\Delta x} \left[ a_{j+\frac{1}{2}} (u_{j+1}^n - u_{j}^n) - a_{j-\frac{1}{2}} (u_{j+1}^n - u_{j}^n) \right].
\]
2. The numerical dissipation for the second-order scheme we derived is $O(\Delta x^3)$. This should be compared with the numerical dissipation in the LxF scheme and the Nessyahu-Tadmor scheme, which are of order $O((\Delta x)^2/\Delta t)$, and $O((\Delta x)^4/\Delta t)$, respectively.

3. The extension to systems is straightforward and is done component-wise. Unlike the “standard” central schemes, the semi-discrete scheme we wrote here is not-staggered. The only additional information we need is an estimation of the local speed of propagation.

4. We can write the scheme in conservation form

$$\frac{du_j(t)}{dt} = -\frac{1}{\Delta x} \left( H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t) \right)$$

where

$$H_{j+\frac{1}{2}}(t) = \frac{1}{2} \left[ f \left( u_{j+\frac{1}{2}}^+(t) \right) + f \left( u_{j+\frac{1}{2}}^-(t) \right) \right] - \frac{1}{2}a_{j+\frac{1}{2}}(t) \left[ u_{j+\frac{1}{2}}^+(t) - u_{j+\frac{1}{2}}^-(t) \right],$$

and

$$u_{j+\frac{1}{2}}^+(t) = u_j^n(t) - \frac{\Delta x}{2} (u_x)_j^n(t),$$

$$u_{j+\frac{1}{2}}^-(t) = u_j^n(t) + \frac{\Delta x}{2} (u_x)_j^n(t).$$

7.2 A Semi-Discrete Scheme for Reaction-Diffusion Equations

We return now to the problem of approximating solutions to

$$u_t + f(u)_x = \frac{\partial}{\partial x} Q(u, u_x).$$

Treating $f$ and $Q$ together and repeating the arguments of §7.1 results with

$$\frac{du_j(t)}{dt} = -\frac{1}{\Delta x} \left( H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t) \right) + \frac{1}{\Delta x} \left( P_{j+\frac{1}{2}}(t) - P_{j-\frac{1}{2}}(t) \right)$$

where $H$ is as above and $P$ is an approximation of the diffusion flux. For example, if we average around $x_{j+\frac{1}{2}}$

$$P_{j+\frac{1}{2}}(t) = \frac{1}{2} \left[ Q \left( u_j(t), \frac{u_j(t) - u_{j-1}(t)}{\Delta x} \right) + Q \left( u_{j+1}(t), \frac{u_{j+1}(t) - u_j(t)}{\Delta x} \right) \right].$$
7.3 Semi-Discrete Schemes in Multiple Dimensions

An example of extending semi-discrete schemes to multiple dimensions is the two-dimensional scheme for

\[ u_t + f(u)_x + g(u)_y = 0, \]

\[
\frac{du_j(t)}{dt} = -\frac{H^x_{j+\frac{1}{2},k}(t) - H^x_{j-\frac{1}{2},k}(t)}{\Delta x} - \frac{H^y_{j,k+\frac{1}{2}}(t) - H^y_{j,k-\frac{1}{2}}(t)}{\Delta y} \\
+ \frac{P^x_{j+\frac{1}{2},k}(t) - P^x_{j-\frac{1}{2},k}(t)}{\Delta x} + \frac{P^y_{j,k+\frac{1}{2}}(t) - P^y_{j,k-\frac{1}{2}}(t)}{\Delta y}
\]

where, \( H^x, H^y \) are the numerical fluxes in the \( x \) and \( y \) directions, and \( P^x, P^y \) are the corresponding diffusion fluxes.