This homework is due Thursday November 7th at the start of class. Remember to write clearly, and justify your solutions. Please make sure to put your name on the first page and to number and staple your pages.

**Problems From The Book:** Axler Chapter 5 problems 1, 8, and 15, Chapter 8 problems 5 and 10. In addition do the following problems:

**Problem 1** (Ch. 5). Show that the sum of invariant subspaces of a linear endomorphism is still invariant.

**Proof.** Suppose that $U_1,\ldots,U_m$ are invariant subspaces of $T \in \mathcal{L}(V)$. We wish to show that $U_1 + \cdots + U_m$ is also an invariant subspace.

Consider any $\vec{v} \in U_1 + \cdots + U_m$. By definition, we can write

$$\vec{v} = \vec{u}_1 + \cdots + \vec{u}_m,$$

for some $\vec{u}_1 \in U_1, \ldots, \vec{u}_m \in U_m$.

Since $U_1,\ldots,U_m$ are invariant subspaces, by definition, we know

$$T\vec{u}_1 \in U_1 \subset U_1 + \cdots + U_m,\ldots,$$

$$T\vec{u}_m \in U_m \subset U_1 + \cdots + U_m.$$

Hence we can conclude, since $U_1 + \cdots + U_m$ is a subspace and therefore closed under addition, that

$$T\vec{v} = T\vec{u}_1 + \cdots + T\vec{u}_m \in U_1 + \cdots + U_m,$$

i.e. $U_1 + \cdots + U_m$ is an invariant subspace. \qed

**Problem 8** (Ch. 5). Find all eigenvalues and eigenvectors of the map

$$T: (z_1, z_2, \ldots) \mapsto (z_2, z_3, \ldots).$$

**Proof.** Suppose that $\vec{v} \neq 0$ is an eigenvector of $T$ with eigenvalue $\lambda$, i.e.

$$T\vec{v} = \lambda\vec{v}.$$

Writing out the components of $\vec{v}$, we have that

$$T\vec{v} = (v_2, v_3, \ldots) = (\lambda v_1, \lambda v_2, \ldots) = \lambda\vec{v}.$$

Equating components,

$$v_2 = \lambda v_1, v_3 = \lambda v_2, \ldots,$$

so

$$v_2 = \lambda v_1, v_3 = \lambda^2 v_1, \ldots, v_n = \lambda^{n-1} v_1, \ldots.$$

Hence every eigenvector of $T$ must be of the form

$$\vec{v} = (v, \lambda v, \lambda^2 v, \ldots),$$

for some $\lambda \in \mathbb{F}$.

Furthermore, we can verify that all vectors of this form are indeed eigenvectors of $T$ with eigenvalue $\lambda$, since in that case,

$$T\vec{v} = (\lambda v, \lambda^2 v, \ldots) = \lambda(v, \lambda v, \ldots) = \lambda\vec{v}.$$

This shows that we have indeed exhibited all eigenvectors and their associated eigenvalues. \qed
\textbf{Problem 15} (Ch. 5). \(p\) a complex polynomial; \(T\) a complex linear transformation. Show that \(a\) is an eigenvalue of \(p(T)\) iff \(a = p(\lambda)\) for some eigenvalue \(\lambda\).

\textit{Proof.} \(\Leftarrow\): Suppose \(\lambda\) is the eigenvalue corresponding to the nonzero eigenvector \(\vec{v} \in V\), i.e.

\[ T\vec{v} = \lambda \vec{v}. \]

Then if \(a = p(\lambda)\), we claim that \(a\) is the eigenvalue of \(p(T)\) corresponding to \(\vec{v}\). Indeed, if we write \(p(x) = anx^n + \cdots + a_0\), we get

\[ p(T)\vec{v} = (anT^n + \cdots + a_0I)\vec{v} = an\vec{v} + \cdots + a_0I\vec{v} = a_n\lambda^n\vec{v} + \cdots + a_0\vec{v} = p(\lambda)\vec{v} = a\vec{v}. \]

\(\Rightarrow\): Note that the statement is obviously true if \(p(x) \equiv 0\), since the only eigenvalues of the 0 map are \(0 = p(\lambda)\) for any \(\lambda\).

Now assume \(p(x)\) is not the zero polynomial, and suppose that \(a\) is an eigenvalue of \(p(T)\), i.e. there exists some nonzero \(\vec{v} \in V\) such that

\[ p(T)\vec{v} = a\vec{v}, \]

i.e.

\[ (p(T) - aI)\vec{v} = 0. \]

By factoring the polynomial \(p(x) - a = c(x - r_1) \cdots (x - r_n), c \neq 0\) we get that

\[ c(T - r_1I)(T - r_2I) \cdots (T - r_nI)\vec{v} = 0. \]

Now we examine this chain of maps starting from the right.

Since \(\vec{v} \neq 0\), either \((T - r_nI)\vec{v} = 0\) (i.e. \(T\vec{v} = r_n\vec{v}\) so \(r_n\) is an eigenvalue of \(T\)), or \((T - r_nI)\vec{v} \neq 0\). In the latter case, if we define \(\vec{v}' := (T - r_nI)\vec{v} \neq 0\), we get

\[ c(T - r_1I)(T - r_2I) \cdots (T - r_{n-1}I)\vec{v}' = 0. \]

By the same logic, now either \(r_{n-1}\) is an eigenvalue of \(T\) (corresponding to the NONZERO eigenvector \(\vec{v}'\)), or \(\vec{v}'' := (T - r_{n-1}I)\vec{v}' \neq 0\) and

\[ c(T - r_1I) \cdots (T - r_{n-2}I)\vec{v}'' = 0. \]

Continuing on in this fashion, we get that

\[ c(T - r_1I)\vec{v}'' \cdots = 0, \]

so in this last case \(r_1\) must be an eigenvalue of \(T\). This shows that at least one of \(r_1, \ldots, r_n\) is an eigenvalue of \(T\); whichever one it is, call it \(\lambda = r_i\).

(Alternatively, to see that at least one of \(r_1, \ldots, r_n\) must be an eigenvalue of \(T\), we can simply note that if this is not the case, i.e. if \(T - r_iI\) is injective for each \(i = 1, 2, \ldots, n\), then their composition must be injective, a contradiction.)

Now, \(\lambda\) is an eigenvalue of \(T\), and by construction it is a root of \(p(x) - a\), i.e. \((x - \lambda)|p(x) - a\) which implies that in fact

\[ p(\lambda) - a = 0, \quad \text{so} \quad p(\lambda) = a, \]

as desired. \(\square\)

\textbf{Problem 5} (Ch. 8). Suppose we have a positive integer \(n\) such that \((ST)^n = 0\). Since multiplying any operator by a zero operator results in the zero operator, we also get:

\[ T(ST)^nS = 0. \]

However this is precisely \((TS)^{n+1} = 0\). And thus \(TS\) is also nilpotent.
Problem 10 (Ch. 8). This is not true. For example, take $V$ to be a two-dimensional complex vector space spanned by $\{\vec{v}_1, \vec{v}_2\}$. Then define $T : V \rightarrow V$ to be
\[
T(\vec{v}_1) = 0, T(\vec{v}_2) = \vec{v}_1,
\]
and extends linearly to all $V$.

Then the null space of $T$ is the subspace $V_1$ spanned by $\{\vec{v}_1\}$, and the range of $T$ is also $V_1$. So the null space and range of $T$ don’t sum up to the whole space. And in fact they have non-zero intersection as well.

Question 1. Let $T_1, T_2 \in \mathcal{L}(\mathbb{C}^2)$ be given by the matrices
\[
T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
What are the invariant subspaces of $T_1$ and $T_2$?

Proof. $T_1$: $\{0\}$ is the only possible 0-dimensional subspace, and it is clearly invariant. Similarly, $\mathbb{C}^2$ is the only possible 2-dimensional subspace and is also clearly invariant.

1-dimensional subspaces are spaces spanned by single eigenvectors (see 5.2, 5.3 in the book). And $T_1$ is already in Jordan normal form, so we can note that the only eigenvector is $[1, 0]$; thus the only 1-dimensional subspace is $\mathbb{C} \times 0 = \{(z, 0) : z \in \mathbb{C}\}$.

$T_2$: Since this is the identity transformation, every vector is fixed by $T_2$, so every subspace is invariant: $\{0\}, \mathbb{C}^2, \{\lambda(z_1, z_2) : \lambda \in \mathbb{C}\}$ for each $(z_1, z_2) \neq (0, 0) \in \mathbb{C}$. \hfill \qed

Question 2. Let $T \in \mathcal{L}(P_n(\mathbb{C}))$ be given by
\[
T(p) = p + p' + p''.
\]
What is the Jordan form of $T$? (Hint: show that $(T - I)^{n+1} = 0$ and that $(T - I)^{n}$ is not)

Ans: It looks like
\[
\begin{bmatrix}
1 & 1 & 0 \\
n & \ddots & \ddots \\
0 & \ddots & 1 \\
0 & \ddots & 1 \\
0 & 1 & 1
\end{bmatrix},
\]
with 1’s on the diagonal and 1 square above the diagonal, and 0’s everywhere else.

Proof. Note that $T - I$ is given by
\[
T - I : p \mapsto p' + p''.
\]
When applied to a polynomial $p$ of degree $n$, we find that $p' + p''$ has degree at most $n - 1$. Hence it follows that $(T - I)^n p$ must be degree at most 0, i.e. a constant function, so $(T - I)^{n+1}$ is the zero map.

On the other hand, $(T - I)^n$ is not the zero map, since $(T - I)^n(x^n) = n! \neq 0$.

(To see this, one can note that $(T - I)$ is like “multiplication” by the operator
\[
\frac{d}{dx} + \frac{d^2}{dx^2} = \frac{d}{dx} \left( 1 + \frac{d}{dx} \right),
\]
and hence we can apply binomial expansion to see
\[
(T - I)^n = \frac{d^n}{dx^n} \left( 1 + \sum_{k=1}^{n} \binom{n}{k} \frac{d^k}{dx^k} \right) + \sum_{k=1}^{n} \binom{n}{k} \frac{d^k}{dx^k} + \frac{d^{n-1}}{dx^{n-1}} + \frac{d^n}{dx^n}.
\]

Alternatively, one can argue by degrees; if $p$ is degree $k$, then $p'$ must be exactly degree $k - 1$, and $p''$ must be exactly degree $k - 2$, so $p' + p''$ must still be degree $k - 1$.)

This implies that $m(x^n) = n$, and thus, by Lemma 8.40, there exists a basis for $V$ of the form
\[
\{ x^n, (T - I)(x^n), (T - I)^2(x^n), \ldots, (T - I)^n(x^n) \}.\]
Using these basis elements as a Jordan basis, we immediately find that $T - I$ looks like
\[
\begin{bmatrix}
0 & 1 & 0 \\
& & \\
& & \\
0 & & 0
\end{bmatrix}.
\]
Adding $I$ to this matrix then finishes the proof.

**Question 3.** Let $S, T \in \mathcal{L}(V)$ be operators on a finite dimensional, non-zero vector space $V$ over $\mathbb{C}$. Suppose that $S$ and $T$ commute, namely that $ST = TS$. Show that there is a non-zero vector $\vec{v} \in V$ so that $\vec{v}$ is an eigenvector of both $S$ and $T$. (Hint: Find a non-zero eigenvector of $T$, show that the eigenspace is $S$-invariant, and find an eigenvector of $S$ within that eigenspace).

**Proof.** Since the vector space is over the algebraically closed field $\mathbb{C}$, the linear operator $T$ has an eigenvalue $\lambda$. Denote by $V_\lambda$ the subspace of all vectors who is an eigenvector of $T$ with eigenvalue $\lambda$. That is, $V_\lambda = \{ \vec{v} \in V : T\vec{v} = \lambda \vec{v} \}$.

Clearly $V_\lambda$ is a subspace of $V$ (You can easily verify $V_\lambda$ is closed under addition and scalar multiplication). We claim that $V_\lambda$ is an invariant subspace of $S$, namely $S(V_\lambda) \subset V_\lambda$. This is obvious: for any element $\vec{v} \in V_\lambda$, since $ST = TS$, we have $T(S(\vec{v})) = S(T(\vec{v})) = S(\lambda \vec{v}) = \lambda S(\vec{v})$.

So $S(\vec{v}) \in V_\lambda$.

We restrict $S$ on the subspace $V_\lambda$ to get $S|_{V_\lambda} \in \mathcal{L}(V_\lambda)$. Since $V_\lambda$ is a complex vector space and $S|_{V_\lambda}$ is a linear operator on it, it has an eigenvalue $\mu \in \mathbb{C}$. That is, there is a vector $\vec{u} \in V_\lambda$ such that $S|_{V_\lambda}(\vec{u}) = \mu \vec{u}$.

This $\vec{u} \in V_\lambda$ is the vector we want, for we have
\[
T(\vec{u}) = \lambda \vec{u}, \quad S(\vec{u}) = S|_{V_\lambda}(\vec{u}) = \mu \vec{u},
\]
as desired.

**Question 4.** Let $T \in \mathcal{L}(\mathbb{C}^n)$ be an operator on $\mathbb{C}^n$. Suppose that $\mathbb{C}^n$ has a basis of eigenvectors of $T$, all of whose eigenvalues have absolute value strictly less than 1. Show that
\[
\lim_{n \to \infty} T^n(\vec{v}) = \vec{0}
\]
for all $\vec{v} \in \mathbb{C}^n$. You may use the fact that $\lim_{n \to \infty} \lambda^n = 0$ if $|\lambda| < 1$.

**Proof.** Suppose $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of eigenvectors of $T$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Now for any vector $\vec{v} \in \mathbb{C}^n$, there are complex numbers $a_1, \ldots, a_n$ such that
\[
\vec{v} = a_1 \vec{v}_1 + \ldots + a_n \vec{v}_n.
\]

By definition we have $T(\vec{v}_j) = \lambda_j \vec{v}_j$ for $j = 1, \ldots, n$. So inductively we have
\[
T^k(\vec{v}_j) = \lambda_j^k \vec{v}_j, \quad j = 1, \ldots, n.
\]

So
\[
T^k(\vec{v}) = T^k(a_1 \vec{v}_1 + \ldots + a_n \vec{v}_n)
= a_1 T^k(\vec{v}_1) + \ldots + a_n T^k(\vec{v}_n)
= a_1 \lambda_1^k \vec{v}_1 + \ldots + a_n \lambda_n^k \vec{v}_n.
\]
With the assumption that $|\lambda_j| < 1$ for $j = 1, \ldots, n$, we have
\[
\lim_{k \to \infty} \lambda_j^k = 0, \quad j = 1, \ldots, n.
\]
So we conclude
\[
\lim_{k \to \infty} a_1\lambda_1^k\vec{v}_1 + \ldots + a_n\lambda_n^k\vec{v}_n = 0,
\]
or equivalently
\[
\lim_{k \to \infty} T^k(\vec{v}) = 0.
\]

**Question 5.** Let $N \in \mathcal{L}(V)$ be a nilpotent operator. Let $p(x)$ be a polynomial. Show that $p(N)$ is nilpotent if and only if $p(0) = 0$.

**Proof.** "$\Leftarrow$" is easier. Suppose we have $p(0) = 0$. Take $n$ to be an integer satisfying $N^n = 0$. Since $p(0) = 0$ we can factor $p$ through $x$, namely there is a polynomial $q(x)$ such that $p(x) = xq(x)$. Therefore $p(x)^n = x^n q(x)^n$, and we have
\[
p(N)^n = N^n q(N)^n = 0,
\]
which means $p(N)$ is nilpotent.

"$\Rightarrow$": we give two proofs.

**Proof 1.** The first proof is based on the fact: a linear operator $T \in \mathcal{L}(V)$ is nilpotent if and only if $T$ has no nonzero eigenvalue (or equivalently 0 is the only eigenvalue of $T$).

To see this, if there is some integer $n$ such that $T^n = 0$, then for any eigenvalue $\lambda$ of $T$, clearly $\lambda^n$ is an eigenvalue of $T^n$, so must be 0. This gives $\lambda^n = 0$, and hence $\lambda = 0$. Conversely, if 0 is the only eigenvalue of $T$, then by proposition 5.16, $T$ has an upper-triangular matrix $A$ with respect to some basis of $V$. Now the eigenvalue of $T$ are all 0 means that the diagonal entries of $A$ are all 0. Now suppose $A$ is an $k$-by-$k$ upper-triangular matrix, then $A^k = 0$. So we have $T^k = 0$, or $T$ is nilpotent.

Now suppose $p(N)$ is nilpotent. From above we see that 0 is the only eigenvalue of $p(N)$.

We prove by contradiction. If $p(0) \neq 0$, then there is some nonzero constant $c$ and some polynomial $q(x)$ such that $p(x) = xq(x) + c$. Suppose $N^m = 0$. Now pick an nonzero eigenvector of $p(N)$, say $\vec{v}$. Then the eigenvalue of $\vec{v}$ is 0, so we get
\[
(Nq(N) + c \cdot \text{Id})(\vec{v}) = 0, \quad Nq(N)\vec{v} = -c\vec{v}.
\]
Also $Nq(N) = -c \cdot \text{Id}$, so
\[
(Nq(N))^m(\vec{v}) = (-c)^m(\vec{v}).
\]
Since $N^m$ is the zero operator, we have
\[
(-c)^m(\vec{v}) = 0, \quad \vec{v} = 0
\]
by the fact that $c \neq 0$. Contradiction.

**Proof 2.** The second proof uses the so-called Bézout theorem:

Let $k$ be a field, and we are given two polynomials $P(x), Q(x) \in k[x]$. Then there exist two polynomials $f(x), g(x) \in k[x]$ such that
\[
P(x)f(x) + Q(x)g(x) = (P,Q)(x),
\]
where $(P,Q)(x)$ is the greatest common divisor of $P$ and $Q$.

Now suppose $p(N)$ is nilpotent. Let $k, m$ be two integers with $p(N)^k = 0, N^m = 0$. We prove by contradiction.

If $p(0) \neq 0$, then $p(x)$ is relatively prime to $x$. Therefore $p(x)^k$ is relatively prime to $x^m$. Therefore by Bézout theorem, there exist two polynomials $f(x), g(x)$ such that
\[
p(x)^k f(x) + x^m g(x) = 1.
\]
Taking the operator $N$ into the above expression, we get

$$p(N)^k f(N) + N^m g(N) = Id.$$  

This gives us a contradiction under the assumption that $p(N)^k = 0$ and $N^m = 0$.  

**Question 6.** *Approximately how much time did you spend working on this homework?*