Warm-up

Problem 1 Simplify the following expressions:

a) $2x + 3x$, and $x^2 \cdot x^3$;
b) $17x + 5x$, and $x^{17} \cdot x^5$;
c) $10x - 2x$, and $\dfrac{x^{10}}{x^2}$.

Do you notice anything?

Problems

Problem 2 Define a function that for each whole number n (the whole numbers are $\{0, 1, 2, \ldots\}$) does $f(n) = x^n$.

a) Compute $f(2)$, $f(3)$ and $f(5)$.
b) Show that for all whole numbers n and m, we have that $f(n + m) = f(n) \cdot f(m)$.

Problem 3 What is $f(0)$? Explain why this makes sense within the context of the mathematical properties of addition and multiplication.

We shall now require a definition. For any number a, there is a unique number b such that $a \cdot b = 1$, and we call this number the *multiplicative inverse* of a. We usually write

$$b = \frac{1}{a}.$$

One implication of this is that given a number a, if we have a number c such that $a \cdot c = 1$, then $c = \frac{1}{a}$. (For bonus points: Given a number a, what is the additive inverse of a?)

Problem 4 We now try to extend our function f to all integers n (the integers are $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$) by still setting $f(n) = x^n$. The problem with doing this is that it might not be clear what x^{-2} or other such expressions might mean. Thankfully, we can use our function above to "show" that $x^{-2} = \dfrac{1}{x^2}$ in the following manner:
We have

$$\begin{aligned}
1 &= f(0) && \text{by definition of } f\\
&= f((-2) + 2) && \text{since } (-2) + 2 = 0\\
&= f(-2) \cdot f(2) && \text{by our proof above}\\
&= x^{-2} \cdot x^2 && \text{by definition of } f
\end{aligned}$$

so that $x^{-2} \cdot x^2 = 1$. By the multiplicative inverse property, this means that $x^{-2} = \dfrac{1}{x^2}$. Use the same technique to "show" that $x^{-3} = \dfrac{1}{x^3}$. Write out all your steps nicely, with justification as above.

Looking Forward

We require now another definition. For any positive number a, there is a unique positive number b such that $b^2 = a$, and we call this number the *principal square root* of a. We usually write

$$b = \sqrt{a}.$$

One implication of this is that given a positive number a, if we have a positive number c such that $c^2 = a$, then $c = \sqrt{a}$.

Problem 5 We now try to extend our function to all rational numbers q (the rational numbers are just all fractions) by still setting $f(q) = x^q$. The problem now is to make sense of $x^{1/2}$ and other such expressions. Just as before, we can use our function above to "show" that $x^{1/2} = \sqrt{x}$. Do it! (Hint: do not think of 1 as $\frac{1}{2} \cdot 2$, but rather think of 1 as $\frac{1}{2} + \frac{1}{2}$.)